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In: Jan Brandts and Sergej Korotov and Michal Křížek and Karel Segeth and Jakub Šístek and Tomáš Vejchodský (eds.): Application of Mathematics 2015, In honor of the birthday anniversaries of Ivo Babuška (90), Milan Práger (85), and Emil Vitásek (85), Proceedings. Prague, November 18-21, 2015. Institute of Mathematics CAS, Prague, 2015. pp. 217–224.

Persistent URL: http://dml.cz/dmlcz/702978

Conference Applications of Mathematics 2015, in honor of the birthday anniversaries of Ivo Babuška (90), Milan Práger (85), and Emil Vitásek (85)
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Institute of Mathematics CAS, Prague 2015

A NOTE ON TENSION SPLINE

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Abstract: Spline theory is mainly grounded on two approaches: the algebraic one (where splines are understood as piecewise smooth functions) and the variational one (where splines are obtained via minimization of quadratic functionals with constraints). We show that the general variational approach called smooth interpolation introduced by Talmi and Gilat covers not only the cubic spline but also the well known tension spline (called also spline in tension or spline with tension). We present the results of a 1D numerical example that show the advantages and drawbacks of the tension spline.

Keywords: smooth interpolation, tension spline, Fourier transform

MSC: 65D05, 65D07, 41A05

1. Introduction

In most practical cases, the minimum curvature (or cubic spline) method produces a visually pleasing smooth curve or surface. However, in some cases the method can create strong artificial oscillations in the curve derivative (surface gradient). A remedy suggested by Schweikert [6] is known as *tension spline*. The functional minimized includes the first derivative term in addition to the second derivative term.

Smooth approximation [10] is an approach to data interpolating or data fitting that employs the variational formulation of the problem in a normed space with constraints representing the approximation conditions. The cubic spline interpolation is also known to be the approximation of this kind.

For the 1D cubic spline, the objective is to minimize the L^2 norm of second derivative of the approximating function. A more sophisticated criterion is then to minimize, with some weights chosen, the integrals of the squared magnitude of some (or possibly all) derivatives of a sufficiently smooth approximating function. In the paper, we are concerned with the tension spline constructed by means of the smooth approximation theory (cf. also [4]), i.e. with the exact interpolation of the data at nodes and, at the same time, with the smoothness of the interpolating curve and its first derivative.

We are mostly interested in the case of a single independent variable in the paper. Assuming the approach of [8] and [10], we introduce the problem to be solved and the tools necessary to this aim in Sec. 3. We also present the general existence theorem for smooth interpolation proven in [8]. We are concerned with the use of basis system $\exp(\mathrm{i}kx)$ of exponential functions of pure imaginary argument for 1D smooth approximation problems in Secs. 4 and 5. We investigate some of its properties suitable for measuring the smoothness of the approximation and for generating the tension spline. We also show results of a 1D numerical experiment and discuss them to illustrate some properties of smooth approximation.

2. Problem of data approximation

Let us have a finite number N of (complex, in general) measured (sampled) values $f_1, f_2, \ldots, f_N \in C$ obtained at N nodes $X_1, X_2, \ldots, X_N \in R^n$. The nodes are assumed to be mutually distinct. We are usually interested also in the intermediate values corresponding to other points in some domain. Assume that $f_j = f(X_j)$ are measured values of some continuous function f while z is an approximating function to be constructed. The dimension n of the independent variable can be arbitrary. For the sake of simplicity we put $n = 1$ and assume that $X_1, X_2, \ldots, X_N \in \Omega$, where either $\Omega = [a, b]$ is a finite interval or $\Omega = (-\infty, \infty)$.

Data interpolation. The interpolating function z is constructed to fulfil the interpolation conditions

$$z(X_j) = f(X_j), \quad j = 1, \ldots, N. \tag{1}$$

Some additional conditions can be considered, e.g. the Hermite interpolation or minimization of some functionals applied to z.

The problem of data interpolation does not have a unique solution. The property (1) of the interpolating function is uniquely formulated by mathematical means but there are also requirements on the *subjective perception* of the behavior of the approximating curve or surface between nodes that can hardly be formalized [11].

3. Smooth approximation

We introduce an inner product space to formulate the additional constraints in the problem of smooth approximation [7, 8, 10]. Let $\widetilde{\mathcal{W}}$ be a linear vector space of complex valued functions g continuous together with their derivatives of all orders on the interval Ω. Let $\{B_l\}_{l=0}^{\infty}$ be a sequence of nonnegative numbers and L the smallest nonnegative integer such that $B_L > 0$ while $B_l = 0$ for $l < L$. For $g, h \in \widetilde{\mathcal{W}}$, put

$$(g, h)_L = \sum_{l=0}^{\infty} B_l \int_{\Omega} g^{(l)}(x)[h^{(l)}(x)]^* \,\mathrm{d}x, \tag{2}$$

$$|g|_L^2 = \sum_{l=0}^{\infty} B_l \int_{\Omega} |g^{(l)}(x)|^2 \,\mathrm{d}x, \tag{3}$$

where * denotes the complex conjugate.

