Václav Kučera

The Bernoulli shift as a basic chaotic dynamical system

In: Jan Chleboun and Pavel Kůs and Petr Přikryl and Miroslav Rozložník and Karel Segeth and Jakub Šístek and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Hejnice, June 24-29, 2018. Institute of Mathematics CAS, Prague, 2019. pp. 89–98.

Persistent URL: `http://dml.cz/dmlcz/703078`

Programs and Algorithms of Numerical Mathematics 19
J. Chleboun, P. Kůs, P. Přikryl, M. Rozložník, K. Segeth, J. Šístek, T. Vejchodský (Eds.)
Institute of Mathematics CAS, Prague 2019

THE BERNOULLI SHIFT AS A BASIC CHAOTIC DYNAMICAL SYSTEM

Václav Kučera

Charles University, Faculty of Mathematics and Physics
Sokolovská 83, 186 75 Praha, Czech Republic
kucera@karlin.mff.cuni.cz

Abstract: We give a brief introduction to the Bernoulli shift map as a basic chaotic dynamical system. We give several examples where the iterates of a mapping can be understood using the Bernoulli shift. Namely, the iteration of real interval maps and iteration of quadratic functions in the complex plain.

Keywords: Bernoulli shift, chaotic dynamics, recurrence
MSC: 37D45, 37F45, 37E05

1. Introduction

The field of chaotic dynamical systems has become a well established part of mathematics with a huge body of work being dedicated in the past decades to the study of chaotic phenomena in various systems ranging from pure mathematics to physics, computer science and even humanities. Due to this diversity, 'chaos theory', rather than being one compact theory with well defined boundaries, is a conglomerate of many sub-theories from different areas which share some common traits.

In this brief note we present an overview of some of the concepts which appear in the theory of chaotic discrete dynamical systems, especially holomorphic dynamics, iteration of real functions and symbolic dynamics. Here the key concept is *iteration* and its qualitative study. This is common ground with the field of numerical mathematics, however in the first case the aim is to build deeper, more qualitative and global theories of iteration. As an example of the intersection of these two fields, we can consider e.g. Newton's method. In this short note we give an overview of the so-called Bernoulli shift mapping which is a basic chaotic dynamical system, which can be easily analyzed and appears in many contexts connected to iteration.

A basic tool that we will use is the concept of *conjugacy*: two mappings $f\colon A \to A$ and $g\colon B \to B$ are (topologically) conjugate if there exists a homeomorphism (i.e.,

DOI: 10.21136/panm.2018.10

a continuous mapping with continuous inverse) $h\colon A \to B$ such that $h \circ f = g \circ h$. Here composition is taken in the order $(h \circ f)(x) = h(f(x))$ and $(g \circ h)(x) = g(h(x))$. Since h has a continuous inverse, we can write $f = h^{-1} \circ g \circ h$. The key observation is that the iterates of f and g are also in the same relation: we have $f^{\circ n} = h^{-1} \circ g^{\circ n} \circ h$. These relations are usually expressed in the form of a *commutative diagram* for clarity:

$$\begin{tikzcd} B \arrow[d, "g"'] & A \arrow[l, "h"'] \arrow[d, "f"] \\ B & A \arrow[l, "h"'] \end{tikzcd}$$

The study of the behavior of f under iteration can thus be investigated by studying the iterates of g. This is advantageous if it is possible for a given 'complicated' f to construct a 'simpler' g for which the behavior of iterates is well understood.

2. The squaring function in $\mathbb{C}$

As a simple first example, we consider the squaring function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$ and its iterates. In other words, we are interested in the behavior of the sequence $\{z_n\}_{n=0}^{\infty}$ given by

$$z_{n+1} = z_n^2 \tag{1}$$

for a given $z_0 \in \mathbb{C}$. In the context of numerical mathematics, the iterative process (1) is simply Newton's method applied to the solution of the nonlinear equation

$$\frac{z}{1-z} = 0. \tag{2}$$

To analyze the behavior of (1) as $n \to \infty$, it is convenient to use the polar form – then if $z = r\mathrm{e}^{2\pi\mathrm{i}\alpha}$, we get $z^2 = r^2\mathrm{e}^{2\pi\mathrm{i}(2\alpha)}$. Hence trivially

$$\begin{aligned} &\text{if } |z_0| < 1 \text{ then } z_n \to 0, \\ &\text{if } |z_0| > 1 \text{ then } z_n \to \infty. \end{aligned} \tag{3}$$

We note that 0 and ∞ are the two roots of equation (2), where the latter is taken in the sense of compactification of $\mathbb{C}$ to the Riemann sphere. Hence (3) gives us the expected quadratic convergence of Newton's method to the two roots.

