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NUMERICAL TREATMENT OF INITIAL VALUE PROBLEMS FOR DELAY DIFFERENTIAL SYSTEMS

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Abstract

This paper deals with the numerical solution of the Cauchy problem for systems of ordinary differential equations with time delay. One-step numerical methods and appropriate interpolation operators are used. Numerical results for a system of three differential equations are presented.

1. Introduction

Delay differential equations (DDE), also called functional differential equations or time delay differential equations, are widely used for describing and mathematical modelling of various processes and systems in various applied problems [1]. Theoretical aspects of the DDE theory are presented as the corresponding parts of the ordinary differential equations (ODE) theory. However, unlike ODEs, even for linear DDEs there are no general methods for finding solutions in explicit forms. Hence, the development of numerical methods for DDEs is a very important problem. Various specific numerical methods have been constructed for solving specific DDEs [2]. Our approach is devoted to the numerical methods for systems with time-varying and constant delays. One-step numerical methods and an interpolation operator are used to solve the Cauchy problem for a system of three ODEs with delay numerically in this paper.

2. Delay differential equations

We first consider a general DDE with bounded delay

$$y'(t) = f(t, y(t), z(.))$$

subject to the initial conditions

$$y(t_0) = y_0, \quad y(t_0 + s) = z_0(s), \quad -p \leq s < 0,$$

where $y_0 \in R$, $z_0(.) \in Q[-p, 0]$. Here, $Q[-p, 0]$ is the space of functions continuous everywhere on $[-p, 0]$ with the possible exception of a finite set of points of discontinuity of the first kind, equipped with the usual sup-norm. The right-hand side of the DDE is a mapping

$$f(t, y(t), z(.)) \,:\, [t_0, t_0 + T] \times R \times Q[-p, 0] \;\rightarrow\; R,$$

where $T > 0$ and $p > 0$ characterizes the delay interval.

For $y \in R$, $z(.) \in Q[-p,0]$, and $d > 0$, we will denote by $E_d[y, z(.)]$ the set of all continuous continuations of $\{y, z(.)\}$ on the interval $[0, d]$, i.e., the set of functions $Z(s) : [-p, d] \to R$ such that:

$$Z(0) = y,$$
$$Z(s) = z(s),\ -p \le s < 0,$$
$$Z(s) \text{ is continuous on } [0, d].$$

To introduce the derivatives of the functional $f(t, y, z(.))$, let us consider the function
$$P_z(e, i, k) = f(t + e, y + i, z_k(.)),$$
where

$$e > 0,\ i \in R,\ k \in [0, d],\ z_k(.) = \{Z(k + s), -p \le s \ < 0\} \in Q[-p, 0].$$

Further, we suppose that the mapping f satisfies the following assumptions:

A1. For any $t \in [t_0, t_0 + T], y \in R, z(.) \in Q[-p, 0]$, there exists $d > 0$ such that for every $Z(.) \in E_d[y, z(.)]$ the corresponding function $P_z = f(e, Z(e - t), Z_{e-t}(.))$ is continuous on $[t, t + d]$.

A2. The mapping f is Lipschitz continuous with respect to y and $z(.)$.

Then for some $\delta > 0$ there exists a unique solution of our problem on $[t_0, t_0 + \delta]$. The proof for systems can be found in [2].

As a test example, we will consider the equation

$$y'(t) = y(t - (e^{-t} + 1)) + \cos(t) - \sin(t - 1 - e^{-t})$$

for $t \in (0, \infty)$. If we take the initial pre-history as

$$y(s) = \sin(s),\ s \in [-2, 0],$$

then by direct substitution one can check that the function $y(t) = \sin(t)$, $t > 0$, is the solution of this initial value problem.

Further, we will use this example to demonstrate some problems in the numerical solution of such IVPs for DDEs.

3. Numerical methods

Let us consider a uniform partition $t_n = t_0 + nh$, $n = 0, 1, \dots, N$, of the interval $[t_0, t_0 + T]$, where $h = T/N$. For the sake of simplicity, we suppose that the ratio $p/h = m$ is a positive integer.

