RESONANCE BEHAVIOUR OF THE SPHERICAL PENDULUM DAMPER*

Cyril Fischer, Jiří Náprstek

Abstract

The pendulum damper modelled as a two degree of freedom strongly non-linear auto-parametric system is investigated using two approximate differential systems. Uni-directional harmonic external excitation at the suspension point is considered. Semi-trivial solutions and their stability are analyzed. The thorough analysis of the non-linear system using less simplification than it is used in the paper [2] is performed. Both approaches are compared and conclusions are drawn.

1. Introduction

Many structures encountered in the civil and mechanical engineering are equipped with various devices for reducing dynamic response component due to external excitations. Among other low cost passive systems the pendulum dampers are still very popular for their reliability and simple maintenance, see e.g. [1]. However the dynamic behaviour of such a pendulum is significantly more complex than it is supposed by a widely used simple linear SDOF model working in the (xz) vertical plane only, see Figure 1. The conventional linear model is satisfactory only if the kinematic excitation $a(t)$ introduced at the suspension point is very small in amplitude and if its frequency remains outside a resonance frequency domain.

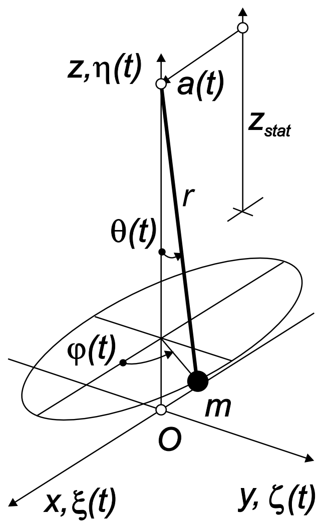

Fig. 1: *Sketch of the pendulum and coordinate systems used.*

2. Mechanical energy balance

Let us consider the kinematic excitation $a(t)$ at the suspension point in the x direction only. The natural choice of the coordinate system suitable for description of the movement of the pendulum would be the spherical coordinate system described by the angles θ (in the xz plane), φ (diversion from the xz plane) and radius $r = const$ (see Fig. 1). However, such a choice does not allow to consider the angle φ as a perturbation of the pure planar motion described by θ, r only. Indeed, even for a small transversal motion (in the y direction), the full range $\varphi \in \langle 0, 2\pi)$ occurs. Thus the mechanical energy balance has to be written in the Cartesian coordinates

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$(\xi(t)=\xi, \zeta(t)=\zeta, \eta(t)=\eta)$. The kinetic and potential energies T, V are described by:

$$
\begin{aligned}
T &= m(\dot{\xi}^2+\dot{\zeta}^2+\dot{\eta}^2+2\dot{a}\dot{\xi}+\dot{a}^2)/2, && (1)\\
V &= mg\eta && (2)
\end{aligned}
$$

and the geometric constraint of the suspension is expressed as:

$$
\xi^2+\zeta^2+(1-\eta)^2=r^2, \tag{3}
$$

where m, r - mass and suspension length of the pendulum

$a=a(t)$ - kinematic excitation at the suspension point

From the relation between the spherical and Cartesian coordinates and the geometric constraint (3) it follows:

$$
\eta=r(1-\cos\theta)\ ;\ \ \dot{\eta}^2=r^2\dot{\theta}^2\sin^2\theta\ ;\ \ \sin\theta=\frac{\varrho}{r}, \quad \text{where} \quad \varrho^2=\xi^2+\zeta^2. \tag{4}
$$

A hypothesis that the amplitude of $\theta(t)$ is small makes acceptable an approximation:

$$
\theta=\arcsin\frac{\varrho}{r}\approx\frac{\varrho}{r}+\frac{1}{6}\frac{\varrho^3}{r^3} \quad \Rightarrow \quad \dot{\theta}^2=\frac{\dot{\varrho}^2}{r^2}\left(1+\frac{\varrho^2}{2r^2}\right)^2. \tag{5}
$$

The equations of the motion follows from the Lagrangian principle:

$$
\partial_t(\partial_{\dot{\chi}}T)-\partial_{\chi}T+\partial_{\chi}V=0\ , \qquad \text{for } \chi\in\{\xi,\varphi\}. \tag{6}
$$

