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AN UNSTEADY NUMERICAL SOLUTION OF VISCOUS COMPRESSIBLE FLOWS IN A CHANNEL*

Petra Punčochářová, Karel Kozel, Jiří Fürst, Jaromír Horáček

Abstract

The work deals with numerical solution of unsteady flows in a 2D channel where one part of the channel wall is changing as a given function of time. The flow is described by the system of Navier-Stokes equations for compressible (laminar) flows. The flow has low velocities (low Mach numbers) and is numerically solved by the finite volume method. Moving grid of quadrilateral cells is considered in the form of conservation laws using ALE (Arbitrary Lagrangian-Eulerian) method.

1. Introduction

This work presents an unsteady numerical solution of the system of Navier-Stokes equations for compressible laminar flow. An unsteady flow is caused by the moving part of the channel wall. The authors investigated flows in two types of channels, in an nonsymmetric channel and in a symmetric channel. The flow in a symmetric channel can represent a very simple model of airflow in a human vocal tract.

The numerical solution was obtained by the explicit central finite volume version of MacCormack scheme on a grid of quadrilateral cells.

2. Mathematical model

The 2D system of Navier-Stokes equations (1) was used as mathematical model to describe an unsteady viscous compressible laminar flow in a channel. The system is expressed in non-dimensional form:

$$W_t + F_x + G_y = \frac{1}{Re}(R_x + S_y), \tag{1}$$

$W = [\rho, \rho u, \rho v, e]^T$ is the vector of conservative variables, F and G are the vectors of inviscid fluxes, R and S are the vectors of viscous fluxes. Variable ρ denotes the density, u and v are the components of the velocity vector, and e is the total energy per unit volume. Static pressure p in the inviscid fluxes is expressed by the equation of state. Reynolds number $Re = \rho_\infty u_\infty H/\eta_\infty$ is computed from the inflow variables: $\rho_\infty = \text{const}$, $u_\infty = \text{const}$, $\eta_\infty = \text{const}$, and H is the inflow width of the channel. Non-dimensional dynamic viscosity $\eta = 1/Re$ is constant in our cases.

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2.1. Mathematical formulation

For the numerical solution, the domain of solution D and the boundary conditions have to be defined. Two channels were tested. The first is an nonsymmetric channel and the second is a symmetric channel. Boundary conditions were considered in the following form:

a) Upstream conditions: three components of W are given, the pressure is extrapolated.

b) Downstream conditions: the pressure is given, the other values are extrapolated or $\partial W/\partial \vec{n} = 0$ where $\vec{n}$ is an outlet normal vector.

c) On the solid wall, the velocity vector and the normal derivative of temperature vanish that is $(u, v)_{\text{wall}} = \vec{0}$ and $\partial T/\partial \vec{n} = 0$.

d) At the axis of symmetry, $(u, v) \cdot \vec{n} = 0$ is considered.

Figure 1 shows D_1, the domain of solution, which is called the nonsymmetric channel. The upper and lower boundary represent solid walls. The lower solid wall of the channel has a time changing part between points A and B that is a given function of time $g_1(t)$.

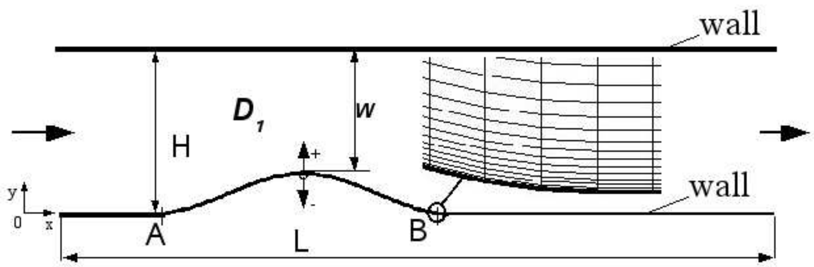


Fig. 1: *Domain of solution D_1 (the nonsymmetric channel).*

Figure 2 shows D_2, the domain of solution in the symmetric channel. The computational domain is only the lower half of the channel. Its upper boundary coincides with the axis of symmetry. The lower boundary represents a solid wall. The part of the wall between points A and B is changing and determined by $g_2(t)$, a given function of time.

