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NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS BY MEANS OF B-SPLINES

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Galerkin method is often used for solving boundary value problems. The most favorite method for solving problems from engineering practice, the finite element method (FEM), corresponds to the Galerkin method, where in particular continuous functions with small support form a basis. By Céa's lemma (see [5]), the error of the Galerkin approximation is bounded by means of the minimal error in the space of test functions. It means that success of the Galerkin method depends on the choice of basis functions. If we focus our attention on approximation theory, then B-splines represent a successful tool for approximation of functions. B-splines are piecewise polynomial functions with compact support that can be computed by means of simple schemes. Their differentiation and integration can be algorithmized. They are closely connected to computational geometry (see [5], [4]).

In this article, we deal with solution of boundary value problems using the Galerkin method, where weighted B-splines form the basis. These splines and their properties are described in the first section. Examples of solutions of 1D boundary value problems using B-spline basis are given in the second section.

1. B-splines and their properties

Definition 1 Let $b^0(x)$ be the characteristic function of the interval $[0,1]$ and

$$b^n(x) = \int_{x-1}^{x} b^{n-1}(\xi)\,\mathrm{d}\xi, \ \ n = 1, 2, \dots . \tag{1}$$

For integer $h > 0$ and number $k \in \mathbb{Z}$, the function

$$b^n_{k,h}(x) = b^n(x/h - k) \tag{2}$$

is the B-spline of order n on the grid of width h.

Remark 1 B-splines $b^n_{k,h}(x)$ have the following useful properties:

- B-spline $b^n_{k,h}(x)$ is positive on the interval $(kh, (k+n+1)h)$ and vanishes outside this interval.
- B-splines $b^n_{k,h}(x)$ are polynomials of order n on each interval $(kh, (k+1)h)$, $k = 0, \dots, n$.

- Recursive formulas enable to compute the derivatives

$$\frac{\mathrm{d}b^n_{k,h}(x)}{\mathrm{d}x} = \frac{1}{h}\left(b^{n-1}_{k,h}(x) - b^{n-1}_{k+1,h}(x)\right) \tag{3}$$

and the scalar products

$$s^n_{k-l} = \int_{\mathbb{R}} b^n_{k,h}(x) b^n_{l,h}(x)\,\mathrm{d}x = hb^{2n+1}(n+1+k-l), \tag{4}$$

$$\int_{\mathbb{R}} (b^n_{k,h}(x))'(b^n_{l,h}(x))'\,\mathrm{d}x = \frac{1}{h}(2s^{n-1}_{k-l} - s^{n-1}_{k-l-1} - s^{n-1}_{k-l+1}), \tag{5}$$

which we may encounter in the weak formulation of certain boundary value problems.

- The identity

$$b^n_{k,h}(x) = 2^{-n}\sum_{l=0}^{n+1}\binom{n+1}{l} b^n_{2k+l,\frac{h}{2}}(x) \tag{6}$$

is useful for mesh refinement.

- It is possible, thanks to Marsden's equality

$$(x-t)^n = \sum_{k\in\mathbb{Z}} h^n(k+1-\frac{t}{h})\dots(k+n-\frac{t}{h})b^n_{k,h}(x), \quad x,t\in\mathbb{R}, \tag{7}$$

to express any polynomial as a linear combination of the B-splines. The relation (7) plays an important role in the stabilization of bases and error estimates.

We receive multivariate B-splines as tensor products of the univariate ones.

Definition 2 For $x\in\mathbb{R}^m$, $k\in\mathbb{Z}^m$, $n\in\mathbb{N}$ and $h>0$ the function

$$b^n_{k,h}(x) = \prod_{i=1}^{m} b^n_{k_i,h}(x_i) \tag{8}$$

is the m-variate B-spline of degree n on the grid of width h.

The nonzero restrictions of B-splines $b^n_{k,h}$ to Ω can be taken as a basis for the solution of Neumann boundary value problem on a bounded domain $\Omega\subset\mathbb{R}^m$. But these B-splines are not suitable for solving any Dirichlet boundary value problem, because their linear combination

$$\sum_{k\in K} b^n_{k,h}(x)u_k,\ \ K = \{k|\operatorname{supp} b^n_{k,h}\cap\Omega\neq 0\}$$

generally does not satisfy essential boundary conditions. It is possible to remove this disproportion if we work with weighted B-splines.