The interesting question however is what happens on the unit circle when $|z_0| = 1$. The answer is that surprisingly complicated behavior occurs.

2.1. Behavior of the iterates on the unit circle

If $|z_0| = 1$, we trivially have $|z_n| = 1$ for all n. Hence we can write the polar form $z_n = \mathrm{e}^{2\pi\mathrm{i}\alpha_n}$, where $\alpha_n \in [0, 1)$. Applying the squaring function gives $z_{n+1} = z_n^2 = \mathrm{e}^{2\pi\mathrm{i}(2\alpha_n)}$. Therefore we see that the action of f on z_n can be viewed as simple doubling of the angle α_n modulo one to ensure that the angles remain in $[0,1)$. We can write this in the form of the commutative diagram

$$\begin{tikzcd} \{|z|=1\} \arrow[d, "z\mapsto z^2"'] & {[0,1)} \arrow[l, "e^{2\pi i\alpha}\mapsfrom\alpha"'] \arrow[d, "\alpha\mapsto 2\alpha\,(\mathrm{mod}\,1)"] \\ \{|z|=1\} & {[0,1)} \arrow[l, "e^{2\pi i\alpha}\mapsfrom\alpha"'] \end{tikzcd} \tag{4}$$

This states that results concerning the iteration of z^2 on the unit circle in the left-hand side can be obtained by analogous results on the iteration of $\alpha \mapsto 2\alpha\,(\mathrm{mod}\,1)$ on the unit interval in the right half of the diagram using the mapping $\alpha \mapsto e^{2\pi i\alpha}$. We will now analyze the angle doubling function in detail.

3. Bernoulli shift

A convenient way how to understand the dynamics of the iterative process $\alpha_{n+1} = 2\alpha_n\,(\mathrm{mod}\,1)$ on $[0,1)$ is writing α in its binary representation. Then if we have e.g. $\alpha_n = 0.11011\ldots$, then $\alpha_{n+1} = 0.1011\ldots$. Thus angle doubling works by omitting the first binary digit after the radix point. Thus we can view the action of angle doubling simply as a left shift on infinite sequences of $\{0,1\}$-symbols. Formally, let

$$\Sigma_2 = \{0,1\}^{\mathbb{N}}$$

be the space of *infinite binary sequences*. Then we define the *Bernoulli shift* as the mapping $\sigma\colon \Sigma_2 \to \Sigma_2$ defined by

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots), \tag{5}$$

where $\mathbf{s} = s_0 s_1 s_2 \ldots \in \Sigma_2$. As we have seen, the mapping σ is conjugate to the function z^2 on the unit circle in $\mathbb{C}$. The advantage is that the basic dynamics of σ are very easy to understand.

3.1. Periodic points

Obviously, the only fixed points of σ are the sequences $0000\ldots$ and $1111\ldots$ corresponding to the binary expansions of $\alpha = 0$ and $\alpha = 1 \equiv 0\,(\mathrm{mod}\,1)$ which gives the only fixed point 1 of z^2 on the unit circle.

It is easy to construct points of various periods. For example if we want *period two* points, i.e., z_0 such that $((z_0)^2)^2 = z_0$, we have two possible choices $\mathbf{s} = \overline{01}$ and $\mathbf{s} = \overline{10}$, where the horizontal line indicates infinite repetition. These sequences correspond to the binary expansion of $1/3$ and $2/3$, respectively. Then e.g. for $\alpha = 1/3$ we have $2\alpha\,(\mathrm{mod}\,1) = 2/3$ and $4\alpha\,(\mathrm{mod}\,1) = 1/3 = \alpha$, as expected. Finally, these angles correspond to

$$z = e^{2\pi i/3} = \frac{1}{2}(-1+\sqrt{3}i),$$
$$z = e^{2\pi i(2/3)} = \frac{1}{2}(-1-\sqrt{3}i),$$

either of which satisfy $(z^2)^2 = z$.

