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ON CONSTRUCTION OF THE COARSE SPACE IN THE BDDC METHOD*

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1. Introduction

Domain Decomposition (DD) methods are getting increasingly popular in various areas of engineering for offering a convenient way to parallelize analysis by the finite element method (FEM). Among the most popular members of this family for symmetric positive definite problems, such as linear elasticity, are the FETI-DP method of Farhat et al. [3] and the BDDC method of Dohrmann [1]. It has been recently proved by Mandel, Dohrmann, and Tezaur [5] that the two methods are spectrally equivalent, which unifies the theory for both methods and allows application of various results already known for FETI-DP to BDDC and vice versa.

Both methods use a coarse space based on a set of selected nodes, called *corners*, in which the continuity of subdomain solutions is required. These nodes assure that the subdomain problems are also positive definite and might be solved by a standard direct method. The set of corners gives rise to an important part of the *coarse space* and the corresponding *coarse problem*.

While corners assure a convenient solvability of subdomain problems, they do not suffice for robust preconditioning with respect to discretization parameter h in three dimensions. This fact was first observed for FETI-DP experimentally in [3], and theoretically in [4]. The theoretical treatment requires adding constraints on equality of averages over subdomain *edges* and *faces* to the coarse problem. This might be done in a uniform way (as was done in [1, 2]), or in a more sophisticated adaptive way, which nearly optimally decreases the condition number of the preconditioned operator (see [6]).

In the present paper, we investigate different and more straightforward approach to the generation of coarse space, that consists of simple addition of more nodes from the interface to the set of corners. Although this is not the optimal case, presented numerical experiments for an industrial application of linear elasticity show that combining both approaches can lead to synergic effect and further reduce the overall computational time.

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2. The Schur complement method

Consider a boundary value problem with self-adjoint operator defined on domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$. If we discretize the problem by means of the standard finite element method (FEM), we arrive at the solution of system of linear equations in the matrix form

$$\mathbf{K}\mathbf{u} = \mathbf{f}, \tag{1}$$

where $\mathbf{K}$ is a large, sparse, symmetric positive definite (SPD) matrix and $\mathbf{f}$ is the vector of right-hand-side.

Let us decompose domain Ω into N non-overlapping subdomains Ω_i, $i = 1, \ldots, N$. Unknowns common to at least two subdomains are called *boundary unknowns* and the union of all boundary unknowns is called the *interface*. Remaining unknowns belong to subdomain *interiors*.

The first step is the reduction of the problem to the interface. Without loss of generality, suppose that unknowns are ordered so that interior unknowns form the first part and the interface unknows form the second part of the solution vector, i.e. $\mathbf{u} = \left[\begin{array}{cc} \mathbf{u}_\mathrm{o} & \widehat{\mathbf{u}} \end{array} \right]^T$, where $\mathbf{u}_\mathrm{o}$ stands for all interior unknowns and $\widehat{\mathbf{u}}$ for unknowns at interface. System (1) can now be formally rewritten to block form

$$\left[\begin{array}{cc} \mathbf{K}_{\mathrm{oo}} & \mathbf{K}_{\mathrm{or}} \\ \mathbf{K}_{\mathrm{ro}} & \mathbf{K}_{\mathrm{rr}} \end{array} \right] \left[\begin{array}{c} \mathbf{u}_\mathrm{o} \\ \widehat{\mathbf{u}} \end{array} \right] = \left[\begin{array}{c} \mathbf{f}_\mathrm{o} \\ \widehat{\mathbf{f}} \end{array} \right]. \tag{2}$$

The hat symbol ($\widehat{\ }$) is used to denote global interface quantities. If we suppose the interior unknowns ordered subdomain after subdomain, then the submatrix $\mathbf{K}_{\mathrm{oo}}$ is block-diagonal with each diagonal block corresponding to one subdomain.