First, our aim is to demonstrate the basic idea of the approach on a simple Euler-like numerical scheme

$$\begin{aligned} x_0 &= y_0, \\ x_{n+1} &= x_n + hf(t_n, x_n, x_{t_n}(.)). \end{aligned}$$

The corresponding numerical scheme for our example with $h = 1$ is

$$\begin{array}{rcl} x_0 & = & y(0) = 0, \\ x_1 & = & x_0 + hf(0, x_0, x_{t0}(.)) = x_0 + y(-2) + \cos(0) - \sin(-2) \\ & = & 0 + \sin(-2) + \cos(0) - \sin(-2) = 1, \\ x_2 & = & x_1 + y(-e^{-1}) + \cos(1) - \sin(-e^{-1}) = 1 + \sin(-e^{-1}) + \cos(1) - \sin(-e^{-1}) \\ & = & 1 + \cos(1), \end{array}$$

but to compute x_3 at the point $t = 3$, it is necessary to make an interpolation (for $y(1 - e^{-2})$) of the approximate solution

$$x_3 = x_2 + y(1-e^{-2}) + \cos(2) - \sin(1-e^{-2}) = 1 + \cos(1) + y(1-e^{-2}) + \cos(2) - \sin(1-e^{-2}).$$

Using simple piecewise constant interpolation we may write

$$[x_{t_n}(.) \approx]\ u(t) = \begin{cases} x_i, & t \in [t_i, t_{i+1}] \\ z_0(t_0 - t), & t \in [-p, 0] \end{cases}$$

and continue with $x_3 = 1 + \cos(1) + \cos(2) - \sin(1 - e^{-2})$.

Our test example leads us thus to the following conclusion: To find the next approximation x_{n+1} at time t_{n+1} by using Euler's scheme, it is necessary to calculate the right-part of the equation on the pre-history $\{x_i, n - m \le i \ \le n\}$. At time t_n, the pre-history is a finite set $x_{n-m}, \dots, x_n$. Hence, to calculate a value of the functional f on the pre-history, it is necessary to make an interpolation of the approximate solution x_n. To obtain more accurate methods, it is necessary to use higher-order one-step methods with higher-order interpolational procedures. The following mapping uses the piecewise linear interpolation as an example of an interpolation operator of the second order:

$$u(t) = \begin{cases} [(t - t_i)x_{i+1} + (t_{i+1} - t)x_i]/h, & t \in [t_i, t_{i+1}] \\ z_0(t_0 - t), & t \in [t_0 - p, t_0] \end{cases} .$$

General interpolation operators can be constructed using splines of a given degree. Some numerical methods for DDEs require to calculate x_{n+1} using the pre-history also for $t \in [t_n, t_{n+1}]$. In this case, it is necessary to use an extrapolation operator on $[t_n, t_n + t]$.

4. Numerical example

Consider the system of nonlinear DDEs of the form

$$\begin{array}{l} x'(t) = (2/\pi)[x(t) + y(t - \pi/2)] - x(t - \pi/2) - \pi y(t)/(2z(t)), \\ y'(t) = (2/\pi)[y(t) - x(t - \pi/2)] - y(t - \pi/2) + \pi x(t)/(2z(t)), \\ z'(t) = \sqrt{x^2(t - \pi/2) + y^2(t - \pi/2)}/z(t - \pi/2). \end{array} \tag{1}$$

If we take the initial functions as

$$\begin{aligned} x(\pi+s) &= (\pi+s)\cos(\pi+s), \\ y(\pi+s) &= (\pi+s)\sin(\pi+s), \\ z(\pi+s) &= (\pi+s) \end{aligned} \tag{2}$$

for $-\pi/2 \le s \le 0$, then the exact solution of this IVP for $t \ge \pi$ has the form

$$\begin{aligned} x(t) &= t\cos(t), \\ y(t) &= t\sin(t), \\ z(t) &= t. \end{aligned}$$

Let $h < \pi/2$ and some interpolation operator I be fixed. An explicit q-stage Runge-Kutta-like method for the equation

$$y'(t) = f(t, y(t), z(.))$$

with the interpolation I is the following numerical scheme:

$$\begin{aligned} &x_0 = y_0, \\ &x_{n+1} = x_n + h\sum_{i=1}^{q} c_i k_i(x_n, x_{t_n}(.)),\ n = 1, 2, \ldots, N-1, \end{aligned}$$

where $k_1(x_n, x_{t_n}(.)) = f(t_n, x_n, x_{t_n}(.))$ and

$$k_i(x_n, x_{t_n}(.)) = f(t_n + a_i h, x_n + h\sum_{j=1}^{i-1} b_{ij} k_j(x_n, x_{t_n}(.)), x_{t_n + a_i h}(.))$$

for $i > 1$. The parameters a_i, b_{ij}, c_i given at the Butcher table

$$\begin{array}{c|lllll} 0 & & & & & \\ a_2 & b_{21} & & & & \\ a_3 & b_{31} & b_{32} & & & \\ \cdots & \cdots & & & & \\ a_q & b_{q1} & b_{q2} & \ldots & b_{q,q-1} & \\ \hline & c_1 & c_2 & \ldots & c_{q-1} & c_q \end{array}$$