To obtain points of *period three*, we can take $\mathbf{s} = \overline{001}$ and $\mathbf{s} = \overline{011}$. These correspond to the binary representations of $1/7$ and $3/7$, respectively, which in turn lead to the two distinct cycles under doubling

$$\frac{1}{7} \to \frac{2}{7} \to \frac{4}{7} \to \frac{1}{7}, \qquad \frac{3}{7} \to \frac{6}{7} \to \frac{5}{7} \to \frac{3}{7}.$$

The corresponding complex numbers do not have a simple closed form, but are approximately $0.623 + 0.782\mathrm{i}$ and $-0.901 + 0.434\mathrm{i}$, respectively. In general we have the following.

Theorem 1. *All angles $\alpha \in [0, 1)$ with period $p \in \mathbb{N}$ under the angle doubling map have the form*

$$\frac{a}{2^p - 1}, \tag{6}$$

where $a \in \{0, \ldots, 2^p - 2\}$.

Proof. Necessarily, α has to have the binary form $0.\overline{s_1 s_2 \ldots s_p}$ where each $s_i \in \{0, 1\}$. Let us define the binary number $a = s_1 s_2 \ldots s_p$. Then $a \in \{1, \ldots, 2^p - 2\}$, since the possibility $a = 2^p - 1$ is excluded, since then $a = 11 \ldots 1$ in binary form, which gives $\alpha = 1$.

Thus we have $\alpha = a \cdot B$, where

$$B = 0.\underbrace{\overline{00 \ldots 01}}_{p}$$

in binary form. An easy computation gives $B = \sum_{i=1}^{\infty} 2^{-pi} = 1/(2^p - 1)$ and $a \in \{0, \ldots, 2^p - 2\}$. This completes the proof. $\square$

In the previous theorem, a periodic point of period p can also have a smaller period corresponding to a divisor of p. If one wishes to construct a point of *exact period p*, one can take e.g. $1/(2^p - 1)$.

Corollary 2. *Periodic points of the mapping $z \mapsto z^2$ are dense in the unit circle.*

3.2. Other interesting points

Similarly, we can construct *preperiodic points*, i.e., points such that $f^{\circ(N+p)}(z) = f^{\circ N}(z)$, i.e. z becomes periodic with period p after N iterations. Clearly, such numbers correspond to sequences in Σ_2 of the form $\mathbf{s} = t_1, \ldots t_N \overline{s_1 \ldots s_p}$, where each $t_i, s_i \in \{0, 1\}$.

More interesting are the points that are not eventually periodic. For example, consider $\mathbf{s} = 01001000100001\ldots$. Then $\sigma(\mathbf{s}) = 1001\ldots$, $\sigma^{\circ 3}(\mathbf{s}) = 10001\ldots$, $\sigma^{\circ 7}(\mathbf{s}) = 100001\ldots$ etc. This corresponds to the binary numbers $0.1001\ldots$, $0.10001\ldots$, $0.100001\ldots$, etc., the limit of which is the binary number 0.1, i.e. 2^{-1}. Thus 2^{-1} is an accumulation point under the angle doubling map for this particular choice of $\mathbf{s}$. In complex numbers, this corresponds to the accumulation point $\mathrm{e}^{2\pi \mathrm{i} 2^{-1}}$ for the sequence z_n. Similarly, $0.01, 0.001, \ldots$ are also accumulation points for this sequence corresponding to the accumulation points for the complex sequence z_n of the form $e^{2\pi i 2^{-p}}$ for all $p \in \mathbb{N}$. Similarly, zero is also an accumulation point. Thus z_n has a countable set of accumulation points but does not have a limit.

It is also easy to construct z_0 such that $\{z_n\}_{n=0}^{\infty}$ is *dense* in the unit circle. One such example is the following 'dictionary' of binary sequences of increasing length:

$$\mathbf{s} = \underbrace{0\ 1}_{1-digit}\ \underbrace{00\ 01\ 10\ 11}_{2-digit}\ \underbrace{000\ 001\ 010\ \ldots}_{3-digit}\ \ldots \in \Sigma_2. \tag{7}$$

This sequence corresponds to the angle $\alpha \approx 0.27638711728$ and the complex number $z \approx -0.1650366 + 0.9862874\mathrm{i}$. It is easy to see that iteration of this angle will fill out $[0, 1)$ densely: If we choose some $\beta \in [0, 1)$ and $N \in \mathbb{N}$, eventually under the doubling map some iteration of α will coincide with β to N binary digits. Choosing N sufficiently large, we can approach β to within an arbitrarily small tolerance.

