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Contribution to construction of global cubic C^1 or C^2-spline on equidistant knotset

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CONTRIBUTION TO CONSTRUCTION OF GLOBAL CUBIC C^1 OR C^2-SPLINE ON EQUIDISTANT KNOTSET *

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1. Problem statement

Given the equidistant knotset $\{x_i,\ i = 0 : n + 1\}$ with prescribed function values (FV), we can find a quadratic interpolating C^1-spline through computing its FV in inserted points of interpolation $t_i = (x_i + x_{i+1})/2$ (see [1], [2]). Cubic C^1 splines based on Hermite interpolation with given function and derivative values in knots have a local character (see e.g. [1], [2]). For cubic C^2-splines with given function values in knots only we can use the B-spline technique or we have to compute the first or second derivatives in knots to obtain C^2-continuity in the local representation. Such splines have a weak localizing property only (the influence of changes in some knot decreases with growing distance).

The aim of this contribution is to discuss the construction of C^1-cubic interpolatory splines based on inserting interval midpoints as additional points of interpolation and to compute the unknown spline function values to obtain C^1-continuity of spline segments in original knots. For simplicity we will discuss the case of the equidistant knotset only. We will also mention an approach with inserting two points of interpolation to obtain C^2-continuity and the solutions which use the B-spline technique (without inserted points).

2. Cubic spline knotset with inserted midpoints

On the four-point equidistant knotset $\{x_i,\ i = 0 : 1 : 3\}$ with the stepsize h and given function values $\{y_i\}$ the cubic Lagrange interpolation polynomial and its derivatives we can write with the local parameter $q = (x - x_0)/h$ and well-known Lagrange interpolation coefficients $l_i(q)$ as

$$L_3^{(k)}(x_0 + qh) = \sum_{i=0}^{3} l_i^{(k)}(q)y_i,\ k = 0, 1, 2; \qquad l_i(q) = \prod_{j\neq i}(q - j)/(i - j). \tag{1}$$

Let us extend given equidistant knotset $\{x_i,\ i = 0 : n + 1\}$ with inserted midpoints $\{t_i = (x_i+x_{i+1})/2,\ i = 0 : n\}$ and denote $\{s_i = s(x_i),\ i = 0 : n+1\}$ the prescribed FV on the original knotset, $\{u_i = s(t_i),\ i = 0 : n\}$ the unknown FV of cubic interpolants over different intervals containing knot x_i and three neighbours on the extended

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knotset. We shall try to compute the values u_i in such a way to obtain C^1-continuity of overlapping segments (one of them have to be defined over the whole interval $[x_i, x_{i+1}]$ or $[x_{i-1}, x_i]$) in the knot x_i . There are several variants of configurations of knots and intervals with common knot x_i. Using the explicit expression for $l_i^{(k)}(q)$ with corresponding values of parameter q , we obtain after some simplifications the C^1, C^2-continuity conditions (CC) in the knot x_i as the recursions between unknown values u_i and prescribed values s_i written in the following table.

Overlapping intervals q used	C^1-continuity condition C^2-continuity condition
$[t_{i-1}, x_{i+1}], [x_i, t_{i+1}]$	$u_{i-1} + 6u_i + u_{i+1} = 4(s_i + s_{i+1})$
1, 0	$u_{i-1} + 6u_i + u_{i+1} = 4(s_i + s_{i+1})$
$[x_{i-1}, t_i], [x_i, t_{i+1}]$	$6u_{i-1} + 16u_i + 2u_{i+1} = s_{i-1} + 14s_i + 9s_{i+1}$
2, 0	$u_{i-1} + 6u_i + u_{i+1} = 4(s_i + s_{i+1})$
$[x_{i-1}, t_i], [t_{i-1}, x_{i+1}]$	$4(u_{i-1} + u_i) = s_{i-1} + 6s_i + s_{i+1}$
2, 1	$u_{i-1} - 2s_i + u_i = u_{i-1} - 2s_i + u_{i+1}$
$[t_{i-2}, x_i], [x_i, t_{i+1}]$	$2u_{i-2} + 18u_{i-1} + 18u_i + 2u_{i+1} = 9s_{i-1} + 22s_i + 9s_{i+1}$
3, 0	$-u_{i-2} - 5u_{i-1} + 5u_i + u_{i+1} = -4s_{i-1} + 4s_i$
$[t_{i-2}, x_i], [x_{i-1}, t_i]$	$2u_{i-2} + 16u_{i-1} + 6u_i = 9s_{i-1} + 14s_i + s_{i+1}$
3, 1	$u_{i-2} + 6u_{i-1} + u_i = 4(s_{i-1} + s_i)$
$[t_{i-2}, x_i], [x_{i-1}, t_i]$	$u_{i-2} + 6u_{i-1} + u_i = 4(s_{i-1} + s_i)$
3, 2	$u_{i-2} + 6u_{i-1} + u_i = 4(s_{i-1} + s_i)$

Tab. 1.

