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ON CONFORMING NONOBTUSE TETRAHEDRALIZATIONS OF SOME CYLINDRICAL-TYPE DOMAINS

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Abstract

We present an algorithm for constructing families of conforming (i.e. face-to-face) nonobtuse tetrahedral finite element meshes for convex 3D cylindrical-type domains. In fact, the algorithm produces only path-tetrahedra.

1. Introduction

Nonobtuse simplicial elements play an important role in the finite element analysis of boundary value problems, since they yield monotone stiffness matrices and thus guarantee the validity of the discrete maximum principle when solving many elliptic boundary value problems (see [2, 3]). Note that even one obtuse simplex in a triangulation may destroy the discrete maximum principle [3]. In [7], we gave a global refinement algorithm which produces only nonobtuse tetrahedra. In [8, 9, 10] several algorithms are designed for various local nonobtuse tetrahedral refinements. However, in all the above mentioned works only the case of polyhedral domains is considered but in practice we may also have domains with curved boundaries, the generation and appropriate refinements of finite element meshes for which can be considerably more dificult, or at least requiring a special treatment, see e.g. [6, 12] and references therein in this respect.

In this paper we present an algorithm for generating and appropriate refining, in a face-to-face manner, nonobtuse tetrahedral meshes for some cylindrical-type domains with curved boundaries. (We notice that the case of isoparametric elements [5] is not considered here.)

Recall that a tetrahedron is said to be *nonobtuse*, if all its six angles between faces, the so-called dihedral angles, are nonobtuse (i.e. not larger than right). The

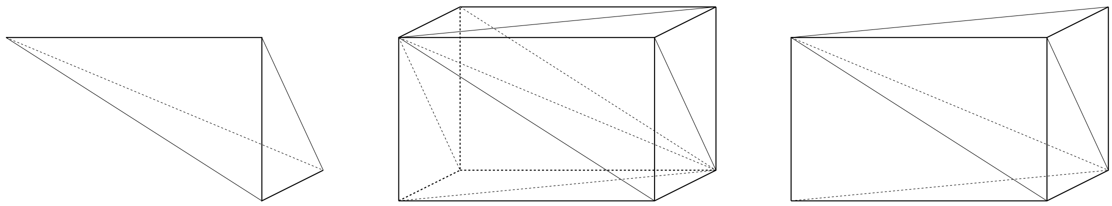

Figure 1: Path-tetrahedron (left), decomposition of a rectangular block in 6 path-tetrahedra (center), and decomposition of a right-angled prism in 3 path-tetrahedra.

path tetrahedron is a special case of nonobtuse tetrahedra, it has three mutually orthogonal edges that form a path (in the sense of graph theory).

In Figure 1 we depict an example of a path-tetrahedron, a decomposition of a rectangular block in 6 path-tetrahedra, and a decomposition of a right-angled prism in 3 path-tetrahedra.

2. Basic idea

To explain the main steps of the algorithm, we assume that Ω is a 3D cylindrical-type domain with a convex planar base S (not necessarily smooth), and let, for simplicity, $\Omega = S \times (\ell_1, \ell_2)$. First, let the initial partition of Ω consist of right-angled prisms face-to-face placed. This presupposes that we are able to construct a conforming triangulation of S into right triangles. Such a triangulation has to be done as follows – we construct some polygon with vertices on the boundary of S (see Figure 2), and then use one of algorithms presented e.g. in [1, 4, 14, 16] for constructing a nonobtuse (or even acute) triangulation of this polygon. To this nonobtuse (or acute) triangulation we apply 2D yellow refinement technique [7], which guarantees that the next (generated by the yellow refinement) triangulation consists of right triangles only. The extra vertices, possibly appearing on the boundary of the initial polygon (marked by the black bullets in the central part of Figure 2), are treated as shown in the right part of Figure 2 – namely, we project each such vertex on the boundary of S orthogonally to the edge to which the vertex belongs and connect the projection point with the end-points of the edge, thus adding to the triangulation in a conforming manner a few extra right-angled triangles. (The last construction is always possible due to the assumption of convexity of S.) On the base of this (final, with all boundary vertices belonging to the boundary of S) right-angled triangulation of S we can, obviously, generate a right-angled face-to-face prismatic partition of Ω by appropriate vertical and horizontal cuts.