4. Chaotic dynamics

The Bernoulli shift, and equivalently the angle doubling mapping and squaring function on the unit circle, are chaotic mappings in the following sense.

Definition 3. *Let X be a metric space. A mapping $f\colon X \to X$ is called* chaotic*, if it has the following properties:*

1. Transitivity*: For all nonempty open sets $U, V \subset X$ there exists $n \geq 0$ such that $f^{\circ n}(U) \cap V \neq \emptyset$.*
2. Sensitive dependence on initial conditions*: There exists $c > 0$ such that for any $x \in X$ and any neighborhood U of x there exists $y \in U$ and $n \in \mathbb{N}$ such that $\|f^{\circ n}(x) - f^{\circ n}(y)\| > c$.*
3. *Periodic points of f are* dense *in X.*

The definition of chaotic mappings has the following interpretation. *Sensitive dependence on initial conditions* essentially means that the system is unpredictable, even though it is fully deterministic. One would need to know the initial condition to within infinite precision to be able to predict the behavior of the iterates. This gives rise to the problem of how to simulate such systems on a computer. *Transitivity* roughly means that the system cannot be decomposed into smaller independent (and hopefully simpler) subsystems, since everything is 'mixed' together. And finally the *density of periodic orbits* states that amongst this unpredictable complicated behavior there is an element of regularity.

Definition 3 is so-called *Devaney's definition of chaos*, cf. [2]. It has been proven in [1] that for continuous mappings it is not necessary to assume sensitive dependence on initial conditions – this follows from the other two requirements. Moreover, if X is an interval, it can be shown that only transitivity is necessary and sufficient for a continuous function to be chaotic in the sense of Definition 3. Finally, we note that there are several other definitions of chaotic mappings (Li-Yorke chaos, Ruelle-Takens, etc.). For an overview and a study of their mutual relations see e.g. [6].

The three requirements of the Definition 3 are easily seen to be satisfied by our considered mapping.

Theorem 4. *The mapping $f\colon z \mapsto z^2$ is chaotic on the unit circle in $\mathbb{C}$.*

Proof. We have already seen that periodic points of f are dense. The other two requirements follow easily from the fact that the mapping is *expansive*: given an arc on the unit circle, its image under f is an image of twice the length. Therefore any arc (open set) will eventually cover the whole circle under repeated application of z^2. Transitivity and sensitive dependence are a trivial consequence. □

5. Recurrence

As we have seen, a key property of the angle doubling map is that it is expansive: each arc on the unit circle is mapped to an arc of twice the length. On the other hand, each point z on the unit circle has two pre-images – those corresponding to the angle $\alpha/2$ and $\alpha/2 + 1/2$. Together this means that the pre-image of an arc consists of two arcs each having half the length of the original. Thus the total length of the pre-images of an arc A is the same as the length of A. This is another key property.

Definition 5. *Let X be a set endowed with a measure μ. We say that a measurable mapping $f\colon X \to X$ is* measure preserving *with respect to μ if*

$$\mu(f^{-1}(A)) = \mu(A) \tag{8}$$

for all μ-measurable A.

As we have indicated, the angle doubling map $\alpha \mapsto 2\alpha$ is measure preserving with respect to the Lebesgue measure on $[0,1)$. The full proof would require proving (8) for all measurable subsets of $[0,1)$, not only intervals which correspond to arcs, however this is only a technicality.

A fundamental property of measure preserving mappings is *recurrence.*

Theorem 6 (Poincaré recurrence theorem)**.** *Let $\mu(X) < \infty$ and let $f\colon X \to X$ be measure preserving with respect to μ. Let $A \subset X$ be measurable with $\mu(A) > 0$. Then for μ-almost every point $x \in A$ there exists $n \in \mathbb{N}$ such that $f^{\circ n}(x) \in A$. Moreover, μ-almost every point in A returns to A infinitely many times.*

Proof. Let $N = \{x \in A : f^{\circ n}(x) \notin A \text{ for all } n \in \mathbb{N}\}$ be the set of points from A that never return to A. We want to show that $\mu(N) = 0$. For simplicity, we use the notation $f^k := f^{\circ k}$ in the following.