Definition 3 Let a weight function[1] w and a B-spline $b^n_{k,h}$ be given, then

$$b_k(x) = w(x) b^n_{k,h}(x) \tag{9}$$

is called the weighted B-spline.

2. Boundary value problems and B-splines

Example 1 Consider the 1D Neumann boundary value problem

$$u''(x) + 16^2 u(x) = x, \quad x \in (0,1), \tag{10}$$

$$u'(0) = u'(1) = 0. \tag{11}$$

Find an approximation of the weak solution using B-splines defined above.
Solution: We find $u \in W^{1,2}(0,1)$ such that

$$-\int_0^1 u'v' \,\mathrm{d}x + 16^2 \int_0^1 uv \,\mathrm{d}x = \int_0^1 xv \,\mathrm{d}x, \ \forall v \in W^{1,2}(0,1). \tag{12}$$

The unknown function $u(x)$ is approximated by

$$\tilde{u}(x) = \sum_{k=1}^{N} b^3_{k-3,h}(x) u_k$$

over N uniformly distributed nodes ($h = \frac{1}{N-1}$). This approximation, in conjunction with the Galerkin method, provides a mesh-free computational formulation of the boundary value problem. The system of linear equations has the form

$$A\tilde{u} = f, \tag{13}$$

$$\tilde{u} = (u_1, \dots, u_N)^T, \quad f = (f_1, \dots f_N)^T, \quad A = \begin{pmatrix} a_{11} & \dots & a_{1N} \\ \dots & \dots & \dots \\ a_{N1} & \dots & a_{NN} \end{pmatrix},$$

$$f_j = \int_\Omega f(x) b^3_{j-3,h}(x) \,\mathrm{d}x,$$

$$a_{i,j} = \int_\Omega \left[-(b^3_{i-3,h}(x))'(b^3_{j-3,h}(x))' + 16^2\, b^3_{i-3,h}(x) b^3_{j-3,h}(x) \right] \mathrm{d}x.$$

Results for $N = 11$ nodes are given in Figure 1 and in Table 1.

[1]Weight function is a nonnegative continuous function on $\overline{\Omega}$ that vanishes on the boundary $\partial\Omega$. For $r > 0$ we can put $w(x) = \mathrm{dist}(x, \partial\Omega)^r,\ x \in \Omega$. If $r = 1$ then w is called the standard weight function.

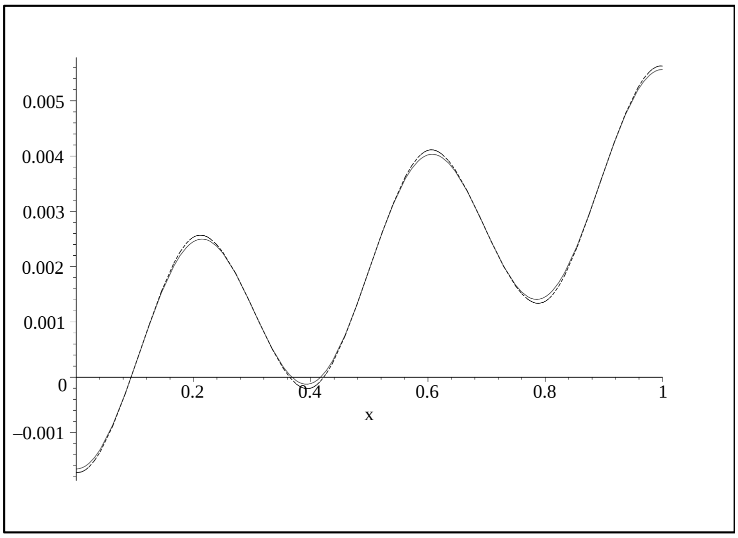

Fig. 1: *The exact solution of the Neumann BVP and its approximation for* $N = 11$.