Using (1), (2), (4), (5) and (6) an approximate Lagrangian system in the x, y coordinates for the components ξ, ζ on the level $O(\varepsilon^6)$; $\varepsilon^2=(\xi^2+\zeta^2)/r^2$ can be obtained. The approximate linear damping with the relative scale ω_b equivalent in both components ξ, ζ will be included, giving the differential system:

$$
\begin{aligned}
&\ddot{\xi}+2\omega_b\dot{\xi}+\xi\left(1+\frac{\xi^2+\zeta^2}{2r^2}\right)\left(\omega_0^2+\frac{((\xi^2+\zeta^2)^{\bullet})^2}{4r^4}+\frac{\left(1+\frac{\xi^2+\zeta^2}{2r^2}\right)(\xi^2+\zeta^2)^{\bullet\bullet}}{2r^2}\right)=-\ddot{a},\\
&\ddot{\zeta}+2\omega_b\dot{\zeta}+\zeta\left(1+\frac{\xi^2+\zeta^2}{2r^2}\right)\left(\omega_0^2+\frac{((\xi^2+\zeta^2)^{\bullet})^2}{4r^4}+\frac{\left(1+\frac{\xi^2+\zeta^2}{2r^2}\right)(\xi^2+\zeta^2)^{\bullet\bullet}}{2r^2}\right)=\ 0,
\end{aligned} \tag{7}
$$

where $\omega_0^2=g/r$. Taking into account the additional simplification,

$$
\left(1+\frac{(\xi^2+\zeta^2)}{2r^2}\right)^2\approx 1, \quad \frac{\chi}{r^4}\left(1+\frac{(\xi^2+\zeta^2)}{2r^2}\right)\left((\xi^2+\zeta^2)^{\bullet}\right)^2\approx 0 \ \text{ for } \chi\in\{\xi,\zeta\}, \tag{8}
$$

the simplified form of the differential system can be obtained (see [2]):

$$
\begin{aligned}
\ddot{\xi}+\frac{1}{2r^2}\xi(\xi^2+\zeta^2)^{\bullet\bullet}+2\omega_b\dot{\xi}+\omega_0^2(\xi+\frac{1}{2r^2}\xi(\xi^2+\zeta^2)) &= -\ddot{a},\\
\ddot{\zeta}+\frac{1}{2r^2}\zeta(\xi^2+\zeta^2)^{\bullet\bullet}+2\omega_b\dot{\zeta}+\omega_0^2(\zeta+\frac{1}{2r^2}\zeta(\xi^2+\zeta^2)) &= 0.
\end{aligned} \tag{9}
$$

In both simplified and complete systems, neglecting the non-linear terms will result in two independent equations. Each of the components ξ, ζ can be considered as arbitrarily small and independently and continuously limited to zero. Therefore the system is auto-parametric and respective procedures can be applied [4].

3. Semi-trivial solution

To investigate the semi-trivial solution let us substitute $\zeta = 0$ into Eqs (7), (9) and specify the excitation to be harmonic (see [3] for details): $a(t) = a_0 \sin \omega t$.

The semi-trivial solution of Eqs (7) or (9) should be searched in the form:

$$\xi_0 = a_c \cos \omega t + a_s \sin \omega t \;;\quad \zeta_0 = 0. \tag{10}$$

The coefficients a_c, a_s in general should be considered as functions of time: $a_c = a_c(t)$, $a_s = a_s(t)$. If a stationary solution exists for a given excitation frequency ω, then a_c, a_s should converge to constants for increasing $t \to \infty$. In such a case the coefficients a_c, a_s can be considered constant. Let us substitute (10) into Eq. (7) and (9), multiply them by $\sin(\omega t)$ or $\cos(\omega t)$ and integrate the resulting expressions over the interval $t \in (0, 2\pi/\omega)$. The described operation (so called *harmonic balance* operation) results for each of the equations (7) or (9) in an algebraic system consisting of two equations. For the simplified case of Eq. (9) it is:

$$\begin{aligned} a_c \left((\omega_0^2 - \omega^2) + \frac{1}{2r^2} \left(\frac{3}{4}\omega_0^2 - \omega^2 \right) (a_c^2 + a_s^2) \right) + 2\omega\omega_b \cdot a_s &= 0, \\ a_s \left((\omega_0^2 - \omega^2) + \frac{1}{2r^2} \left(\frac{3}{4}\omega_0^2 - \omega^2 \right) (a_c^2 + a_s^2) \right) - 2\omega\omega_b \cdot a_c &= a_0 \cdot \omega^2. \end{aligned} \tag{11}$$

If both equations are raised to the second power and summed together, then, finally, the equation for the amplitude of the response arises ($R_0^2 = a_c^2 + a_s^2$):

$$R_0^2 \left[4\omega^2\omega_b^2 + \left((\omega^2 - \omega_0^2) + \frac{R_0^2}{2r^2} \left(\omega^2 - \frac{3}{4}\omega_0^2 \right) \right)^2 \right] - 4\omega^4 a_0^2 \;=\; 0. \tag{12}$$