1. The sets $f^{-k}(N)$ and $f^{-l}(N)$ are disjoint for all $k \neq l \in \mathbb{N}$. To see this, suppose there exists $x \in f^{-k}(N) \cap f^{-l}(N)$ for some $k > l$. Denoting $y = f^l(x)$, we have $y \in N$ and $f^{k-l}(y) = f^k(x) \in N$. Hence $y \in N$ returns to N after $k - l$ iterations. This is a contradiction, as $y \in N$.

2. Assume that $\mu(N) > 0$. Since f is measure preserving and $f^{-k}(N)$ and $f^{-l}(N)$ are disjoint,

$$\infty > \mu(X) \geq \mu\left(\bigcup_{k=1}^{\infty} f^{-k}(N)\right) = \sum_{k=1}^{\infty} \mu(f^{-k}(N)) = \sum_{k=1}^{\infty} \mu(N) = \infty,$$

which is a contradiction. Hence $\mu(N) = 0$, which is the first statement of the theorem.

3. Let $F = \{x \in A : f^n(x) \in A \text{ for finitely many } n \in \mathbb{N}\}$. By definition, for each $x \in F$ there exists $n \in \mathbb{N}$ such that $f^n(x) \in N$, i.e. after the n-th iteration, the iterates of x never return to N. Thus

$$F \subset \bigcup_{k=1}^{\infty} f^{-k}(N).$$

Since f is measure preserving and $\mu(N) = 0$,

$$\mu(F) \leq \mu\left(\bigcup_{k=1}^{\infty} f^{-k}(N)\right) \leq \sum_{k=1}^{\infty} \mu(f^{-k}(N)) = \sum_{k=1}^{\infty} \mu(N) = 0.$$

Therefore, $\mu(F) = 0$. $\square$

6. Dynamical systems conjugate to the Bernoulli shift

The Bernoulli shift mapping σ serves as a basic prototype of a chaotic dynamical system which is easy to analyze while having all of the basic ingredients. Therefore many discrete dynamical systems have been analyzed by constructing a conjugacy with σ as in (4). Here we provide several examples where this is possible.

6.1. Logistic map

Define the *logistic map* as $L_4(x) = 4x(1-x)$ on $[0,1]$. Then L_4 is a two-to-one mapping of $[0,1]$ onto itself. Defining $h_1(t) = (1-t)/2$ and $q(x) = 2x^2 - 1$, it is easy to verify that $L_4 \circ h_1 = h_1 \circ q$, thus the dynamics of L_4 and q are conjugate, similarly as was the case for the Bernoulli shift and the squaring map on the unit circle in $\mathbb{C}$.

Furthermore, let us take $h_2(\theta) = \cos(2\pi\theta)$ and $g(\alpha) = 2\alpha \pmod 1$. Then

$$h_2(g(\theta)) = \cos(4\pi\theta) = 2\cos^2(2\pi\theta) - 1 = q(h_2(\theta)), \tag{9}$$

thus the dynamics of q and the angle doubling mapping are also conjugate. We can write this in the form of the commutative diagram

$$\begin{array}{ccccc} [0,1] & \xleftarrow{h_1} & [-1,1] & \xleftarrow{h_2} & [0,1) \\ {\scriptstyle L_4}\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} \\ [0,1] & \xleftarrow{h_1} & [-1,1] & \xleftarrow{h_2} & [0,1) \end{array} \tag{10}$$

which allows us to conjugate the dynamics of L_4 to that of the angle doubling map, hence the Bernoulli shift.

We note that the functions g and q are, strictly speaking, not conjugate in the sense we have used so far, since $h_2(\theta) = \cos(2\pi\theta)$ is not a bijection on $[0, 1)$, hence it does not have an inverse. Such a situation is called *semi-conjugacy*. Nevertheless for our purposes this is sufficient to investigate the dynamics of q using that of g. For example, if θ is a fixed point of the angle doubling map g, i.e. $\theta = g(\theta)$ then $h_2(\theta)$ is a fixed point of q:

$$h_2(\theta) = h_2(g(\theta)) = q(h_2(\theta))$$

due to (9). Similarly, h_2 maps (pre)periodic points of g to (pre)periodic point of q, etc.