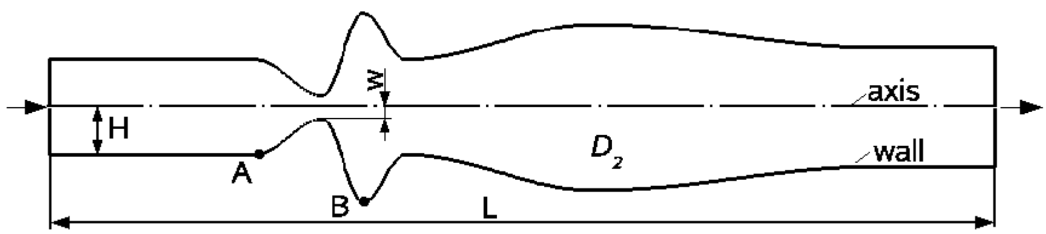


Fig. 2: *Domain of solution D_2 (the symmetric channel).*

3. Numerical solution

The numerical solution of the above two-dimensional problems is obtained by the finite volume method in the cell centered form (FVM) on a grid of quadrilateral cells.

The bounded domain D is divided into mutually disjoint sub-domains $D_{i,j}$ (e.g. quadrilateral cells). Equations (1) are integrated over subdomain $D_{i,j}$. By using the Green formula and the Mean Value Theorem, we can write the integral form of FVM:

$$W_t|_{i,j} = \frac{-1}{\mu_{i,j}}\left[\oint_{\partial D_{i,j}}(Fdy - Gdx) - \oint_{\partial D_{i,j}}(Rdy - Sdx)\right], \tag{2}$$

where $\mu_{i,j} = \int\int_{D_{i,j}} dxdy$ stand for the volumes of the cells. We get FVM in the differential form:

$$\frac{W_{i,j}^{n+1} - W_{i,j}^{n}}{\Delta t} = \frac{-1}{\mu_{i,j}}\sum_k [(\tilde{F}_k - \tilde{R}_k)\Delta y_k - (\tilde{G}_k - \tilde{S}_k)\Delta x_k], \tag{3}$$

where $\Delta t = t^{n+1} - t^n$ is the time step. Physical fluxes F, G, R, S on edge k of cell $D_{i,j}$ are replaced by numerical fluxes $\tilde{F}, \tilde{G}, \tilde{R}, \tilde{S}$. The particular choice of numerical fluxes and of the time derivative approximation depend on a chosen numerical scheme.

3.1. Numerical scheme

The explicit MacCormack (MC) scheme in the predictor-corrector form is used to approximate system (1). This scheme is 2^{nd} order accurate in time and space.

$$\begin{aligned}
W_{i,j}^{n+1/2} &= W_{i,j}^{n} - \frac{\Delta t}{\mu_{i,j}}\sum_{k=1}^{4}[(\tilde{F}_k^n - s_{1k}W_k^n - \tilde{R}_k^n)\Delta y_k - (\tilde{G}_k^n - s_{2k}W_k^n - \tilde{S}_k^n)\Delta x_k],\\
\bar{W}_{i,j}^{n+1} &= \frac{1}{2}(W_{i,j}^{n} + W_{i,j}^{n+1/2}) - \frac{\Delta t}{2\mu_{i,j}}\sum_{k=1}^{4}[(\tilde{F}_k^{n+1/2} - s_{1k}W_k^{n+1/2} - \tilde{R}_k^{n+1/2})\Delta y_k\\
&\quad -(\tilde{G}_k^{n+1/2} - s_{2k}W_k^{n+1/2} - \tilde{S}_k^{n+1/2})\Delta x_k].
\end{aligned} \tag{4}$$

Equation (4) represents the MC scheme for a viscous flow in a domain with a moving grid of quadrilateral cells. The moving grid in an unsteady domain is described by using the Arbitrary Lagrangian-Eulerian (ALE) method which defines the projection of reference domain D_0 to a time-dependent domain D_t [1]. Consequently, additional fluxes $\vec{s}_k W_k$ appear in the MC scheme, where vector $\vec{s}_k$ represents the speed of edge k. The approximations of conservative variable W_k and diffusive components R_k, S_k on edge k are central. The second derivatives (dissipative terms) on an edge are approximated using dual volumes [2] as is shown in Figure 3.