Applying the same procedure to the original system (7), one can get a similar equation for the amplitude:

$$R_0^2 \left[4\omega^2\omega_b^2 + \left((\omega^2 - \omega_0^2) + \frac{R_0^2}{2r^2} \left(\omega^2 - \frac{3}{4}\omega_0^2 \right) + \omega^2 \frac{R_0^4}{8r^4} \left(3 + \frac{5R_0^2}{8r^2} \right) \right)^2 \right] - \omega^4 a_0^2 = 0. \tag{13}$$

The Eqs (12) and (13) are known as *resonance curves.* They express the dependence of the amplitude R_0^2 of the solution (response) on the excitation frequency. Both curves are demonstrated in Figure 2. Depending on the parameters a_0, ω_b and ω, this relations can lose their unique character in some intervals of ω.

4. Perturbation of the semi-trivial solution

To assess the stability of the semi-trivial solution we will endow the semi-trivial solution (10) with small (in the meaning of a norm) perturbations u, v in both coordinates:

$$\begin{aligned} \xi &= \xi_0 + u, \qquad & u = u(t) = u_c \cos \omega t + u_s \sin \omega t. \\ \zeta &= 0 + v, \qquad & v = v(t) = v_c \cos \omega t + v_s \sin \omega t. \end{aligned} \tag{14}$$

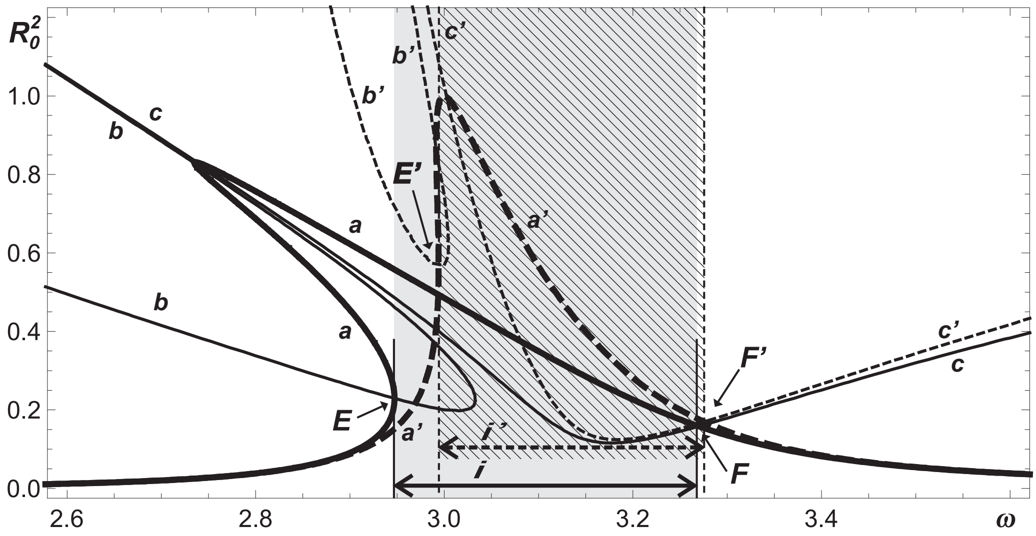

Fig. 2: *Resonance curves (thick lines a, a') and stability limits (thin lines b, b', c, c') of the semi-trivial solution computed using the original (solid lines a, b, c) and simplified (dashed lines a', b', c') equations.*
Curves (b, b'): in (xz) plane – ξ stability limit, Eqs (16), (18).
Curves (c, c'): out of (xz) plane – ζ stability limit, Eqs (17), (19).
Interval **i** *corresponds to the non-stability interval of the original formulas (16–17).*
Interval **i'** *corresponds to the non-stability interval of the simplified formulas (18–19).*
Values used: $r = 1$, $g = 9.81$, $\omega_b = 0.075$, $a_0 = 0.05$.

