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A SHOCK-CAPTURING DISCONTINUOUS GALERKIN METHOD FOR THE NUMERICAL SOLUTION OF INVISCID COMPRESSIBLE FLOW*

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1 Introduction

A specific wide class of problems of fluid mechanics is formed of inviscid compressible flow, which is described by the system of the compressible Euler equations. The solutions of such problems usually contain subdomains, where steep gradients or discontinuities are presented (e.g., shock waves or contact discontinuities). To solve these problems in a sufficiently robust, efficient and accurate way, the *discontinuous Galerkin method* (DGM) is popularly used. DGM is based on a piecewise polynomial but discontinuous approximation, for a survey, see, e.g., [2], [3]. However when DGM is applied to the compressible inviscid fluid flow, the resulting solutions suffer from Gibbs-type oscillations, which arise in the vicinage of discontinuities, spread into the computational domain and corrupt the solution. In order to suppress these non-physical oscillations and improve a prediction of crucial flow phenomena the standard DGM is treated with a *shock-capturing* technique, see, e.g., [5], [7].

This article extends a shock capturing approach from [7], which is based on adding the artificial diffusion term to the original system, in a view of technique presented in [6], where the amount of added artificial viscosity is abided by the residual of the entropy equation. The resulting scheme denoted by SC-DGM is applied to a classical benchmark problem of inviscid steady-state flow.

2 Compressible Euler equations

We consider the compressible Euler equations in an open domain $Q_T = \Omega \times (0,T)$, where $T > 0$ is the final time and $\Omega \subset \mathbb{R}^2$ is the flow domain. We denote the boundary of Ω by $\partial\Omega$, it consists of several disjoint parts — inlet, outlet and impermeable walls. The system of the Euler equations describing a motion of inviscid compressible fluids can be written in conservative variables $\boldsymbol{w} = (\rho,\, \rho v_1,\, \rho v_2,\, e)^T$ in the dimensionless form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \vec{\boldsymbol{f}}(\boldsymbol{w}) = 0 \quad \text{in } Q_T, \tag{1}$$

where $\vec{\boldsymbol{f}} = (\vec{f_1}, \vec{f_2})^T$ are the *inviscid (Euler) fluxes*, defined by

$$\vec{f_s}(\boldsymbol{w}) = (\rho v_s,\, \rho v_s v_1 + \delta_{s1} p,\, \rho v_s v_2 + \delta_{s2} p,\, (e+p)\, v_s)^T,\ \ s = 1, 2. \tag{2}$$

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We use a notation: ρ - density, $\boldsymbol{v} = (v_1, v_2)^T$ - velocity field, e - total energy, p - pressure and δ_{sk} - Kronecker delta. The system (1) is closed with the equation of state of a perfect gas and equipped with the initial condition and the set of boundary conditions on appropriate parts of boundary, see [3].

3 DG discretization

Let $\mathcal{T}_h$ $(h > 0)$ represents a partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed elements K with mutually disjoint interiors. We call $\mathcal{T}_h = \{K\}_{K \in \mathcal{T}_h}$ a *triangulation* of Ω and do not require the conforming properties from the finite element method. By $\mathcal{F}_h$ we denote the set of all open edges of all elements $K \in \mathcal{T}_h$. Further, the symbol $\mathcal{F}_h^I$ stands for the set of all $\Gamma \in \mathcal{F}_h$ that are contained in Ω (inner edges). Finally, for each $\Gamma \in \mathcal{F}_h$, we define a unit normal vector $\vec{n}_\Gamma = (n_1, n_2)^T$. We assume that $\vec{n}_\Gamma$, $\Gamma \subset \partial\Omega$, has the same orientation as the outer normal of $\partial\Omega$. For $\vec{n}_\Gamma$, $\Gamma \in \mathcal{F}_I$, the orientation is arbitrary but fixed for each edge.

DGM allows to treat with different polynomial degrees over elements. Therefore, we assign a positive integer p_K as a *local polynomial degree* to each $K \in \mathcal{T}_h$. Then we set the vector $\mathsf{p} = \{p_K, K \in \mathcal{T}_h\}$. Over the triangulation $\mathcal{T}_h$ we define the finite dimensional space of discontinuous piecewise polynomial functions

$$S_{h\mathsf{p}} = \{v; v|_K \in P_{p_K}(K)\ \forall\, K \in \mathcal{T}_h\}, \tag{3}$$

where $P_{p_K}(K)$ denotes the space of all polynomials of degree $\leq p_K$ on K, $K \in \mathcal{T}_h$. Then we seek the approximate solution of the system (1) in the space of vector-valued functions $\mathbf{S}_{h\mathsf{p}} = [S_{h\mathsf{p}}]^4$.