The inviscid numerical fluxes are approximated as follows:

$$\begin{aligned}
&\tilde{F}_1^n = F_{i,j}^n, \quad \tilde{F}_1^{n+1/2} = F_{i+1,j}^{n+1/2}, \quad \tilde{F}_3^n = F_{i-1,j}^n, \quad \tilde{F}_3^{n+1/2} = F_{i,j}^{n+1/2},\\
&\tilde{G}_2^n = G_{i,j}^n, \quad \tilde{G}_2^{n+1/2} = G_{i,j+1}^{n+1/2}, \quad \tilde{G}_4^n = G_{i,j-1}^n, \quad \tilde{G}_4^{n+1/2} = G_{i,j}^{n+1/2}, \quad \text{etc.}
\end{aligned} \tag{5}$$

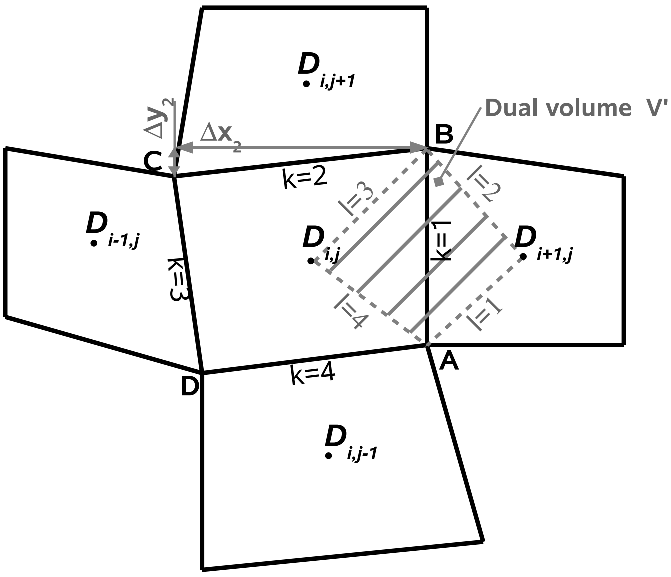


Fig. 3: *Finite volume $D_{i,j}$, dual volume V'_k.*

The last term of the MC scheme is the Jameson artificial dissipation $AD(W_{i,j})^n$, which is added to schemes with higher order of accuracy to stabilize the numerical solution:

$$AD(W_{i,j})^n = C_1\gamma_1(W^n_{i+1,j} - 2W^n_{i,j} + W^n_{i-1,j}) + C_2\gamma_2(W^n_{i,j+1} - 2W^n_{i,j} + W^n_{i,j-1}), \quad (6)$$

where $C_1, C_2 \in R$ are constants and the normed pressure gradients have the form:

$$\gamma_1 = \frac{|p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}|}{|p^n_{i+1,j}| + 2|p^n_{i,j}| + |p^n_{i-1,j}|}, \quad \gamma_2 = \frac{|p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}|}{|p^n_{i,j+1}| + 2|p^n_{i,j}| + |p^n_{i,j-1}|}. \quad (7)$$

Then we can compute a vector of conservative variables W at a new time level t^{n+1}:

$$W^{n+1}_{i,j} = \bar{W}^{n+1}_{i,j} + AD(W_{i,j})^n. \quad (8)$$

Stability condition of the scheme (on a regular orthogonal grid) limits the time step

$$\Delta t \leq CFL\left(\frac{|u_{\max}| + c}{\Delta x_{\min}} + \frac{|v_{\max}| + c}{\Delta y_{\min}} + \frac{2}{Re}\Big(\frac{1}{\Delta x^2_{\min}} + \frac{1}{\Delta y^2_{\min}}\Big)\right)^{-1}, \quad (9)$$

where c denotes the local speed of sound, $CFL < 1$, and the minimal step of the grid in the y-direction is $\Delta y_{\min} \approx 1/\sqrt{Re}$ due to boundary layer.