are called the coefficients of the method. Thus, for Heun's method of the second order we have $q = 2$, $c_1 = c_2 = 1/2$, $a_1 = 0$, $a_2 = 1$, $b_{21} = 1$. The Dormand–Prince method ($q = 6$) of the fifth order has the coefficients in the following table:

$$\begin{array}{r|cccccc} 0 & & & & & & \\ 1/5 & 1/5 & & & & & \\ 3/10 & 3/40 & 9/40 & & & & \\ 4/5 & 44/45 & -56/15 & 32/9 & & & \\ 8/9 & 19372/6561 & -25360/2187 & 64448/6561 & -212/729 & & \\ 1 & 9017/3168 & -355/33 & 46732/5247 & 49/176 & -5103/18656 & \\ \hline & 35/384 & 0 & 500/1113 & 125/192 & -2187/6784 & 11/84 \end{array}$$

For the method of Dormand and Prince, a 4-th order continuous extension is possible even without an extra function evaluation. The solution, which for $\theta = 1$ becomes the fifth order solution, is given by the following formulas:

$$
\begin{aligned}
c_1(\theta) &= \theta(1+\theta(-1337/480+\theta(1039/360+\theta(-1163/1152)))) \\
c_2(\theta) &= 0 \\
c_3(\theta) &= 100\theta^2(1054/9275+\theta(-4682/27825+\theta(379/5565)))/3 \\
c_4(\theta) &= -5\theta^2(27/40+\theta(-9/5+\theta(83/96)))/2 \\
c_5(\theta) &= 18225\theta^2(-3/250+\theta(22/375+\theta(-37/600)))/848 \\
c_6(\theta) &= -22\theta^2(-3/10+\theta(29/30+\theta(-17/24)))/7 \\
x(t_i+\theta h) &= x(t_i)+h\sum_{j=1}^{6} c_j(\theta)k_j, \quad 0 \le \theta \le 1.
\end{aligned}
$$

These formulas together with the coefficients a_i, b_{ij} of the Butcher table for the Dormand–Prince method provide means for an excellent treatment of the numerical solution of DDEs as an interpolation (also extrapolation) operator.

Heun's method of second order with a given interpolation operator of second order (B), then the Dormand–Prince method of the fifth order with the same in-

t	**Exact value x**	**abs.err. (A)**	**abs.err. (B)**	**abs.err. (C)**
$\pi+1$	-2.237712	0.029251	0.041258	5.23E-14
$\pi+3$	6.080131	0.256435	0.039100	0.007798
$\pi+5$	-2.309462	2.974648	0.284654	0.020482
$\pi+7$	-7.645770	2.698746	0.407037	0.025842
$\pi+9$	11.062572	5.953736	0.140169	0.002473
t	**Exact value y**	**abs.err. (A)**	**abs.err. (B)**	**abs.err. (C)**
$\pi+1$	-3.485030	0.030911	0.028471	2.81E-14
$\pi+3$	-0.866702	1.159457	0.125410	0.003816
$\pi+5$	7.807171	2.292112	0.242516	0.017214
$\pi+7$	-6.662890	4.340690	0.196576	0.002628
$\pi+9$	-5.003775	2.937441	1.083522	0.104271
t	**Exact value z**	**abs.err. (A)**	**abs.err. (B)**	**abs.err. (C)**
$\pi+1$	4.141593	0.010000	0.010000	2.15E-14
$\pi+3$	6.141593	0.009898	0.007855	7.16E-05
$\pi+5$	8.141593	0.012984	0.006108	0.001400
$\pi+7$	10.141593	0.620172	0.042969	0.004394
$\pi+9$	12.141593	1.034933	0.069802	0.006185

Tab. 1:

terpolation operator (A) and finally the Dormand–Prince method with the above given formulas for c_i (C) have been used to the numerical solution of (1), (2) for $t \in [\pi, \pi + 10]$ with the step $h = 0.1$. Some results are summarized in the table 1.

This table shows the three components of the solution subsequently. One can easily observe that a proper choice of a suitable order of the interpolation operator to a given one-step method leads to improved accuracy of the numerical solution.

References

[1] R.D. Driver: *Ordinary and delay differential equations.* New York, Springer–Verlag 1977.

[2] A.V. Kim: *Functional differential equations. Application of i-smooth calculus.* Dordrecht, Kluwer Academic Publishers 1999.