If $L = 0$ (i.e. $B_0 > 0$), consider functions $g \in \widetilde{\mathcal{W}}$ such that the value of $|g|_0$ exists and is finite. Then $(g,h)_0 = (g,h)$ has the properties of *inner product* and the expression $|g|_0 = \|g\|$ is *norm* in a normed space $W_0 = \widetilde{\mathcal{W}}$.

Let $L > 0$. Consider again functions $g \in \widetilde{\mathcal{W}}$ such that the value of $|g|_L$ exists and is finite. Let $P_{L-1} \subset \widetilde{\mathcal{W}}$ be the subspace whose basis $\{\varphi_p\}$ consists of monomials

$$\varphi_p(x) = x^{p-1}, \quad p = 1,\dots,L.$$

Then $(\varphi_p, \varphi_q)_L = 0$ and $|\varphi_p|_L = 0$ for $p,q = 1,\dots,L$. Using (2) and (3), we construct the *quotient space* $\widetilde{\mathcal{W}}/P_{L-1}$ whose zero class is the subspace P_{L-1}. Finally, considering $(\cdot,\cdot)_L$ and $|\cdot|_L$ in every equivalence class, we see that they represent the inner product and norm in the normed space $W_L = \widetilde{\mathcal{W}}/P_{L-1}$.

W_L is the normed space where we minimize functionals and measure the smoothness of the interpolation. For an arbitrary $L \geq 0$, choose a *basis system* of functions $\{g_k\} \subset W_L$, $k = 1,2,\dots$, that is complete and orthogonal (in the inner product in W_L), i.e., $(g_k, g_m)_L = 0$ for $k \neq m$, $(g_k, g_k)_L = |g_k|_L^2 > 0$. If $L > 0$ then it is, moreover, $(\varphi_p, g_k)_L = 0$ for $p = 1,\dots,L$, $k = 1,2,\dots$. The set $\{\varphi_p\}$ is empty for $L = 0$.

Smooth data interpolation. The *problem of smooth data interpolation* [10] consists in finding the coefficients A_k and a_p of the expression

$$z(x) = \sum_{k=1}^{\infty} A_k g_k(x) + \sum_{p=1}^{L} a_p \varphi_p(x) \tag{4}$$

such that

$$z(X_j) = f_j, \quad j = 1,\dots,N, \tag{5}$$

and

$$\text{the quantity } \; |z|_L^2 \; \text{ attains its minimum.} \tag{6}$$

Introduce the *generating function*

$$R_L(x,y) = \sum_{k=1}^{\infty} \frac{g_k(x) g_k^*(y)}{|g_k|_L^2}. \tag{7}$$

We state in Theorem 1 that a finite linear combination of the values of the generating function R_L at particular nodes is used for the practical interpolation instead of the infinite linear combination in (4). Further put

$$R = [R_L(X_i, X_j)], \quad i,j = 1,\dots,N,$$

where R is an $N \times N$ square Hermitian matrix, and if $L > 0$ introduce an $N \times L$ matrix

$$\varPhi = [\varphi_p(X_j)], \quad j = 1,\dots,N, \; p = 1,\dots,L.$$

Theorem 1. *Let $X_i \neq X_j$ for all $i \neq j$. Assume that the generating function (7) converges for all $x, y \in \Omega$. If $L > 0$ let* rank $\Phi = L$. *Then the problem of smooth interpolation (4) to (6) has the unique solution*

$$z(x) = \sum_{j=1}^{N} \lambda_j R_L(x, X_j) + \sum_{p=1}^{L} a_p \varphi_p(x), \tag{8}$$

where the coefficients λ_j, $j = 1, \ldots, N$, and a_p, $p = 1, \ldots, L$, are the unique solution of a nonsingular system of $N + L$ linear algebraic equations.

Proof. The proof is given in [8]. □

The *problem of smooth curve fitting (data smoothing)*, where the interpolation condition (1) is not applied, is treated in more detail in [8], [10].

4. A particular choice of basis function system

Recall that we have put $n = 1$. Let the function f to be approximated be 2π-periodic in $[0, 2\pi]$. We choose exponential functions of pure imaginary argument for the periodic basis system $\{g_k\}$ in W_L. The following theorem shows important properties of the system.

Theorem 2. *Let there be an integer s, $s \geq L$, such that $B_l = 0$ for all $l > s$ in W_L. The system of periodic exponential functions of pure imaginary argument*

$$g_k(x) = \exp(-\mathrm{i}kx), \quad x \in [0, 2\pi],\ k = 0, \pm 1, \pm 2, \ldots, \tag{9}$$

is complete and orthogonal in W_L.