Further, we assume that each prism from the prismatic partition is refined in 3 path-tetrahedra as shown in Figure 3. Obviously, the issue of providing the overall conformity of the tetrahedral partition obtained in this way appears (cf. [11]) as some common rectangular faces of adjacent prisms might be not necessarily split in the same way. Therefore, using the ideas from [11] we shall propose a suitable strategy how to conformly tetrahedralize our right-angled prismatic partition. Observe first

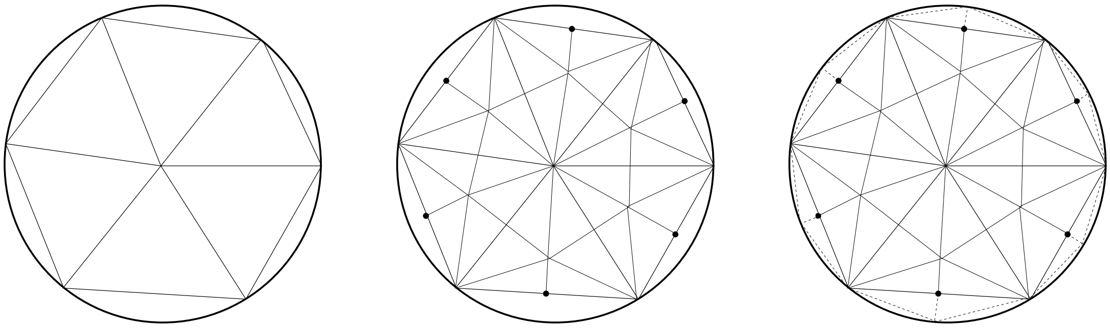

Figure 2: Construction of a conforming triangulation of a convex base S, which consists of right triangles only.

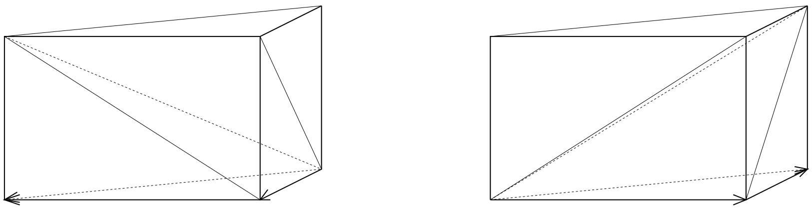

Figure 3: Two possible ways of splitting a right-angle prism in 3 path-tetrahedra with corresponding orientations for the edges of the base.

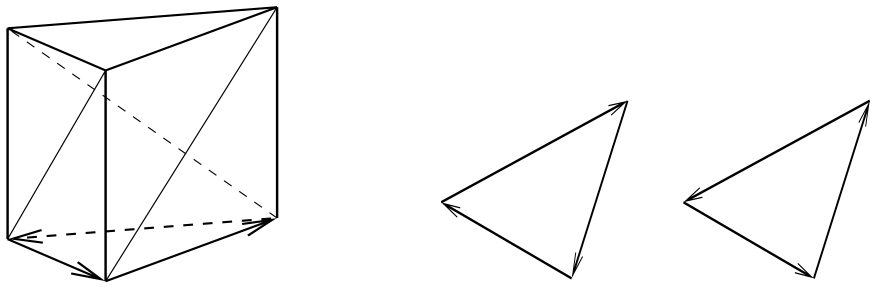

Figure 4: Orientation of edges not leading to any tetrahedral splitting of a prism (left), and two non-allowed orientations (center and right).

that the splitting of a prism in 3 tetrahedra can be, in principle, "suggested" by the orientation for edges of the base of the prism (cf. Figure 3). However, as Figure 4 (left) demonstrates, the orientation cannot be prescribed arbitrarily – there is no tetrahedral splitting of the prism for the partition of rectangular faces associated with certain orientations. Nevertheless, only two special cases for the orientation, illustrated in Figure 4 (center and right), are not allowed, and we have to avoid them in practice in order to guarantee the overall conformity of the final tetrahedral mesh. This can be done, for example, as in [11].

