Baire class of completely additive systems of Baire sets

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Let X be any set and let $\mathcal{B}_0$ be any family of subsets of X. Let $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha ; \alpha < \omega_1\}$ be defined as follows:

$\mathcal{B}_\alpha$ is the collection of all countable unions or intersections of elements of $\bigcup\{\mathcal{B}_\beta ; \beta < \alpha\}$ according to as α is odd or even.

For $B \in \mathcal{B}$ put $\text{class}(B) = \min\{\alpha ; B \in \mathcal{B}_\alpha\}$.

As an example we can have X a topological space and $\mathcal{B}_0$ the family of zero sets.

It was proved in [P] that if a disjoint family $\{X_a ; a \in A\}$ of subsets of X is completely additive (i.e. if the union of each subfamily of $\{X_a\}$ belongs to $\mathcal{B}$) then the family $\{X_a\}$ ranges in some $\mathcal{B}_\alpha$. This, together with a result of Hansel, implies that the family of all unions of $\{X_a\}$ ranges in some $\mathcal{B}_\alpha$ provided that X is a metrizable absolute Souslin space and $\mathcal{B}_0$ is the family of zero sets (cf. [F]).

In this note we prove that the last result holds, under some cardinality assumptions, even without the hypotheses that X is an absolutely Souslin space.

Recall that accessible cardinal numbers are defined as follows

$\aleph_0$ is accessible

the successor of an accessible number is accessible

the supremum of an accessible number of accessible numbers is accessible.

<u>Theorem.</u> Let $\{X_a;\ a \in A\}$ be a completely additive family of disjoint subsets of X. If card$A$ is accessible then the family of all unions of subfamilies of $\{X_a\}$ ranges in some $\mathcal{B}_\alpha$.

Proof. For $C \subset A$ put $\mu(C) = 0$ if the family of all unions of subfamilies of $\{X_a;\ a \in C\}$ ranges in some $\mathcal{B}_\alpha$ and $\mu(C) = 1$ otherwise. The map μ fulfils the countable chain condition (for disjoint C_ε ($\varepsilon < \omega_1$) with $\mu(C_\varepsilon) = 1$ one could choose $a_\varepsilon \in C_\varepsilon$ with class$(X_{a_\varepsilon}) \geq \varepsilon$; the existence of the family $\{X_{a_\varepsilon};\ \varepsilon < \omega_1\}$ contradicts the result of [P]) and $\mu(\bigcup_{i=1}^{\infty} C_i) \leq \sum \mu(C_i)$. Now the method of Ulam's proof [U] shows that $\mu(A) = 0$.

<u>Corollary.</u> Assume $2^{\aleph_1}$ accessible. If $\{X_a;\ a \in A\}$ is a disjoint completely additive family of subsets of X then the family of all unions of subfamilies of $\{X_a\}$ ranges in some $\mathcal{B}_\alpha$.

Proof. Suppose, on the contrary, that for some completely additive disjoint family of subsets of X the family of all unions of subfamilies of $\{X_a\}$ ranges in no $\mathcal{B}_\alpha$. For every $\varepsilon < \omega_1$ let $C_\varepsilon \subset A$ such that class$(\bigcup_{a \in C_\varepsilon} X_a) \geq \varepsilon$. For $a,b \in A$ let $a \sim b$ <u>if</u> $a \in C_\varepsilon$ is equivalent to $b \in C_\varepsilon$ and let D be the set of all classes of this equivalence. Then card $D \leq 2^{\aleph_1}$ and one obtains a contradiction with the preceding Theorem considering the family $\{Y_d;\ d \in D\}$ where $Y_d = \bigcup_{a \in d} A_d$.

References

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