As the perturbations are expected to be small, only the first powers of u, v and their derivatives are kept after inserting expressions (14) into Eqs (7) and (9). After the harmonic balance operation and some algebra one obtains two linear algebraic systems for u_c, u_s and v_c, v_s. For the simplified case (9) it reads:

$$\begin{pmatrix} w_1 & w_2 \\ w_3 & w_1 \end{pmatrix} \begin{pmatrix} u_c \\ u_s \end{pmatrix} = 0 \; ; \quad \begin{pmatrix} z_1 & z_2 \\ z_3 & z_1 \end{pmatrix} \begin{pmatrix} v_c \\ v_s \end{pmatrix} = 0 \; ; \tag{15}$$

where it has been denoted:

$$w_1 = \left[2\omega\omega_b + \tfrac{1}{4r^2}\Omega_1 a_c a_s\right]; \; w_2 = \left[2\omega\omega_b + \tfrac{1}{4r^2}\Omega_1 a_c a_s\right]; \; w_3 = \left[\tfrac{1}{4r^2}\Omega_1 a_c a_s - 2\omega\omega_b\right]$$

$$z_1 = \left[\Omega_2 + \tfrac{1}{8r^2}(\Omega_1 a_c^2 + \Omega_3 a_s^2)\right]; \; z_2 = \left[\tfrac{1}{4r^2}\Omega_4 a_c a_s + 2\omega\omega_b\right]; \; z_3 = \left[\tfrac{1}{4r^2}\Omega_4 a_c a_s - 2\omega\omega_b\right]$$

$$\Omega_1 = 3\omega_0^2 - 4\omega^2; \; \Omega_2 = \omega_0^2 - \omega^2; \; \Omega_3 = \omega_0^2 + 4\omega^2; \; \Omega_4 = \omega_0^2 - 4\omega^2.$$

The both systems (15) are homogeneous and independent of excitation amplitude. Consequently to receive a non-trivial solution for u_c, u_s or v_c, v_s, the determinant of the systems (15) must equal zero. This rationale leads to two independent equations:

$$\frac{1}{2r^2}\Omega_1 R_0^2 \left(\Omega_2 + \frac{3}{32r^2}\Omega_1 R_0^2\right) + \Omega_2^2 + 4\omega^2\omega_b^2 = 0, \tag{16}$$

$$\frac{1}{2r^2} R_0^2 \left(\omega_0 \Omega_2 + \frac{1}{32r^2}\Omega_1 \Omega_3 R_0^2\right) + \Omega_2^2 + 4\omega^2\omega_b^2 = 0. \tag{17}$$

Similar equations can also be formulated for the original system (7)

$$\begin{aligned}\frac{175\omega^4 R_0^{12}}{4096 r^{12}} + \frac{45\omega^4 R_0^{10}}{128 r^{10}} + \frac{5\omega^2\left(56\omega^2 - 15\omega_0^2\right)R_0^8}{256 r^8} + \frac{\omega^2\left(17\omega^2 - 14\omega_0^2\right)R_0^6}{8r^6} + \\ + \frac{3\left(-3\omega^2\Omega_2 + \left(\frac{1}{4}\Omega_1\right)^2\right)R_0^4}{4r^4} + \frac{\Omega_2\Omega_1 R_0^2}{2r^2} + \Omega_2^2 + 4\omega^2\omega_b^2 = 0,\end{aligned} \tag{18}$$

$$\begin{aligned}-\frac{5\omega^4 R_0^{12}}{4096 r^{12}} - \frac{\omega^4 R_0^{10}}{64 r^{10}} - \frac{\omega^2\left(24\omega^2 + \omega_0^2\right)R_0^8}{256 r^8} - \frac{\omega^2\left(3\omega^2 + \omega_0^2\right)R_0^6}{16r^6} + \\ + \frac{\omega_0^2\left(3\omega_0^2 - 8\omega^2\right)R_0^4}{64r^4} + \frac{\omega_0^2\Omega_2 R_0^2}{2r^2} + \Omega_2^2 + 4\omega^2\omega_b^2 = 0.\end{aligned} \tag{19}$$

Eqs (16)–(17) and (18)–(19) can be interpreted as limits dividing the plane (R_0^2, ω) into the stable and unstable domains. For given parameters r, ω_b, a_0 the unstable interval of excitation frequency is defined by the position of the intersections of the resonance curve with the corresponding stability limits (points **E**, **F** in Figure 2).