We can see that with $q = [2, 1]$ the C^2- CC is identically fulfilled and for $q = [1, 0], [3, 2]$ the conditions for C^1, C^2 -continuity are identical. We can also consider the relevant problem with given values u_i and free parameters s_i.

3. C^1- continuity conditions for cubic splines

We can find various systems of overlapping intervals with four points of the extended mesh such that each knot $x_i, i = 1 : n$ belongs just to two such intervals. To obtain the cubic spline on the original knotset we need to have for each interval $[x_j, x_{j+1}]$ some cubic interpolant defined over the whole such interval. When we then write down the corresponding C^1-CC in all internal knots x_i, we obtain system of linear equations for unknown parameters u_i. Then we choose for each interval $[x_i, x_{i+1}]$ the corresponding cubic polynomial and we obtain the interpolating cubic C^1-spline on the original knotset.

3.1. The intervals with q=[2,0] or q=[3,1] connected

When we use for the cubic interpolants the knots $[x_i, t_i, x_{i+1}, t_{i+1}],\ i = 0 : n-1$, then these interpolants have just the common knots $x_i,\ i = 1 : n$ with the C^1- CC from the case with $q = [2, 0]$ ($q = [2, 1]$ for the last couple). Using Table 1, we obtain so the system of n linear relations between parameters $\mathbf{u} = [u_0, \cdots, u_n]$ and $\mathbf{s} = [s_0, \cdots, s_{n+1}]$

$$\mathbf{A_u u} = \mathbf{A_s s} \tag{2}$$

with the tridiagonal matrices $\mathbf{A_u}, \mathbf{A_s}$ of the sizes $(n, n+1)$, $(n, n+2)$ and coefficients

$$\begin{bmatrix} 6 & 16 & 2 & & & \\ & 6 & 16 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 6 & 16 & 2 \\ & & & & 4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 14 & 9 & & & & \\ & 1 & 14 & 9 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 14 & 9 \\ & & & & & 1 & 6 & 1 \end{bmatrix}. \tag{3}$$

Both these matrices are of the full rank and with given values $\{s_i, i = 0 : n+1\}$ we have one free parameter u_i – our problem has the solution depending on one parameter, which can be chosen or used for some optimization purposes. The cubic C^1-spline then consists of corresponding parts of the cubic interpolants. Let us mention some most interesting cases in these variants.

1. We can choose arbitrary free parameter u_0. The tridiagonal matrix of the reduced system is then regular and diagonally dominant (exception - the last row) and we obtain the unique solution for arbitrary values $\{u_0; s_i, i = 0 : n\}$. The corresponding spline will have the mentioned weak localising property.

2. For similar proper system of overlapping intervals for cubic interpolation corresponding to $q = [2, 1], [3, 1], [3, 1], ...$ we obtain the C^1-CC with slightly modified matrices $\mathbf{A_u}, \mathbf{A_s}$. With the choice of the free parameter u_n we obtain now uniquely solvable system with weak localising property.

3. In both mentioned cases we can obtain the unique solution of the relevant problem with given parameters $s_0, s_{n+1}; u_i, i = 0 : n$ with diagonally dominant reduced (n,n)-matrix $\mathbf{A_u}$.

4. To obtain the solution with minimal 2-norm we can use pseudoinverse for the solution of the original system - but we obtain usually the solution with some oscillations.

Example 1: For the data $x = 0 : 1 : 9$, $y(x) = 1 - x * cos(x)/2$ we can see plotted in Fig. 1 the original function and the values s_i of the corresponding C^1-cubic spline in midpoints, computed with pseudoinverse (circles) and with $u_0 = y(0.5)$ from (2). The 2-norms of the vector $\mathbf{u}$ are here [5.69, 5.73]. When we use as free parameter $y(8.5)$, we obtain the oscillating solution with norm equal to 127.25 !

3.2. Another configurations

We can find also another possible variants with still worse properties.

1. In the configuration of two families of local intervals with repeating values of $q = [2, 1], [3, 1], [3, 1], [3, 0], ...$ we recognize from Table 1 the C^2-continuity of overlapping segments in knots x_{4k+1}. The system of CC consists of (4,6)-blocks, some proper choice of free parameter is u_n.