4. Numerical results

For numerical computation, domains D_1 and D_2 (see Figures 1, 2) are covered with a grid of quadrilateral cells. The cells near the wall boundary have successive refinement in the y-direction (due to the existing boundary layer) as shown in detail in Figure 1. The results are depicted as Mach number isolines and as the velocity vectors.

4.1. Numerical results in domain D_1

The length and width of domain D_1 are $L = 12$ and $H = 0.5$, and D_1 contains 600×50 cells. Parametres considered for computation: the outflow pressure is $p_2 = 0.9p_\infty$ and it corresponds to the inflow Mach number $M_\infty = 0.120$ and $Re = 5 \cdot 10^5$. Figure 4 shows the steady solution of viscous laminar flow in the nonsymmetric channel where the moving part of the solid wall (see Figure 1) is fixed. The maximum Mach number in the domain was computed to be $M_{\max} = 0.345$. Figure 5 (a, b, c, d, e) shows the development of unsteady viscous compressible laminar flows in domain D_1 at several time layers starting by the second period. For the computation of the unsteady solution, the steady solution was used as the initial state.

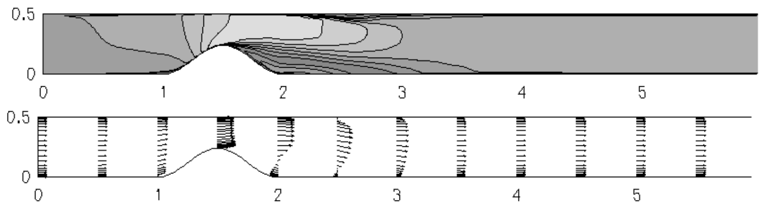
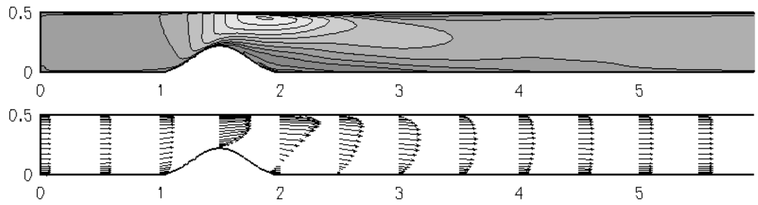


Fig. 4: *The steady solution of a viscous laminar flow in the nonsymmetric channel,* $p_2 = 0.9p_\infty$*,* $Re = 5 \cdot 10^5$*,* $M_{\max} = 0.345$*,* 600×50 *cells.*