For each $\Gamma \in \mathcal{F}_h^I$ there exist two elements K_L, $K_R \in \mathcal{T}_h$ such that $\Gamma \subset K_L \cap K_R$. We use a convention that K_R lies in the direction of $\vec{n}_\Gamma$ and K_L in the opposite direction of $\vec{n}_\Gamma$. For $v \in S_{h\mathsf{p}}$, by $v|_\Gamma^{(L)} =$ trace of $v|_{K_L}$ on Γ, $v|_\Gamma^{(R)} =$ trace of $v|_{K_R}$ on Γ we denote the *traces* of v on edge Γ, which are different in general. Moreover, $[v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}$ and $\langle v \rangle_\Gamma = \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\right)$ denote the *jump* and *mean value* of function v over the edge Γ, respectively. For $\Gamma \in \partial\Omega$ there exists an element $K_L \in \mathcal{T}_h$ such that $\Gamma \subset K_L \cap \partial\Omega$. Then for $v \in S_{h\mathsf{p}}$, we put: $v|_\Gamma^{(L)} =$ trace of $v|_{K_L}$ on Γ, $\langle v \rangle_\Gamma = [v]_\Gamma = v|_\Gamma^{(L)}$.

Now, we recall the space semi-discrete DG scheme presented in [3]. The crucial item of the DG formulation of the Euler equations is the treatment of the inviscid terms. We employ the concept of numerical flux $I\!\!H(\cdot,\cdot,\cdot)$, namely the Vijayasundaram numerical flux, see [5].

Therefore, a function $\boldsymbol{w}_h \in C^1([0,T]; \mathbf{S}_{h\mathsf{p}})$ is called the *semi-discrete solution* of (1) if

$$\left(\frac{\partial \boldsymbol{w}_h(t)}{\partial t}, \boldsymbol{\varphi}_h\right) + \boldsymbol{b}_h(\boldsymbol{w}_h(t), \boldsymbol{\varphi}_h) = 0 \qquad \forall\, \boldsymbol{\varphi}_h \in \mathbf{S}_{h\mathsf{p}},\ \forall\, t \in (0,T), \tag{4}$$

where $(\cdot,\cdot)$ denotes the L^2-scalar product and

$$\boldsymbol{b}_h(\boldsymbol{w}_h,\boldsymbol{\varphi}_h)=\sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma I\!H(\boldsymbol{w}_h|_\Gamma^{(L)},\boldsymbol{w}_h|_\Gamma^{(R)},\vec{n}_\Gamma)\,[\boldsymbol{\varphi}_h]_\Gamma\,\mathrm{d}S-\sum_{K\in\mathcal{T}_h}\int_K\vec{\boldsymbol{f}}(\boldsymbol{w}_h)\cdot\nabla\boldsymbol{\varphi}_h\,\mathrm{d}x. \quad (5)$$

The problem (4) represents a system of ordinary differential equations (ODEs) for $\boldsymbol{w}_h(t)$ which has to be discretized in time by a suitable method. Since these ODEs belong to the class of stiff problems whose solutions by an explicit scheme are rather inefficient, it is advantageous to use a *semi-implicit* approach.

According to [3], we define the semi-implicit time discretization of (4) by

$$(\boldsymbol{w}_h^{k+1},\boldsymbol{\varphi}_h)+\tau_k\boldsymbol{b}_h^L(\boldsymbol{w}_h^k,\boldsymbol{w}_h^{k+1},\boldsymbol{\varphi}_h)=(\boldsymbol{w}_h^k,\boldsymbol{\varphi}_h)\qquad\forall\boldsymbol{\varphi}_h\in\mathbf{S}_{h\mathsf{p}},\ k=0,1\dots,r, \quad (6)$$

where $\boldsymbol{w}_h^k\in\mathbf{S}_{h\mathsf{p}}$, $k=0,\dots,r$, denote approximate solutions at time levels t_k, $k=0,\dots,r$, $\tau_k=t_{k+1}-t_k$ is the size of the time step and $\boldsymbol{b}_h^L(\cdot,\cdot,\cdot)$ formally represents a linearization of the DG discretization of the inviscid fluxes (5), see [3].

4 Shock-capturing scheme

We have proposed a *viscosity limiter* approach, which is based on adding artificial diffusion term to the system (1) in the form which corresponds to the viscous part of the compressible Navier-Stokes equations but with the variable Reynolds number Re in the whole computational domain as in [7]. This variable choice of Re plays a role as an artificial viscosity μ_{art}, which depends on the solution of the system (1) in a special way, i.e., $Re^{-1}\approx\mu_{art}(\boldsymbol{w}_h)$.