After eliminating all the interior unknowns from (2), we arrive to *Schur complement problem* for the interface unknowns

$$\widehat{\mathbf{S}}\,\widehat{\mathbf{u}} = \widehat{\mathbf{g}}, \tag{3}$$

where $\widehat{\mathbf{S}} = \mathbf{K}_{\mathrm{rr}} - \mathbf{K}_{\mathrm{ro}}\mathbf{K}_{\mathrm{oo}}^{-1}\mathbf{K}_{\mathrm{or}}$ is a *Schur complement* of (2) with respect to interface and $\widehat{\mathbf{g}} = \widehat{\mathbf{f}} - \mathbf{K}_{\mathrm{ro}}\mathbf{K}_{\mathrm{oo}}^{-1}\mathbf{f}_\mathrm{o}$ is sometimes called *condensed right hand side*. Interior unknowns $\mathbf{u}_\mathrm{o}$ are determined by interface unknowns $\widehat{\mathbf{u}}$ as

$$\mathbf{K}_{\mathrm{oo}}\mathbf{u}_\mathrm{o} = \mathbf{f}_\mathrm{o} - \mathbf{K}_{\mathrm{or}}\widehat{\mathbf{u}}. \tag{4}$$

The solution can now be divided into three steps: (i) construction of problem (3), (ii) solution of problem (3), and (iii) resolution of interior unknowns by (4). Because problem (4) represents N independent subdomain problems with Dirichlet boundary condition prescribed on the interface, steps (i) and (iii) are performed in parallel and are very fast. Thus, the main concern represents the solution of problem (3) in step (ii). This problem is solved by the preconditioned conjugate gradient method (PCG). Since only matrix-vector multiplications are necessary in PCG, the Schur complement matrix $\widehat{\mathbf{S}}$ need not be constructed explicitly. Instead, we only need to

factorize the block $\mathbf{K}_{\mathrm{oo}}$ and perform the multiplications with $\widehat{\mathbf{S}}$ as $\widehat{\mathbf{S}}\widehat{\mathbf{v}} = \mathbf{K}_{\mathrm{rr}}\widehat{\mathbf{v}} - \mathbf{K}_{\mathrm{ro}}\mathbf{w}$, where $\mathbf{K}_{\mathrm{oo}}\mathbf{w} = \mathbf{K}_{\mathrm{or}}\widehat{\mathbf{v}}$. Also this process may be performed subdomain by subdomain in parallel, using blocks of local subdomain matrices $\mathbf{K}_i$ defined in the next section.

3. The BDDC method

The BDDC method may be viewed as a preconditioner for problem (3) when it is solved by the PCG method. The main idea of the preconditioner is to split the problem into independent subdomain problems and the global coarse problem. This process is described in this section.

Let $\mathbf{K}_i$ be the local subdomain matrix obtained by the sub-assembling of element matrices of elements contained in subdomain Ω_i. The global stiffness matrix $\mathbf{K}$ might be obtained by further assembling of these matrices on the interface. We introduce the *coarse space basis functions* on each subdomain Ω_i represented by columns of matrix $\mathbf{\Psi}_i$, which is the solution to the saddle point problem with multiple right hand sides

$$\begin{bmatrix} \mathbf{K}_i & \mathbf{C}_i^T \\ \mathbf{C}_i & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi}_i \\ \mathbf{\Lambda}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}. \tag{5}$$

This is a key problem in the BDDC method and deserves a careful explanation. Matrix $\mathbf{\Lambda_i}$ is a block of Lagrange multipliers, $\mathbf{I}$ is an identity block.

Matrix $\mathbf{C}_i$ represents constraints on functions $\mathbf{\Psi}_i$, one row per each. These constraints enforce prescribed values of the *coarse degrees of freedom* on subdomain. They guarantee continuity of solution in selected points (*corners*) or equality of averages over some subsets of interface (*edges* or *faces*) of adjacent subdomains. While the former type of constraints corresponds to exactly one nonzero entry in a row of $\mathbf{C}_i$, the latter leads to several nonzeros in a row. Each column of matrix $\mathbf{\Psi}_i$ defined by (5) represents one coarse space basis function on subdomain Ω_i and corresponds to one local coarse degree of freedom.

Using the coarse basis functions $\mathbf{\Psi}_i$, we define the *local coarse matrix* $\mathbf{K}_{Ci} = \mathbf{\Psi}_i^T \mathbf{K}_i \mathbf{\Psi}_i$ on each subdomain. This matrix has the dimension equal to the number of constraints for each subdomain.