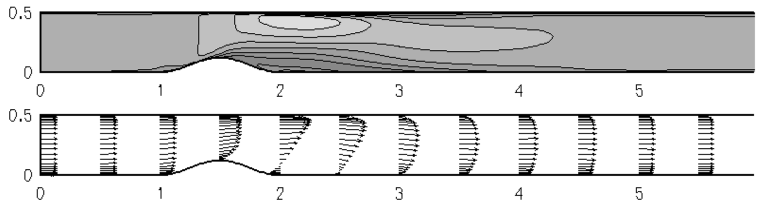
4.2. Numerical results in domain D_2

The length and width of domain D_2 are $L = 8$ and $H = 0.4$, and D_2 contains 400×50 cells. Parametres considered for computation: the inlet Mach number $M_\infty = 0.02$, the dimension frequency of the solid wall between points A, B (see Figure 2) is $f_{\mathrm{dim}} = 20\,\mathrm{Hz}$ and $Re = 9 \cdot 10^3$. These values approximately correspond to the real flow in the human vocal tract. Figure 6a) shows the steady solution of viscous laminar flow in the symmetric channel where the moving part of the solid wall is fixed. The maximum Mach number in the domain was computed, $M_{\max} = 0.096$. Figure 6b) shows convergence to a steady solution that is observed using L_2 norm of momentum residuals (ρu). It seems to be relatively good for this case with a very low Mach number. Figure 7 (a, b, c, d, e) shows development of unsteady viscous compressible laminar flows in domain D_2 at several time layers starting by the third period. For computation of the unsteady solution, the steady solution was used as the initial state. In Figure 7b), one can see typical behaviour with choking flows in a very narrow part of the channel and with the time development of flow including separation domains. The geometry of domain D_2 and the boundary conditions represent a simple model of flow in the human vocal tract [3, 4].

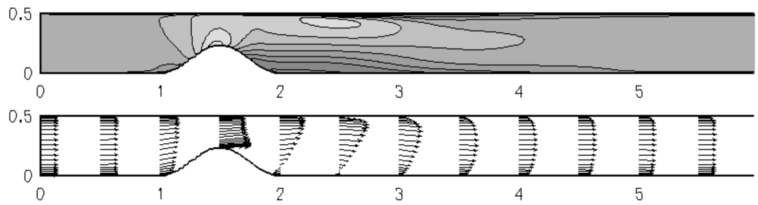
We also tried to compute both cases without the artificial dissipation $AD(W_{i,j})^n$. In this case, however, the convergence to the steady state was not satisfactory.



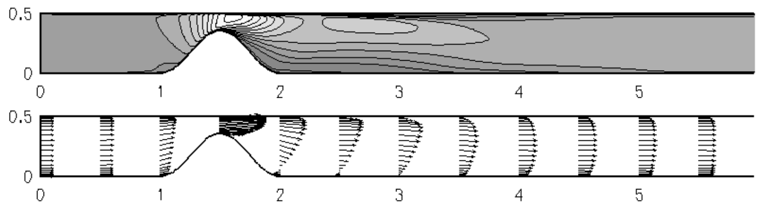
a) $t = 2\pi, M_{\max} = 0.455$



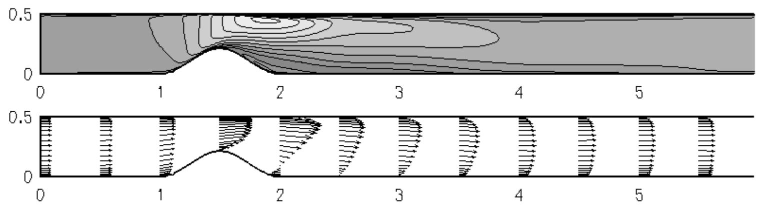
b) $t = 2\pi + \frac{\pi}{2}, M_{\max} = 0.338$



c) $t = 3\pi, M_{\max} = 0.374$



d) $t = 2\pi + \frac{3\pi}{2}, M_{\max} = 0.568$



e) $t = 4\pi, M_{\max} = 0.464$

Legend to Mach numbers:

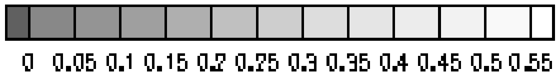
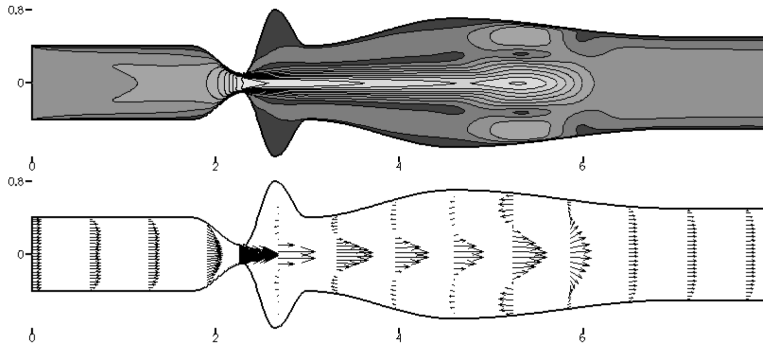
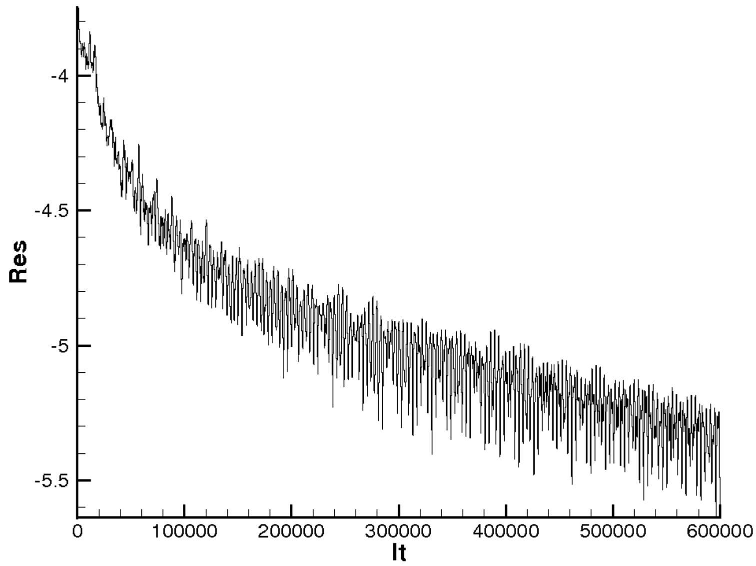


Fig. 5: *The unsteady solution of a viscous laminar flow in the nonsymmetric channel,* $p_2 = 0.9p_\infty$, $Re = 5 \cdot 10^5$, 600×50 *cells.*



a) Numerical result

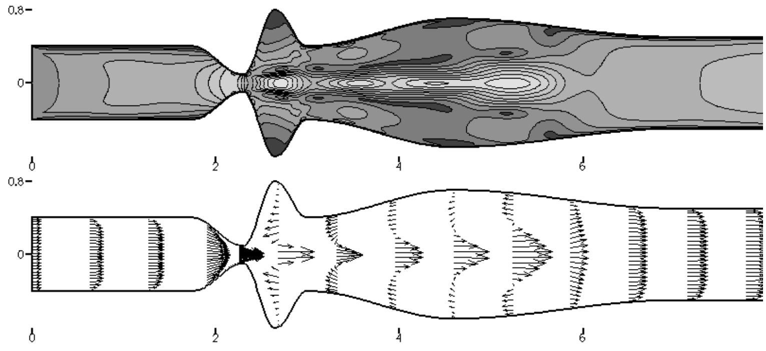


b) Convergence to a steady solution – residual vs. number of iterations

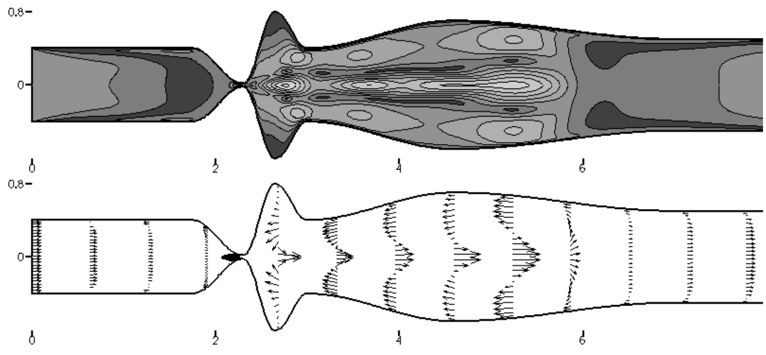
Fig. 6: *The steady solution of a viscous laminar flow in the symmetric channel,* $M_\infty = 0.02$*,* $Re = 9 \cdot 10^3$*,* $M_{\max} = 0.096$*,* 400×50 *cells.*