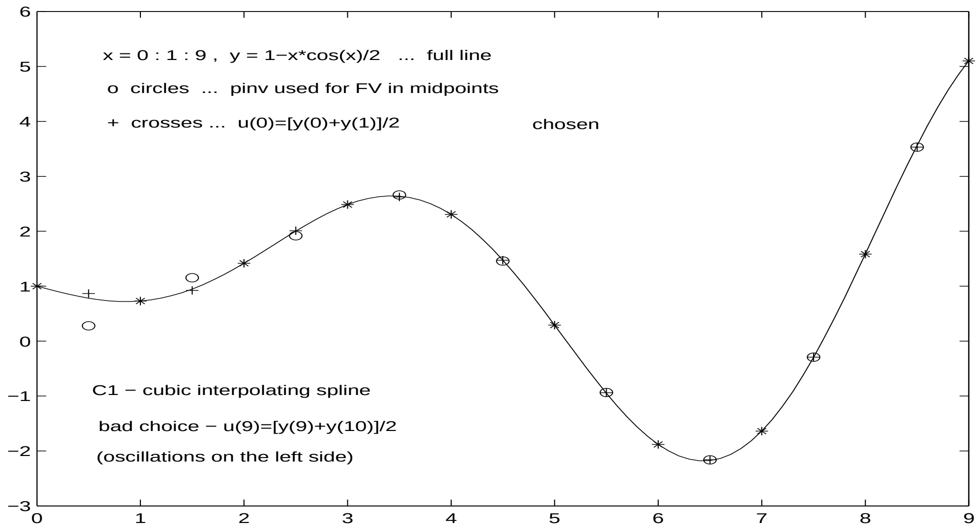

Fig. 1.

Similarly we find another union of two families of intervals - the first starting with the knots x_{3k} only, the second family of connected intervals starting with the inserted knot t_0 where the repeated triples of corresponding values of the local parameter q for CC in internal knots x_i are $[2,1],[3,0],[2,0]$ for $i = 3k+1, 3k+2, 3k+3$ (with possible changes in the last CC). The whole system of C^1-CC we can write now again using Table 1 as $\mathbf{A_u u} = \mathbf{A_s s}$ with block diagonal matrices with (3,4),(3,5)-blocks

$$\begin{bmatrix} 4 & 4 & & \\ 2 & 18 & 18 & 2 \\ & 6 & 16 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 6 & 1 & & \\ & 9 & 22 & 9 & \\ & & 1 & 14 & 9 \end{bmatrix} \tag{4}$$

and to choose u_n as free parameter. We find now also the C^2-continuity of the resulting cubic spline in knots x_{3k+1}.

We have found in both such cases as the most proper free parameter u_n, the unique solution and diagonal dominancy in some rows. But in rows with coefficients [4,4] and [2,18,18,2] the diagonal dominancy is lost and we can find some oscillations in the solution. The structure of the matrix $\mathbf{A_u}$ shows that the choice of the free parameter u_0 will result in more oscillating solution. When we are interested to use u_0 as the free parameter, then we can use some more appropriate variant of two families of covering intervals with q=[2,0],[2,1],3,0] and changes in row orderings in blocks mentioned in (4).

2. When we try to use another possible variants with corresponding repeating values q=[2,0],[2,1],[3,1],[3,0] or q=[2,1],[3,0], then the structure of the full rank matrix $\mathbf{A_u}$ shows the danger of strong oscillations of computed values u_i with the choice of any free parameter u_i - as we can justify it in computations.

3. In all cases discussed till now we find for the relevant problem with given values $u_i,\ i = 0:n$ and parameters s_0, s_n the reduced matrix $\mathbf{A_s}$ to be diagonally dominant – *all such problems have the unique solution with weak localising property!*

4. The configurations with q=[1,0],[2,1],[3,2] or q=[1,0],[3,2] seem to result in C^2-cubic spline - but we have not unique cubic polynomials for each interval $[x_i, x_{i+1}]$ in such cases.

Statement 1: *On the equidistant knot mesh $\{x_i\}$ with given function values $\{s_i\}$ we can find (with one free parameter) function values u_i in each of interval midpoints $t_i = (x_i + x_{i+1})/2$ such that the corresponding parts of the local cubic interpolants on intervals $[x_i, x_{i+1}]$ will form the cubic interpolating C^1- spline with knots $\{x_i\}$ for the original data $\{x_i, s_i\}$. One free parameter we can use for optimization purposes, or to choose u_0, u_n (according to the structure of the matrix $\mathbf{A_u}$) to obtain weak localising property of the spline.*
The relevant problem with given values $s_0, s_n; \{u_i, i = 0 : n\}$ has in all cases discussed above the unique solution with weak localising property.

