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Baire class of completely additive systems of Baire sets

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SEMINAR UNIFORM SPACES (1977)

Paire class of completely additive systems of Baire sets Devid-Preiss

Let \mathfrak{T} be any set and let $\mathcal{O}_{\mathfrak{p}}$ be any family of subsets of \mathfrak{T} . Let $\mathcal{O} = \mathcal{V}(\mathcal{O}_{\mathfrak{p}}) \times \mathcal{O}_{\mathfrak{p}}$ be defined as follows:

 \mathcal{B}_{∞} is the collection of all countable unions or intersections of elements of $\bigcup \{\mathcal{B}_{\mathcal{B}}, \mathcal{B}_{\infty} \alpha\}$ according to as ∞ is odd or even.

For Bess put class(B) = min $\{\alpha ; B \in \mathcal{B}_{\alpha} \}$

As an example we can have X a topological space and \mathcal{S}_{\wp} the family of zero sets.

It was proved in [P] that if a disjoint family $\{X_B; a \in A\}$ of subsets of X is completely additive (i.e. if the union of each subfamily of $\{X_B\}$ belongs to \mathcal{B}) then the family $\{X_B\}$ ranges in some \mathcal{B}_X . This, together with a result of Hansel, implies that the family of all unions of $\{X_B\}$ ranges in some \mathcal{B}_X provided that X is a metrizable absolute Souslin space and \mathcal{B}_X is the family of zero sets (cf. [F]).

In this note we prove that the last result holds, under some cardinality assumptions, even without the hypotheses that X is an absolutely Souslin space.

Recall that accessible cardinal numbers are defined as foll-

the successor of an accessible number is accessible the supremum of an accessible number of accessible numbers accessible.

Theorem. Let $\{X_a; a \in A\}$ be a completely additive family of disjoint subsets of X. If cardA is accessible then the family of all unions of subfamilies of $\{X_a\}$ ranges in some \mathcal{S}_{α} .

Proof. For $C \subset A$ put $\mathcal{M}(C) = 0$ if the family of all unions of subfamilies of $\{X_{\underline{a}}; a \in C\}$ ranges in some $G_{\underline{a}}$ and $\mathcal{M}(C) = 1$ otherwise. The map \mathcal{M} fulfils the countable chain condition (for disjoint $C_{\underline{c}}$ ($\underline{c} < \omega_{1}$) with $\mathcal{M}(C_{\underline{c}}) = 1$ one could choose $a_{\underline{c}} \in C_{\underline{c}}$ with class $(X_{\underline{a}_{\underline{c}}}) \ge \underline{c}$; the existence of the family $\{X_{\underline{a}_{\underline{c}}}; \underline{c} = \omega_{1}\}$ contradicts the result of [P]) and $\mathcal{M}(\bigcup_{\underline{c}} C_{\underline{i}}) \le \underline{c}$; $\mathcal{M}(C_{\underline{i}})$. Now the method of Ulam's proof [U] shows that $\mathcal{M}(A) = 0$.

Corollary. Assume 2^{N_4} accessible. If $\{X_a; a \in A\}$ is a disjoint completely additive family of subsets of X then the family of all unions of subfamilies of $\{X_a\}$ ranges in some \mathcal{B}_{∞} .

Proof. Suppose, on the contrary, that for some completely additive disjoint family of subsets of X the family of all unions of subfamilies of $\{X_a\}$ ranges in no \mathcal{B}_{α} . For every $\epsilon < \omega_{\alpha}$ let $C_{\epsilon} \subset A$ such that $\operatorname{class}(\bigcup_{\alpha \in C_{\epsilon}} X_{\alpha}) \geq \epsilon$. For $a,b \in A$ let $a \sim b$ if $a \in C_{\epsilon}$ is equivalent to $b \in C_{\epsilon}$ and let D be the set of all classes of this equivalence. Then $\operatorname{card} D \leq 2^{\aleph_1}$ and one obtains a contradiction with the preceding Theorem considering the family $\{Y_a; d \in D\}$ where $Y_d = \bigcup_{\alpha \in G} A_d$.

References

- [F] Frolik Z.: Baire sets and uniformities on complete metric spaces, Comment.Math.Univ.Carolinae 13(1972),137-147
- [P] Preiss D.: Completely additive disjoint system of Baire sets is of bounded clas, Comment.Math.Univ.Carolinae 15(1974),341-344 [U] Ulam S.: Zur Majtheorie in der allgemeinen Mengenlehre, Fund. Math. 16(1930),140-150