Proof. The proof is given in [9]. □

The range of k implies a minor change in the notation introduced above. For the basis system (9), notice that the generating function

$$R_L(x, y) = \sum_{k=-\infty}^{\infty} \frac{g_k(x) g_k^*(y)}{|g_k|_L^2} = \sum_{k=-\infty}^{\infty} \frac{\exp(-\mathrm{i}k(x-y))}{|g_k|_L^2} \tag{10}$$

is the Fourier series in $L^2(0, 2\pi)$ with the coefficients $|g_k|_L^{-2}$, where

$$|g_k|_L^2 = 2\pi \sum_{l=L}^{\infty} B_l k^{2l} \tag{11}$$

according to (3).

Let now the function f to be approximated be nonperiodic on $(-\infty, \infty)$ and $f^{(l)}(\pm\infty) = 0$ for all $l \geq 0$. Let us define the generating function $R_L(x, y)$ as the Fourier transform of the function $|g_k|_L^{-2}$ of continuous variable k,

$$R_L(x, y) = \int_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}k(x - y))}{|g_k|_L^2} \, \mathrm{d}k, \tag{12}$$

if the integral exists. Using the effect of transition from the Fourier series (10) with the coefficients $|g_k|_L^{-2}$ to the Fourier transform (12) of the function $|g_k|_L^{-2}$ of continuous variable k (cf., e.g., [3]), we have transformed the basis functions, enriched their spectrum, and released the requirement of periodicity of f. Moreover, if the integral (12) does not exist, in many instances we can calculate $R_L(x, y)$ as the Fourier transform $\mathcal{F}$ of the generalized function $|g_k|_L^{-2}$ of k.

Tension spline. Choosing now a particular sequence $\{B_l\}$, we finish the definition of the inner product and norm (2), (3) in a particular space W_L and set, therefore, the minimization properties of the smooth interpolant. Let us thus put (cf. [4])

$$B_l = 0 \text{ for all } l \text{ with the exception of } B_1 = \alpha^2, \ \alpha > 0, \text{ and } B_2 = 1. \tag{13}$$

It means that we have $L = 1$ and minimize the L^2 norm of the first derivative (characterizing the oscillations) multiplied by α^2 plus the L^2 norm of the second derivative (characterizing the curvature) of the interpolant (4) in the form (8), i.e.

$$z(x) = \sum_{j=1}^{N} \lambda_j R_1(x, X_j) + a_1. \tag{14}$$

We get

$$|g_k|_1^2 = 2\pi(\alpha^2 k^2 + k^4)$$

from (11). Putting $r = |x - y|$, we arrive at

$$\begin{aligned} R_1(x, y) &= \mathcal{F}\left(\frac{1}{2\pi k^2(\alpha^2 + k^2)}\right) = \frac{1}{2\pi}\mathcal{F}\left(\frac{1}{\alpha^2 k^2} - \frac{1}{\alpha^2(k^2 + \alpha^2)}\right) \\ &= -\frac{1}{2\alpha^3}(\alpha|r| + \exp(-\alpha|r|)), \end{aligned} \tag{15}$$

where $\mathcal{F}$ denotes the Fourier transform of a generalized function (see [2], p. 375, formula 14 and p. 377, formula 29; and [1], formula 8.469.3). It is easy to find out that this version of smooth approximation is, in fact, equivalent to the tension spline interpolation [6] but introduced in a way different from [4].

There are further practical examples of smooth approximation where the integral (12) that defines the generating function can be calculated with the help of the Fourier transform.

Cubic spline. Choosing another particular sequence $\{B_l\}$, i.e. $B_l = 0$ for all l with the exception of $B_2 = 1$ (cf. [9], [10]), we have $L = 2$ and minimize the usual L^2 norm of the second derivative (curvature) of the interpolant (8) only, i.e.

$$z(x) = \sum_{j=1}^{N} \lambda_j R_2(x, X_j) + a_1 + a_2 x.$$

This sequence $\{B_l\}$ differs from (13) only by the condition $B_1 = 0$ that replaced the tension spline condition $B_1 = \alpha^2 > 0$. We get $|g_k|_2^2 = 2\pi k^4$ from (11) and arrive at (see [2])

$$R_2(x, y) = \mathcal{F}(1/(2\pi k^4)) = \tfrac{1}{12} r^3.$$

Apparently, this version of smooth approximation is, in fact, the *cubic spline interpolation* [5] considered to be a classical interpolation method and known for the above mentioned minimization property.