Let $\mathbf{R}_{Ci}$ realize the restriction of global coarse degrees of freedom to local coarse degrees of freedom on subdomain Ω_i. Using this matrix, we can construct the global *coarse matrix* by the assembling procedure, formally written as

$$\mathbf{K}_C = \sum_{i=1}^{N} \mathbf{R}_{Ci}^T \mathbf{K}_{Ci} \mathbf{R}_{Ci}. \tag{6}$$

We are ready to describe the algorithm of the BDDC method. Suppose $\widehat{\mathbf{r}} = \widehat{\mathbf{g}} - \widehat{\mathbf{S}}\,\widehat{\mathbf{u}}$ is a residual within the PCG method. Let us define matrices $\mathbf{E}_i^T$ for distribution of $\widehat{\mathbf{r}}$ to subdomains. Each matrix selects the interface unknowns of subdomain Ω_i and weights them so that the decomposition of unity applies to the residual across subdomains. It puts zeros to unknowns interior to subdomains. This corresponds

to computing with Schur complement (see [7] for details). The residual assigned to subdomain Ω_i is computed as $\mathbf{r}_i = \mathbf{E}_i^T \widehat{\mathbf{r}}$. The subdomain correction from Ω_i is now defined as the solution to system

$$\begin{bmatrix} \mathbf{K}_i & \mathbf{C}_i^T \\ \mathbf{C}_i & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_i \\ \lambda_i \end{bmatrix} = \begin{bmatrix} \mathbf{r}_i \\ \mathbf{0} \end{bmatrix}. \tag{7}$$

The residual for the coarse problem is constructed using the coarse basis functions subdomain by subdomain and assembling the contribution as

$$\mathbf{r}_C = \sum_{i=1}^{N} \mathbf{R}_{Ci}^T \mathbf{\Psi}_i^T \mathbf{E}_i^T \widehat{\mathbf{r}}. \tag{8}$$

The coarse correction is defined as the solution to problem

$$\mathbf{K}_C \, \mathbf{z}_C = \mathbf{r}_C. \tag{9}$$

Both corrections are then added together and averaged on the interface by matrices $\mathbf{E}_i$ to produce the preconditioned residual

$$\widehat{\mathbf{z}} = \sum_{i=1}^{N} \mathbf{E}_i \left(\mathbf{\Psi}_i \mathbf{R}_{Ci} \mathbf{z}_C + \mathbf{z}_i \right). \tag{10}$$

Properties of the coarse space are fully determined by constraints in matrices $\mathbf{C}_i$. The more constraints are prescribed, the more efficient preconditioner is constructed, but the larger coarse space is obtained, making factorization of matrix $\mathbf{K}_C$ more expensive.

4. Numerical results

We investigate the two ways of constructing the coarse space on a problem of elasticity analysis of a turbine nozzle, through which the steam enters the turbine blades. The geometry is discretized using 2 696 quadratic elements, which leads to 13 418 nodes and 40 254 unknowns. The mesh was divided into 16 and 32 subdomains, respectively. The division into 16 subdomains is depicted in Figure 1. Presented calculations were performed on 16 processors of SGI Altix 4700 computer of Supercomputing Centre of Czech Technical University in Prague.

The first experiment consists in adding more interface nodes to the set of corners. It should be noted that the initial set of corners is already sufficient for all subdomain problems to be nonsingular, as well as the coarse problem. In Figure 2, we present the plot of number of PCG iterations with respect to the number of corners. The effect on condition number is presented in Figure 3.

We can observe from these plots that some initial amount of corners is necessary for fast decrease of these values, while from some amount, these values behave quite

linearly in dependence on number of corners. These points of break are quite important, because they correspond to the optimal values of wall clock times presented in Figure 4. This is caused by the fact that the number of PCG iterations decreases rapidly at this point, while for adding more corners, the factorization of the growing coarse problems starts to dominate the time. Thus, it is desirable to set-up the preconditioner in such a way, that it works around this point or slightly to the right. The problem is that this point is unknown a priori. For the turbine nozzle, it corresponds to approximately 20 percent of all interface nodes for 16 subdomains and to 25 percent for the case of 32 subdomains. Similar results were also observed for other industrial problems. Although this value highly depends on problem topology and division into subdomains, from the practical point of view it seems that putting as much as a quarter of interface nodes into the set of corners is a reasonable set-up.

In the second experiment, we vary the size of the coarse problem in a conceptually different way – besides the continuity at corners, we enforce the equality of arithmetic averages of the approximate solution on all edges, on all faces, and on both edges and faces.