5. Post-critical response in the resonance domain

Let us try to assume a more general expressions as the basic solution:

$$\xi(t) = a_c(t)\cos\omega t + a_s(t)\sin\omega t\ ; \qquad \zeta(t) = b_c(t)\cos\omega t + b_s(t)\sin\omega t. \tag{20}$$

Increasing the number of unknown functions to four, one can exploit a possibility to formulate two arbitrarily selectable additional conditions. Then the following expressions for the first derivatives of the general solution (20) can be stated:

$$\dot{\xi}(t) = -a_c\omega\sin\omega t + a_s\omega\cos\omega t\ ; \qquad \dot{\zeta}(t) = -b_c\omega\sin\omega t + b_s\omega\cos\omega t, \tag{21}$$

where $a_c = a_c(t), a_s = a_s(t), b_c = b_c(t), b_s = b_s(t)$. Let us insert expressions (20), (21) in the simplified differential system (9) and apply the operation of the harmonic balance once again. After dull routine work one obtain the differential system for amplitudes a_c, a_s, b_c, b_s, whose system matrix $\mathbf{A}$ depends only on $a_c, a_s, b_c, b_s, \omega$:

$$\mathbf{A}\begin{bmatrix}\dot{a}_c\\ \dot{a}_s\\ \dot{b}_c\\ \dot{b}_s\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}a_c(8\Omega_2 r^2 + R_A^2\Omega_1) + 2b_s S_A^2\Omega_4 + 4a_s\omega_b\omega r^2 \\ a_s(8\Omega_2 r^2 + R_A^2\Omega_1) + 2b_c S_A^2\Omega_4 + 4a_c\omega_b\omega r^2 \ +8\omega^2 a_0 r^2 \\ b_c(8\Omega_2 r^2 + R_A^2\Omega_1) + 2a_s S_A^2\Omega_4 + 4b_s\omega_b\omega r^2 \\ b_s(8\Omega_2 r^2 + R_A^2\Omega_1) + 2a_c S_A^2\Omega_4 + 4b_c\omega_b\omega r^2\end{bmatrix}, \tag{22}$$

where it has been denoted

$$R_A^2 = a_c^2 + a_s^2 + b_c^2 + b_s^2\ ; \qquad S_A^2 = a_s b_c - a_c b_s. \tag{23}$$

The explicit solution of Eqs (22) is generally not possible in the resonance interval. However, from the numerical analysis can be seen that at least part of the resonance interval can be described by a steady state solution. The other part of the resonance interval, where the transient solution takes place, will not be discussed here.

The steady state response is characterized by constant amplitudes (for $t \to \infty$). This means that the time derivatives $\dot{a}_c, \dot{a}_s, \dot{b}_c, \dot{b}_s$ vanish for large t. The left-hand side of Eq. (22) vanishes and Eq. (22) reduces itself into the algebraic system. After tedious work, the relation between R_A^2, S_A^2 and ω can be deduced from Eq. (22), where the left hand side was substituted by the zero vector:

$$\begin{aligned} R_A^2 \left((8\Omega_2 r^2 + R_A^2 \Omega_1)^2 + 4\left(4\omega^2\omega_b^2 r^4 + S_A^4 \Omega_4^2\right)\right) - 8S_A^4 \left(8\Omega_2 r^2 + R_A^2\Omega_1\right)\Omega_4 &= 64 r^4 a_0^2 \omega^4, \\ S_A^2 \left(2R_A^2 \left(8\Omega_2 r^2 + R_A^2\Omega_1\right)\Omega_4 - \left(8\Omega_2 r^2 + R_A^2\Omega_1\right)^2 - 16\omega^2\omega_b^2 r^4 - 4S_A^4\Omega_4^2\right) &= 0. \end{aligned} \tag{24}$$

Parameter R_A^2 can be interpreted as a generalized total or effective amplitude including both components (20). As regards the S_A^2, it represents a certain characteristics of their phase shift. If $S_A^2 = 0$ the vectors $[a_c, a_s]$, $[b_c, b_s]$ are co-linear. It represents the motion in the vertical plane. Indeed, putting $S_A^2 = 0$ into the first equation of (24) one obtains the formula for the semi-trivial resonance curve (12). The case $S_A^2 \neq 0$ implies motion out of the vertical plane. For this case, an analysis of the system (24) was carried out, but it is beyond the scope of this contribution.

Using the procedure described above, a similar relation was also derived for the original system (7). The resulting formulas are rather complicated which fact makes an analysis of the individual components hardly feasible. On the other hand, it brings no new qualitative results comparing to the simplified version (24).

6. Conclusion

Analytical and numerical investigations have shown that the widely used linear model of the damping pendulum is acceptable only in a very limited extent of parameters concerning pendulum characteristics and excitation properties. In the case of a harmonic kinematic external excitation at the suspension point, it is necessary to thoroughly investigate the dynamic stability limits and post-critical behaviour. To investigate the stability of the semi-trivial solution, it is necessary to use the approximate equations in the Cartesian coordinates. Using the harmonic balance method the resonance curves of a planar stationary response as well as the stability limits of the semi-trivial solution in both response components have been determined. Omitting the simplification (8) results in very complicated formulas and brings only quantitative specification.

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