5. Computational comparison

We present results of a simple numerical experiment with the tension spline for $n = 1$. We employ the complete and orthogonal system (9) and the sequence (13) to introduce the space W_1. We use the interpolant (14), where R_1 is given by (15). The function to be interpolated is

$$f(x) = 5 + \frac{2}{1 + 16x^2}. \tag{16}$$

Apparently, it has "almost a pole" at $x = 0$. The tension spline interpolation of the function (16) has been constructed in several equidistant grids of N nodes on $[-1, 1]$ and for several values of α^2 including also $\alpha^2 = 0$, i.e. the cubic spline.

Some of the results of interpolation are in Fig. 1. We put $N = 9$ and compare tension splines with $\alpha^2 = 0$ (i.e. the cubic spline, solid line), $\alpha^2 = 1\ 000$ (dashed line), and $\alpha^2 = 10\ 000$ (dotted line). The interpolants are in the upper part of the figure, their first derivatives in the lower part along the x axis.

We see that the tension splines do not differ substantially from each other but their derivatives are very unlike. The derivative of the cubic spline is a smooth function while the derivative of the tension spline with $\alpha^2 = 10\ 000$ is similar to a piecewise constant function with smooth changes (not jumps) between the constant levels. This corresponds to the behavior of this tension spline if examined in a different scale: it resembles a piecewise linear curve but it is smooth, not sharp-cornered also at nodes, i.e. its derivative is continuous.

A proper choice of the parameter α^2 can provide a compromise interpolation solution with both tension spline and its derivative so smooth that they give a good, pleasing subjective impression.

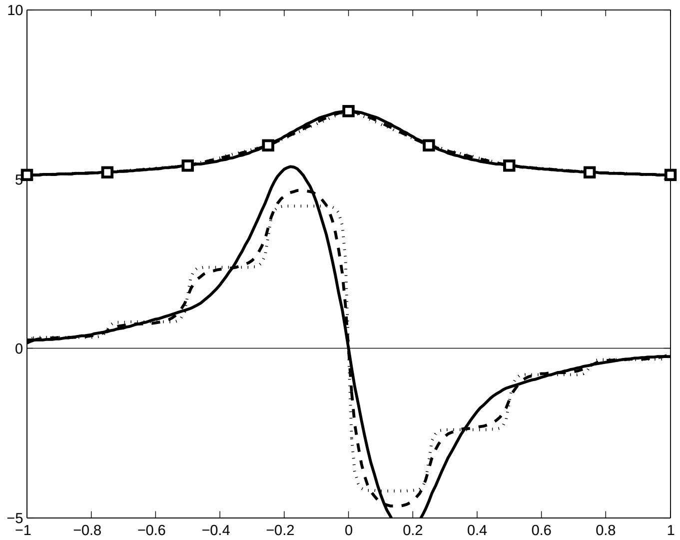

Figure 1: $N = 9$. The horizontal axis: independent variable, the vertical axis: interpolant (in the upper part of the figure) and its derivative (in the lower part). Cubic spline (tension $\alpha^2 = 0$): solid line, tension spline ($\alpha^2 = 1\ 000$): dashed line, tension spline ($\alpha^2 = 10\ 000$): dotted line.

6. Conclusion

The aim of this paper was to show that the generating function for the tension spline interpolation can be obtained by means of the Fourier transform of generalized functions. The Fourier transform can be successfully used to determine the generating function also in several other cases including $n = 2$ and $n = 3$. The example in Fig. 1 is a very simple illustration.

Acknowledgements

This work has been supported by project RVO 67985840 and by Czech Science Foundation grant P101/14-02067S.

References

[1] Gradshteyn, I. S. and Ryzhik, I. M.: *Table of integrals, series, and products.* Academic Press, Boston, 1994.

[2] Kreĭn, S. G. (Ed.): *Functional analysis* (Russian). Nauka, Moskva, 1964.

[3] Lanzos, C.: *Discourse on Fourier series.* Oliver & Boyd, Edinburgh, 1966.

[4] Mitáš, L. and Mitášová, H.: General variational approach to the interpolation problem. Comput. Math. Appl. **16** (1988), 983–992.

[5] Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T.: *Numerical recipes. The art of scientific computing.* Cambridge University Press, Cambridge, 1986.

[6] Schweikert, D. G.: An interpolation curve using a spline in tension. J. Math. and Phys. **45** (1966), 312–317.

[7] Segeth, K.: Smooth approximation of data with applications to interpolating and smoothing. In: *Programs and Algorithms of Numerical Mathematics* 16, pp. 181–186. Institute of Mathematics AS CR, Prague, 2013.

[8] Segeth, K.: Some computational aspects of smooth approximation. Computing **95** (2013), S695–S708.

[9] Segeth, K.: A periodic basis system of the smooth approximation space. Appl. Math. Comput. **267** (2015), 436–444.

[10] Talmi, A. and Gilat, G.: Method for smooth approximation of data. J. Comput. Phys. **23** (1977), 93–123.

[11] Ueberhuber, C. W.: *Numerical computation*, vol. 1. Springer, Berlin, 1997.