The considerations above therefore allow to investigate the properties of the iteration of the function $4x(1-x)$ on $[0, 1]$ using the theory of Bernoulli shifts. Specifically, in our case one obtains density of periodic points, points with dense orbit and chaotic behavior of the iterates.

We note that the function L_4 is a special case of the family of logistic functions $L_a(x) = ax(1-x)$, the dynamics of which has been extensively studied. Here we only mention that for $a > 4$ we have a basic dichotomy: for each $x \in \mathbb{R}$ either the sequence of iterates of x under L_a tends to $-\infty$, or else its entire orbit lies in a bounded set Λ, which is a Cantor set and the dynamics of L_a restricted to Λ is conjugate to the Bernoulli shift map. For more details we refer to [2].

6.2. Tent map

Another function whose dynamics is well studied due to its relative simplicity is the *tent map*

$$T(x) = \begin{cases} 2x, & 0 \le x < 1/2, \\ 2 - 2x, & 1/2 \le x \le 1. \end{cases}$$

A straightforward calculation shows that T can be conjugated to the logistic map L_4 using the function $h_3(\theta) = (1 - \cos(\pi\theta))/2$. Therefore, all the mentioned results hold for T as well.

6.3. Quadratic functions in $\mathbb{C}$

Now we return to the example of the squaring function in $\mathbb{C}$ considered in Section 2. The dynamics of iteration of a general quadratic function (or more generally a holomorphic function) in $\mathbb{C}$ is a wide and deep field of research starting in the early 1900s by the work of Julia and Fatou and revived by the work of Mandelbrot, Douady and Hubbard, see e.g. [4]. Here we only mention the connection to the dynamics of shift mappings.

Let us consider the function $f_c : \mathbb{C} \to \mathbb{C}$ defined by $f_c(z) = z^2 + c$, where $c \in \mathbb{C}$. It can be easily shown that any quadratic map can be conjugated by a linear mapping to f_c for some c. The special case f_0 is the simple squaring function we considered in Section 2. In that case, points outside of the unit disk converged to infinity under iteration, while the orbits of those from the unit disk remained bounded. On the boundary between these two regions, on the unit circle, the dynamics was complicated, chaotic, being conjugated to the Bernoulli shift on two symbols. It can be shown that similar phenomena occur for general c, cf. [2], [4].

Let us define the *filled-in Julia set* by

$$K_c = \{z \in \mathbb{C} : \; f_c^{\circ k}(z) \nrightarrow \infty\}.$$

Thus $K_0 = \{|z| \leq 1\}$. Then the basic dichotomy of Julia and Fatou holds, [2]: Either K_c is a connected compact set or Cantor set. The latter case holds if and only if the iterates of $z_0 = 0$ tend to infinity and in this case f_c restricted to the Cantor set K_c is conjugate to the Bernoulli shift map.

In the case when K_c is connected, Douady and Hubbard introduced a useful tool for the study of the dynamics of f_c, cf. [2]. This conjugates f_c on the exterior of K_c to z^2 on the outside of the unit disk.

Lemma 7 (Uniformization)**.** *Let K_c be connected. Then there exists an analytic homeomorphism $\phi_c : \mathbb{C} \setminus K_c \to \{|z| > 1\}$ which conjugates f_c to f_0, i.e. we have the commutative diagram*

$$\begin{array}{ccc} \mathbb{C} \setminus K_c & \xrightarrow{\phi_c} & \{|z| > 1\} \\ {\scriptstyle f_c}\big\downarrow & & \big\downarrow{\scriptstyle z^2} \\ \mathbb{C} \setminus K_c & \xrightarrow{\phi_c} & \{|z| > 1\} \end{array} \tag{11}$$

As we have seen, the iteration of z^2 is closely related to the Bernoulli shift. To specify the connection in this setting, we define the *external ray* R_θ^c of argument θ corresponding to f_c as the pre-image of the set $\{re^{2\pi i\theta}, \; r > 1\}$. We note that since we are dealing with quadratic functions, there exist two pre-images and one must choose appropriately.