The results for the division into 16 subdomains are summarized in Table 1 for the initial set of 30 corner nodes, and in Table 2 for 280 corner nodes, the optimal number determined in the first experiment.

We can observe that adding constraints on averages can significantly improve the preconditioner and decrease the computational time. However, averages on faces might be too expensive considering time of computation. It could also be seen from the tables that using the optimal number of corner nodes can lead to improvement of computational times after addition of averages on edges.

5. Conclusion

We have presented a comparison of two ways for generating the coarse space in the BDDC method. We can conclude that while adding averages over edges and faces

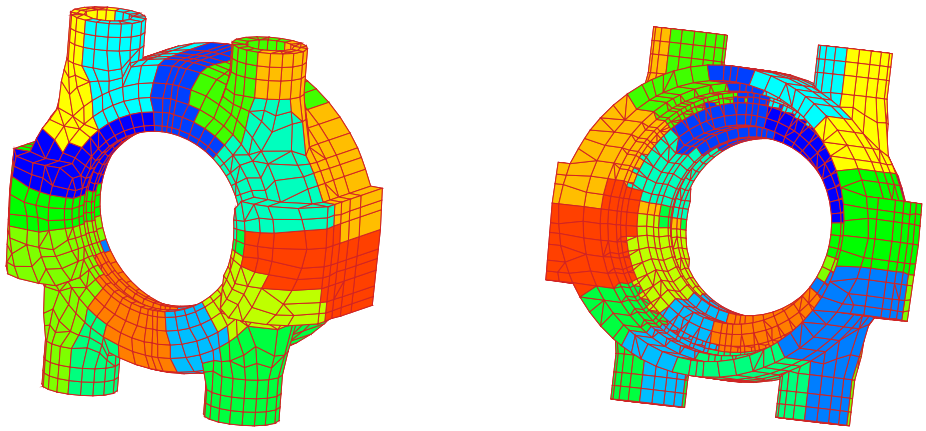

Fig. 1: *Turbine nozzle, division into 16 subdomains.*

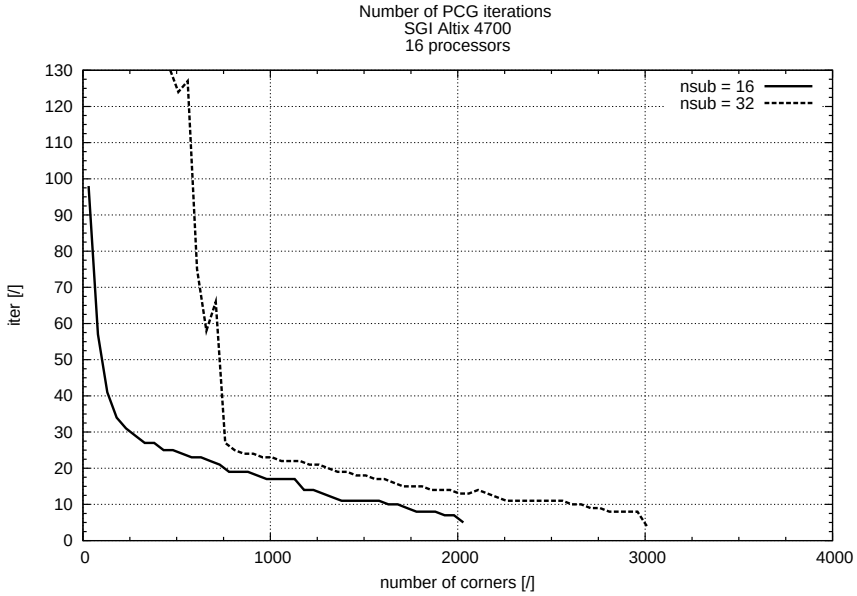

Fig. 2: *Turbine nozzle, number of iterations in dependence on number of corners, 16 and 32 subdomains.*