N	8	11	14	21	31
$\max\lvert u-\tilde{u}\rvert$	1.8×10^{-3}	8×10^{-5}	14×10^{-6}	16×10^{-7}	24×10^{-8}

Tab. 1: *Dependence of the error of the approximation on the number of nodes.*

Example 2 Solve the 1D Dirichlet boundary value problem

$$u''(x) + 16^2 u(x) = x, \quad x \in (0,1), \tag{14}$$

$$u(0) = u(1) = 0 \tag{15}$$

using B-splines.
Solution: We find $u \in W_0^{1,2}(0,1)$ such that

$$-\int_0^1 u'v' \,\mathrm{d}x + 16^2 \int_0^1 uv \,\mathrm{d}x = \int_0^1 xv \,\mathrm{d}x, \ \forall v \in W_0^{1,2}(0,1). \tag{16}$$

We suppose that nodes $x_1, \ldots, x_N$, at which the approximate values are computed, are uniformly distributed.

i) We replace the original set $\{b^3_{k-3,h}(x)\}_{k=1}^N$ by the set of weighted B-splines. We consider the Galerkin approximation in the form

$$\tilde{u}(x) = \sum_{k=1}^N w_1(x) b^3_{k-3,h}(x) u_k, \text{ where } w_1(x) = \begin{cases} x/s, & \text{if } 0 < x < s \\ 1, & \text{if } s \le x \le 1-s \\ (1-x)/s, & \text{if } 1-s < x < 1 \end{cases}$$

and the parameter s represents the width of the strip inside $[0,1]$, where the function $w_1 \neq 1$. Results for $N = 11$ and $s = 0.2$ are given in Figure 2. The dependence of

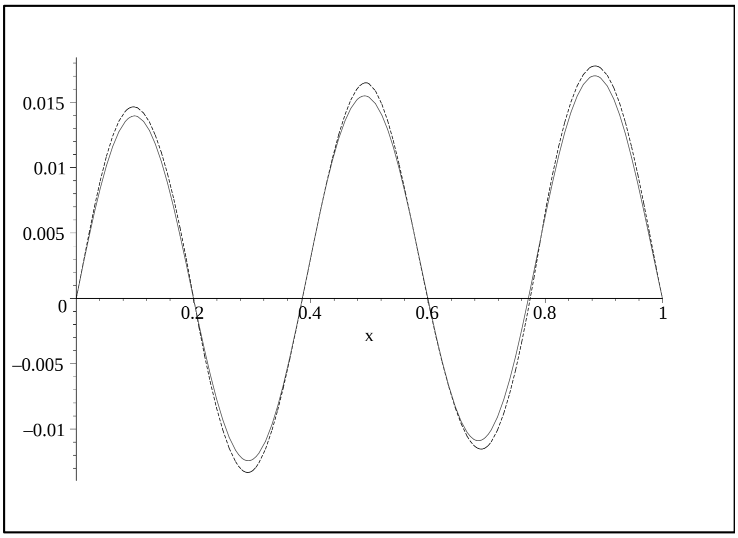

Fig. 2: *The exact solution of the Dirichlet BVP and its approximation for $N = 11$, linear weight function and $s = 0.2$.*

N	8	11	14	21	31
$s = 0.2$	1.5×10^{-2}	1.1×10^{-3}	4×10^{-4}	2×10^{-4}	10^{-4}
$s = 0.3$	1.2×10^{-3}	1.7×10^{-3}	1.2×10^{-3}	8×10^{-4}	4.8×10^{-4}
$s = 0.4$	2.4×10^{-3}	4×10^{-4}	3×10^{-4}	2×10^{-4}	1.5×10^{-4}

Tab. 2: *The error of the approximation for different number of nodes and for different widths of the strip.*

the error $\max|u - \tilde{u}|$ on the number N of nodes and on the width of the strip s can be seen in Table 2.

Not only the width s of the strip affects the quality of the approximate solution, but also the choice of the proper weight function is important. The errors of approximation for the linear weight function w_1 and the quadratic weight function

$$w_2(x) = \begin{cases} \left(2 - \dfrac{x}{s}\right)\dfrac{x}{s}, & \text{if } 0 < x < s \\ 1, & \text{if } s \le x \le 1 - s \\ \left(2 - \dfrac{1-x}{s}\right)\dfrac{1-x}{s}, & \text{if } 1 - s < x < 1 \end{cases}$$

for $N = 11$ and $s = 0.2$ are compared in Table 3. The quadratic weight function produces more accurate results than the linear weight function.