The basic observation is that due to the (11), we immediately have

$$f_c(R_\theta^c) = R_{g(\theta)}^c, \tag{12}$$

where $g(\theta) = 2\theta \pmod 1$ is the angle doubling map. Thus the Bernoulli shift enters the picture.

As an example, we consider the case $c = -1$, where the filled-in Julia set is called the *Basilica*, Figure 1. It is known that the point $z_0 = (1 + \sqrt{5})/2$ is a fixed point, where K_{-1} is 'pinched', i.e. this point separates K_{-1} into two separate components. Thus there are two external rays 'landing' at z_0 and due to the conjugacy with z^2, their arguments must have period two under angle doubling. As we have seen in Section 3.1 these arguments must be $1/3$ and $2/3$. By taking pre-images of these arguments under the angle doubling map, one can show in general that two external rays of arguments of the form $(6k - 1)/3.2^n$ and $(6k + 1)/3.2^n$ land at a common point which separates K_{-1} into two components. These and related concepts lead to

Thurston's theory of laminations, [7], which deals with the combinatorial behavior of the mappings f_c. For a more general overview and expository accounts we refer the interested reader to [3], [4] and for a historical overview we recommend [5].

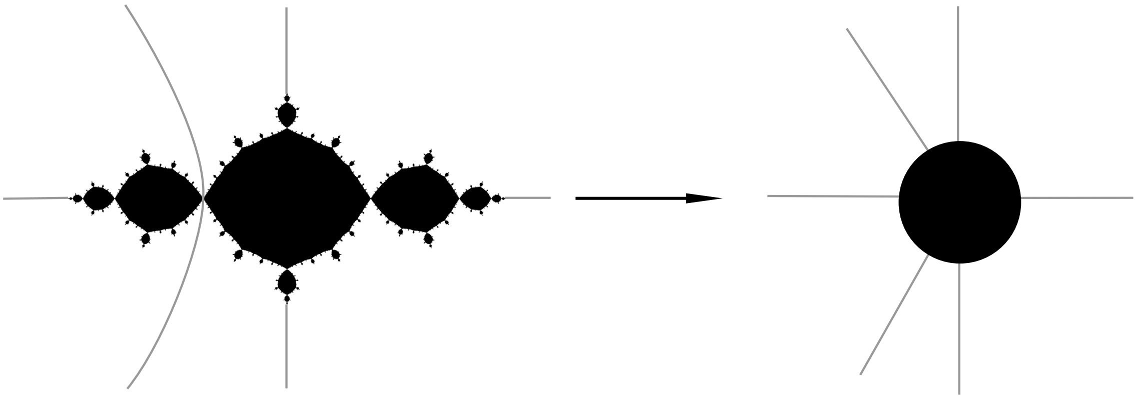

Figure 1: The filled-in Julia set for $c = -1$ and the mapping ϕ_{-1} of its exterior onto the exterior of the unit disk. The external rays with $\theta = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ are indicated along with their images under ϕ_{-1}.

Acknowledgements

This work was supported by project No. 32/2017 of the Neuron Benevolent Fund for Support of Science.

References

[1] Banks, J., Brooks, J., Cairns, G., Davis, G., and Stacey, P.: On Devaney's Definition of Chaos. Amer. Math. Monthly **99** (1992), 332–334.

[2] Devaney, R. L.: *An introduction to chaotic dynamical systems.* Westview Press, Boulder, 2003.

[3] McMullen, C.: Frontiers in complex dynamics. Bull. AMS **31** (1994), 155–172.

[4] Milnor, J.: *Dynamics in one complex variable.* Princeton University Press, 2006.

[5] Rees, M.: One hundred years of complex dynamics. Proc. R. Soc. A **472** (2016).

[6] Ruette, S.: *Chaos on the interval.* University Lecture Series Vol. 67. AMS, Providence, 2017.

[7] Thurston, W. P.: On the combinatorics of iterated rational maps. Complex dynamics: families and friends, J.H. Hubbard 60th birthday volume, A.K. Peters, 3109, Princeton University and IAS, 2009.

[8] Vellekoop, M., and Berglund, R.: On Intervals, Transitivity = Chaos. Amer. Math. Monthly **101** (1994), 353–355.