5. Summary

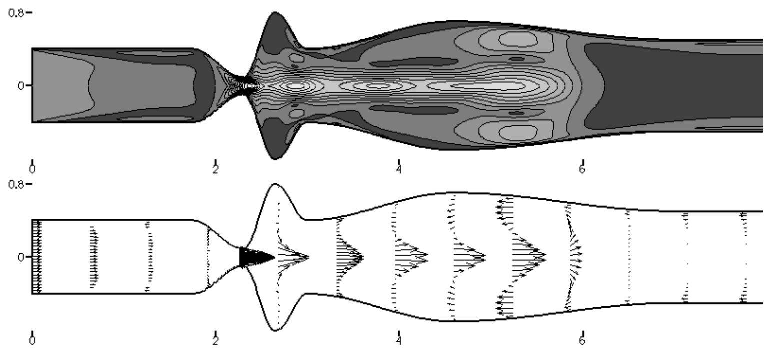
The calculation numerical approximations of steady state solutions for inviscid compressible flows with very low Mach numbers is a very difficult task and special methods have to be used. For viscous compressible problems, the method described above can be successfully used for the steady as well as unsteady numerical solutions of flows with low Mach numbers.



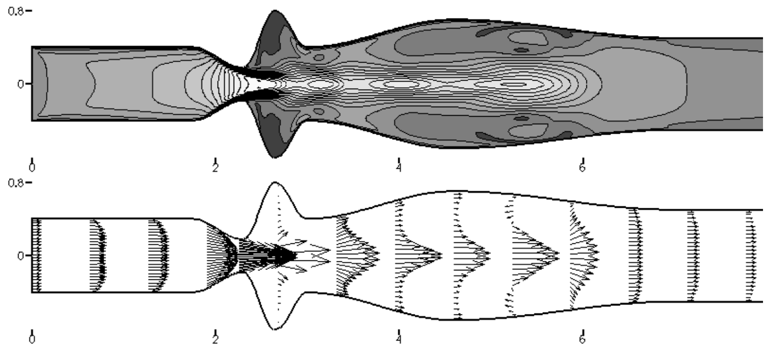
a) $t = 4\pi, M_{\max} = 0.094$



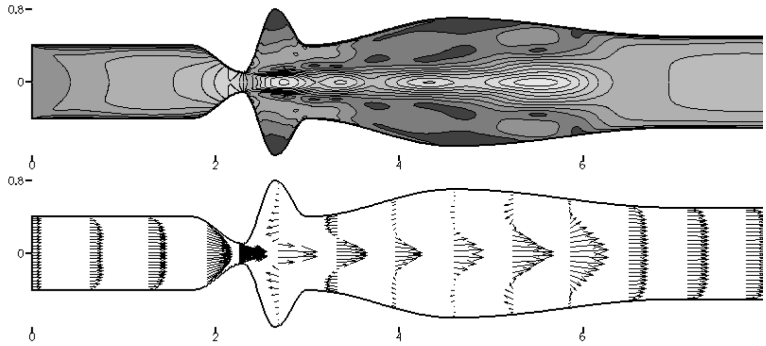
b) $t = 4\pi + \frac{\pi}{2}, M_{\max} = 0.077$



c) $t = 5\pi, M_{\max} = 0.129$



d) $t = 4\pi + \frac{3\pi}{2}, M_{\max} = 0.145$



e) $t = 6\pi, M_{\max} = 0.102$

Legend to Mach numbers:

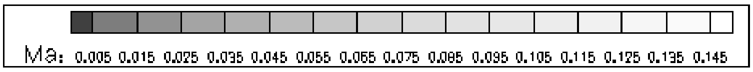


Fig. 7: *The unsteady solution of a viscous laminar flow in the symmetric channel,* $M_\infty = 0.02$*,* $Re = 9 \cdot 10^3$*,* 400×50 *cells.*

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