4. Two inserted knots per interval for C^2-continuity

Let us insert into each interval of the original equidistant knotset $\{x_i,\ i = 0 : n+1\}$ two uniformly situated knots $x_i < t_i^1 < t_i^2 < x_{i+1},\ i = 0 : n$. We want try to find function values in knots t_i^1, t_i^2 to obtain C^2-cubic spline on the original knotset. We can write the C^1, C^2- CC in knots x_i ($q = [3, 0]$ used from Table 1) with the notation $u_i = s(t_i^1), v_i = s(t_i^2), s_i = s(x_i)$ as

$$\begin{aligned} -9u_{i-1} + 18v_{i-1} + 18u_i - 9v_i &= -2s_{i-1} + 22s_i - 2s_{i+1}, \\ -4u_{i-1} + 5v_{i-1} - 5u_i + 4v_i &= -s_{i-1} + s_{i+1}, \quad i = 1 : n. \end{aligned} \tag{5}$$

With $n + 1$ intervals $[x_i, x_{i+1}]$, $2(n + 1)$ inserted knots $t_i^1, t_i^2,\ i = 0 : n$ and given values $s_i = s(x_i)$ we obtain so the system of $2n$ C^1, C^2- CC with the block structure and full rank matrix. So we have two free parameters, similarly as for classical cubic C^2-splines. We again can use free parameters to various purposes.
1. We can solve the system of CC (5) with the pseudoinverse, to obtain the spline with minimal norm of parameters $[u_i, v_i]$.
2. We can choose the values u_0, v_n and compute the remaining parameters from the regular reduced system (5).
3. We can prescribe boundary conditions for the spline computed - the first or second derivatives as with classical cubic splines (details in [3]).
4. There are also solutions with free parameters u_0, v_0 or u_n, v_n which can be computed recursively. But in both such cases we obtain strongly oscillating solutions.

Statement 2: *On the equidistant mesh with given function values in knots and two inserted points of interpolation we can compute the function values in inserted points to obtain cubic C^2-spline. Two free parameters can be prescribed, used for boundary conditions with the first (second) derivative or for optimization purposes.*

In Fig. 2 we can see the FV in inserted knots computed with pseudoinverse (circles) and from the system of CC extended with boundary conditions (natural spline - zero values of the second derivative at boundaries).

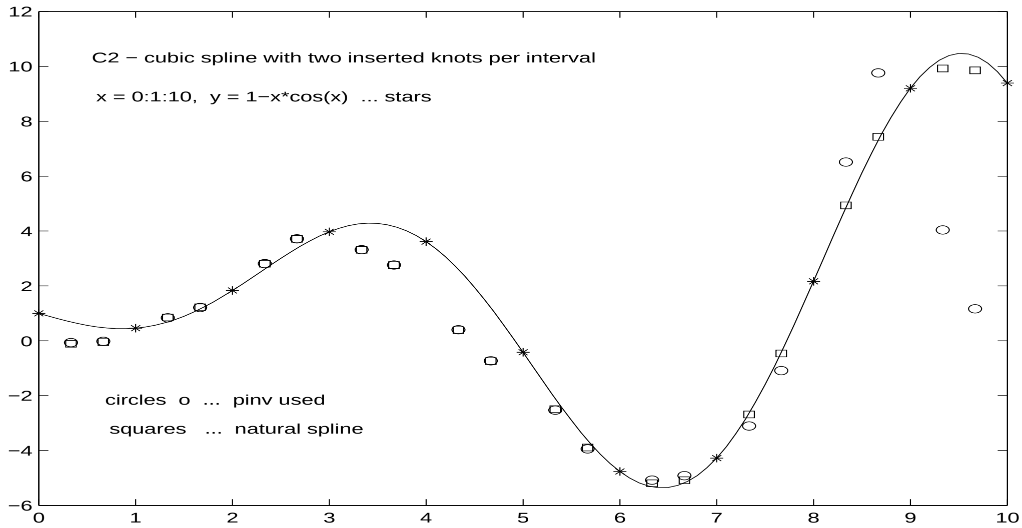

Fig. 2.

Remarks:
1) We can use two inserted points per interval to find cubic C^2-spline interpolating the given mean values (see [3]).
2) We have to use B-splines with coinciding knots to find cubic C^1-spline interpolating in knots x_i (several possible variants).

References

[1] C. De Boor: *A practical guide to splines.* Springer 2001.

[2] J. Kobza: *Splines.* Textbook UP Olomouc, 1993 (in Czech).

[3] J. Kobza: *Global cubic C^1-splines on equidistant knotset.* Preprint FS UP Olomouc, 2004, 15 pp.

[4] B.I. Kvasov: *Shape-preserving spline approximation.* World Sci.Publ. 2000.

[5] Gh. Micula, S. Micula: *Handbook of splines.* Kluwer 1999.

[6] L. Schumaker: *Spline functions: basic theory.* Wiley 1981.

[7] V.M. Schumilov: *Lagrange cubic spline interpolation with inserted knots.* Spline Approximations (Comput. Systems 106), 29–40 (in Russian); Inst. Math. Acad. Sci., Novosibirsk 1984, ISSN 0568-661X.

[8] J.S. Zavjalov, B.I. Kvasov, V.L. Miroschnichenko: *Methods of spline functions.* Nauka, Moscow 1980 (in Russian).