The errors for the linear weight function w_1 and for the quadratic weight function w_2 for $N = 11$ and for different values of the parameter s are given in Table 4.

Note that the weighted B-splines $w_1 b^3_{k,h}$ have not the first derivative and $w_2 b^3_{k,h}$ have not the second derivative in some points of the interval $[0, 1]$. Considering that

N	8	11	14	21	31
e_{w_1}	1.5×10^{-2}	1.1×10^{-3}	4×10^{-4}	2×10^{-4}	10^{-4}
e_{w_2}	8×10^{-3}	5.3×10^{-4}	2×10^{-4}	8.8×10^{-5}	6.4×10^{-5}

Tab. 3: *The error* e_{w_i} *=max*$|u - \tilde{u}_{w_i}|$ *of the approximation for different form of weight functions and different values of* s*.*

s	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$e_{w_1} \times 10^4$	22	11	16	17	7.4	4	4.3	11
$e_{w_2} \times 10^4$	13	5.3	5.1	6.1	5.4	4	3.1	3.4

Tab. 4: *The error* e_{w_i} *=max*$|u - \tilde{u}_{w_i}|$ *of the approximation for different forms of weight functions and different values of* s*.*

these functions are elements of the space $W_0^{1,2}$, they can be advantageously used in our problem, in spite of the fact that the smoothness of these basis functions is lower than the smoothness of original cubic splines. It can be seen from Figure 2 and Table 3 that this fact has only small influence on approximation properties of the used weighted basis. The amplitude and frequency of the approximation received for $N = 11$ and w_1 are in accordance with the analytic solution. The errors received in the case of piecewise linear and quadratic weight functions are similar. The quadratic weight function gives a bit better results, very similar to the case when we use e.g. the perfectly smooth weight function $w(x) = \sin(\pi x)$. (For more information about weight functions see [4].)

ii) If the basis contains B-splines that only have a small part of their support in the considered interval, then the system of linear equations (13) is ill-conditioned and convergence of the iterative process can be slow. This can be improved if we modify the B-splines whose supports intersect the boundary.

Consider again the uniformly distributed nodes $0 = x_1 < \cdots < x_N = 1$, cubic splines $b^3_{k,h}$, and the weight function w_1. Let

$$\begin{aligned}\tilde{u}(x) = &\sum_{k=5}^{N-4} w_1(x) b^3_{k-3,h}(x) u_k \\ &+ 4w_1(x)\left(b^3_{-2,h}(x)u_1 + b^3_{0,h}(x)u_3 + b^3_{N-3,h}(x)u_N + b^3_{N-5,h}(x)u_{N-2}\right) \\ &- 6w_1(x)\left(b^3_{-1,h}(x)u_2 + b^3_{N-4,h}(x)u_{N-1}\right) - w_1(x)\left(b^3_{1,h}(x)u_4 + b^3_{N-6,h}(x)u_{N-3}\right).\end{aligned}$$

The errors for $s = 0.2$ and for different values of N are provided in Table 5.

N	8	11	14	21	31
$\max_{w_1}\lvert u - \tilde{u}\rvert$	1.5×10^{-2}	1.4×10^{-3}	3×10^{-4}	2×10^{-4}	8×10^{-5}

Tab. 5: *Dependence of the error of approximation on the number of nodes.*

3. Conclusion

In this contribution, we presented methods of solving boundary value problems using the Galerkin method with B-spline basis. This method belongs to the meshless methods, because no explicitly given mesh is required for its realization. (For more information about the meshless methods see [1], [2], [3]).

The weighted B-splines are a simple and comfortable tool from the computational point of view (recursive formulas enable to compute derivatives and scalar products of B-splines easily, see Remark 1). The size of support and smoothness of B-splines depend on the parameters n and h, which we choose at the beginning of the computation. In case of the Neumann boundary value problem it suffices to work with B-splines only, whereas for the Dirichlet boundary value problem it is necessary to use the weighted B-splines.

The error of any approximation depends not only on the number of nodes, but on another factors, too. Example 2 showed that in the case of the Dirichlet problem the error of the resulting approximation can become smaller if a proper weight function is chosen. The influence of the choice of the weigh function and of the width s of the the strip on the approximate solution can be the subject of a further study.

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