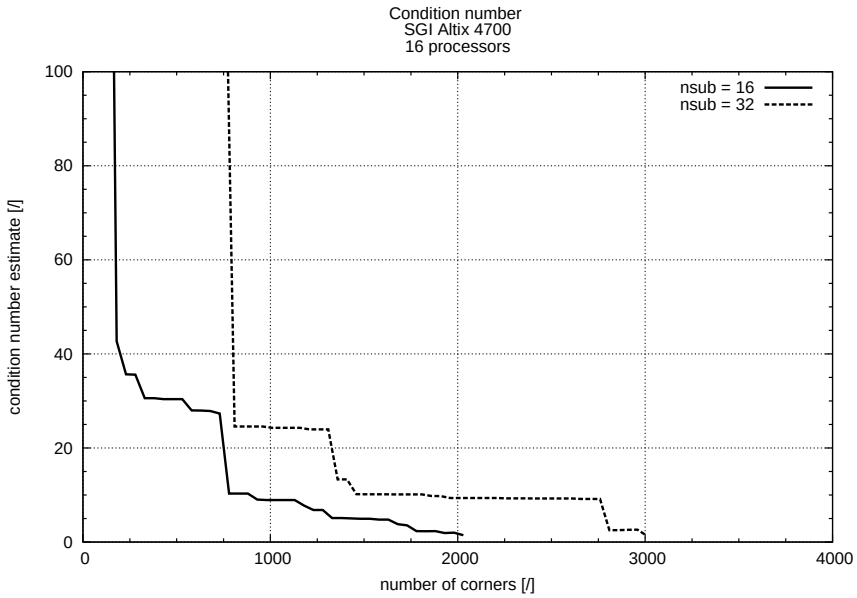

Fig. 3: *Turbine nozzle, condition number in dependence on number of corners, 16 and 32 subdomains.*

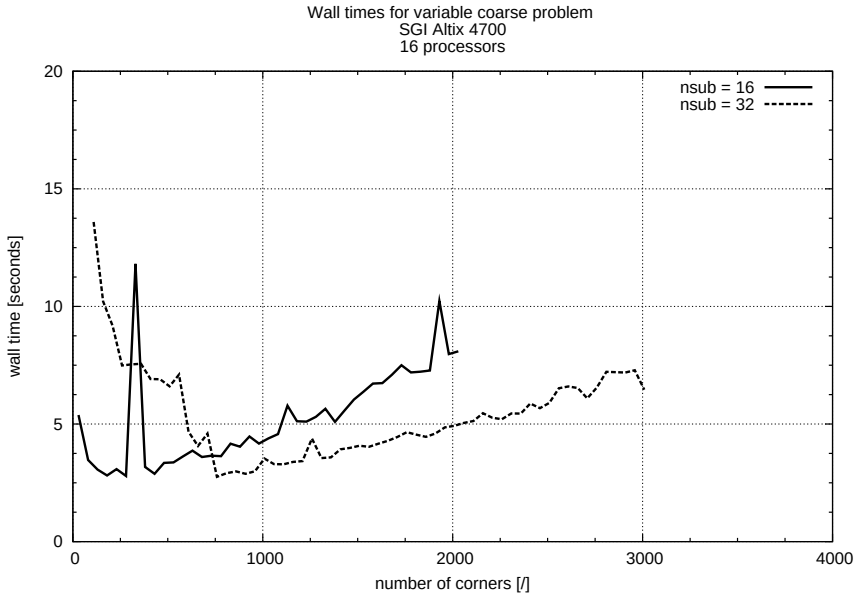

Fig. 4: *Turbine nozzle, wall clock time in dependence on number of corners, 16 and 32 subdomains.*

coarse problem	corners	corners+edges	corners+faces	corners+edges+faces
iterations	98	78	41	36
cond. number est.	2 933	1 546	164	142
factorization (sec)	0.5	0.6	0.6	0.8
pcg iter (sec)	4.6	3.7	1.9	1.9
total (sec)	5.3	4.5	2.8	2.9

Tab. 1: *Turbine nozzle, 16 subdomains, 30 corners, adding averages.*

coarse problem	corners	corners+edges	corners+faces	corners+edges+faces
iterations	29	26	26	23
cond. number est.	36	23	25	13
factorization (sec)	1.2	1.1	1.5	1.8
pcg iter (sec)	1.9	1.6	1.7	1.8
total (sec)	3.6	2.9	3.5	3.8

Tab. 2: *Turbine nozzle, 16 subdomains, 280 corners, adding averages.*

is often applied in literature, addition of more corners might be also contributive and the best results are likely to be obtained by combination of both approaches. According to our observations, there is an optimal number of corner nodes, with a steep decrease of number of iterations followed by the lowest computational time. Constraints on averages over faces might be too expensive compared to averages over edges. These observations advocate the adaptive approach for selecting constraints described in [6].

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