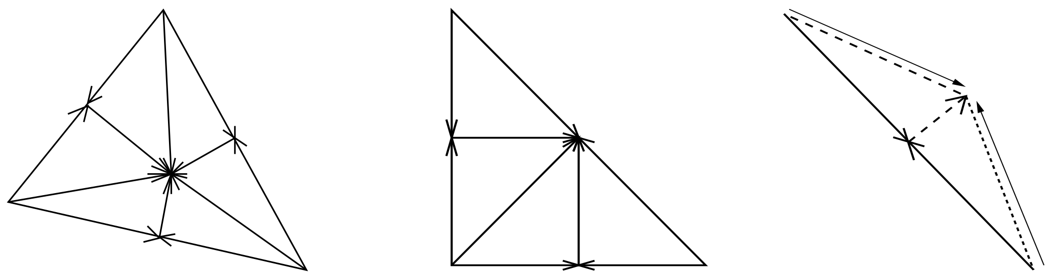

Figure 5: On orientation of edges of right sub-triangles generated within each nonobtuse triangle of the initial triangulation of S – the case of an acute triangle (left) and the case of a right triangle (center). In the right part of the figure we sketch how we should orient edges of (extra) right triangles possibly appearing near the boundary of S.

However, as we want to generate nonobtuse tetrahedra (more precisely – only path-tetrahedra), among all allowed edge orientations in what follows we shall only accept those two used in Figure 3 (i.e. two arrows emmanating from the same vertex should not be orthogonal). Therefore we propose to prescribe the orientations of edges within each triangle of the initial nonobtuse (or acute) triangulation (cf. Figure 2 (left)) as presented in Figure 5, where three possible cases are illustrated. It is clear that the resulting tetrahedral mesh will be then conforming and it will consist of path-tetrahedra only.

In addition to the nonobtuse tetrahedral mesh, in the neigbourhood of the curved boundary $\partial\Omega$ we get a number of “slice-domains” sketched in Figure 6 (left), each of which is formed by two neigbouring planes parallel to S (used to generate horizontal layers for a prismatic mesh of Ω), an “outward” rectangular face of some prism from the initial prismatic partition (with the face cut by one of its diagonals in the process of tetrahedralization), and a curved part of $\partial\Omega$.

As the tetrahedral mesh generated by now consists of path-tetrahedra only, we can use 3D yellow refinement technique from [7] in order to generate the next (finer) tetrahedral mesh which will be again consisting of path-tetrahedra. However, due to the presence of a curved boundary we get something like “hanging vertices” (marked by black bullets in Figure 6 (right)) in the rectangular faces of slice-domains, and in what follows we explain how to get rid of them in the spirit of Figure 2 (right).

First we project orthogonally these hanging vertices to $\partial\Omega$ (cf. Figure 7 (left)) and connect the projections to the neighbouring vertices and also along $\partial\Omega$ (see Figure 7 (right) for details). Further, we mark another pair of vertices to be used in our next constructions by empty bullets. Now, a cut by a plane parallel to S through those vertices marked by empty bullets is done (see Figure 8 (left)), which immediately suggests a natural construction of four new (smaller) prisms inside the slice-domain. Each of four new prisms is further decomposed (using the given splitting of one of its rectangular faces due to the neighbouring elements) in 3 path-tetrahedra in the manner of Figure 8 (right). The resulting tetrahedral mesh is, obviously, conforming and consists of path-tetrahedra only.

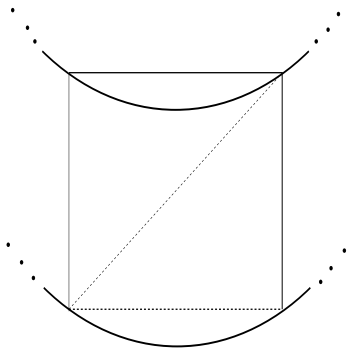

Figure 6: A slice-domain near the curved boundary $\partial\Omega$ (left) and the yellow refinements of two flat triangular faces associated with this slice-domain (right).

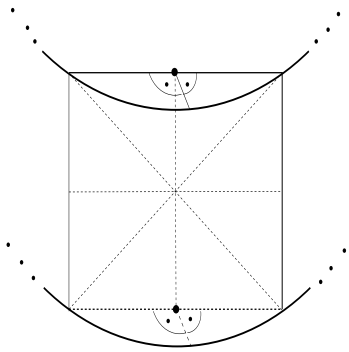

Figure 7: A slice-domain with hanging vertices and their projections.

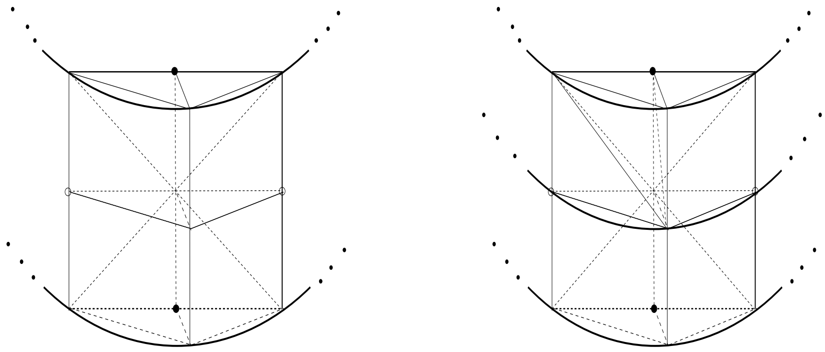

Figure 8: Construction of four new prisms (left) and partition of one of them (right).

Finally, after the last divisions we are in a position sketched in Figure 6 (left) for each "outward face" of four newly generated right-angled prisms, so that the whole algorithm can be repeated.

3. Final remarks

For domains with curved boundaries the following definition (cf. [15, 12]) is useful.

Definition 1 *Let Ω be a bounded domain in $\mathbf{R}^3$, let $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ be a set of tetrahedral face-to-face partitions of Ω_h, where*

$$\overline{\Omega}_h = \bigcup_{K\in\mathcal{T}_h} K.$$

The set $\mathcal{F}$ is said to be a family of tetrahedral finite element meshes of Ω if for any $x \in \overline{\Omega}$ there exists a sequence $\{x_h\}$, $x_h \in \overline{\Omega}_h$, such that $x_h \to x$ as $h \to 0$, and for any convergent sequence $\{x_h\}$, $x_h \in \overline{\Omega}_h$, there exists $x \in \overline{\Omega}$ such that $x_h \to x$ as $h \to 0$.

According to the above definition, $\mathcal{F}$ is a family of tetrahedral meshes of Ω if the associated polyhedra Ω_h "converge" to Ω. The definition "does not contradict" the standard definition of a family of (conforming) tetrahedral meshes for polyhedral domains [5] as we can simply define $\{x_h\} = \{x\}$ then. It is clear that our algorithm produces a family of nonobtuse conforming tetrahedral meshes of Ω in the sense of Definition 1.

Remark 1 *Notice that nonobtuse tetrahedral meshes (whose tetrahedral elements are known to have nonobtuse triangular faces* [3]*) satisfy the maximum angle condition* [13]*, which is a popular (sufficient) condition for various convergence proofs in finite element analysis.*

Remark 2 *To the author's knowledge it is still unclear how the usage of isoparametric elements can influence the validity of discrete maximum principles.*

Remark 3 *The main idea of the algorithm can be used even in more general situations – for more complicated 3D domains. This issue will be considered in the forthcoming journal paper.*

Acknowledgements

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