We modify (1) and get new system

$$\frac{\partial\boldsymbol{w}}{\partial t}+\nabla\cdot\vec{\boldsymbol{f}}(\boldsymbol{w})=\mu_{art}(\boldsymbol{w})\nabla\cdot\vec{\boldsymbol{R}}(\boldsymbol{w},\nabla\boldsymbol{w})\quad\text{in }Q_T, \quad (7)$$

where

$$\vec{\boldsymbol{R}}(\boldsymbol{w},\nabla\boldsymbol{w})=(\vec{R}_1,\vec{R}_2)\text{ and }\vec{R}_s=\begin{pmatrix}0\\ \left(\frac{\partial v_s}{\partial x_1}+\frac{\partial v_1}{\partial x_s}\right)-\frac{2}{3}\mathrm{div}(\boldsymbol{v})\,\delta_{s1}\\ \left(\frac{\partial v_s}{\partial x_2}+\frac{\partial v_2}{\partial x_s}\right)-\frac{2}{3}\mathrm{div}(\boldsymbol{v})\,\delta_{s2}\\ \sum_{k=1}^2\vec{R}_s^{(k)}v_k+\frac{\gamma}{Pr}\frac{\partial\theta}{\partial x_s}\end{pmatrix},\ s=1,2, \quad (8)$$

with a notation: γ - Poisson adiabatic constant, Pr - Prandtl number and θ - temperature. Finally, the system (7) has to be closed by the equation of the total energy, see, e.g., [4].

Let us note that this proposed artificial viscosity approach corresponds to the solution of the compressible Navier-Stokes equations with "do-nothing" boundary condition.

According to (6) and (7) we obtain a shock-capturing scheme (SC):

$$\begin{aligned}(\boldsymbol{w}_h^{k+1},\boldsymbol{\varphi}_h)&+\tau_k\Big(\boldsymbol{b}_h^L(\boldsymbol{w}_h^k,\boldsymbol{w}_h^{k+1},\boldsymbol{\varphi}_h)+\boldsymbol{a}_h^L(\boldsymbol{w}_h^k,\boldsymbol{w}_h^{k+1},\boldsymbol{\varphi}_h)+\boldsymbol{J}_h(\boldsymbol{w}_h^k,\boldsymbol{w}_h^{k+1},\boldsymbol{\varphi}_h)\Big)\\ &=(\boldsymbol{w}_h^k,\boldsymbol{\varphi}_h)\qquad\forall\boldsymbol{\varphi}_h\in\mathbf{S}_{h\mathsf{p}},\ k=0,1\dots,r, \quad (9)\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{a}_h^L(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) \quad &= \sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^{2} \mu_{art}(\boldsymbol{w}_h^k) \left(\sum_{k=1}^{d} \boldsymbol{K}_{s,k}(\boldsymbol{w}_h^k) \frac{\partial \boldsymbol{w}_h^{k+1}}{\partial x_k} \right) \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s}\, \mathrm{d}x \qquad (10)\\
&- \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \sum_{s=1}^{2} \left\langle \mu_{art}(\boldsymbol{w}_h^k) \left(\sum_{k=1}^{d} \boldsymbol{K}_{s,k}(\boldsymbol{w}_h^k) \frac{\partial \boldsymbol{w}_h^{k+1}}{\partial x_k} \right) \right\rangle_\Gamma n_s \cdot [\boldsymbol{\varphi}_h]_\Gamma \, \mathrm{d}S\\
&- \Theta \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \sum_{s=1}^{2} \left\langle \mu_{art}(\boldsymbol{w}_h^k) \sum_{k=1}^{d} \boldsymbol{K}_{s,k}^T(\boldsymbol{w}_h^k) \frac{\partial \boldsymbol{\varphi}_h}{\partial x_k} \right\rangle_\Gamma n_s \cdot [\boldsymbol{w}_h^{k+1}]_\Gamma \, \mathrm{d}S
\end{aligned}$$

represents the linearized viscous fluxes (8). The detailed description of the matrices $\mathbb{K}_{s,k} \in \mathbb{R}^{4\times 4},\ k = 1, 2, s = 1, 2$, can be found in [5] and Θ is a stabilization parameter which can take the values $\{-1; 0; 1\}$ according to the chosen variant of stabilization. In order to replace inter-element discontinuities of the scheme (SC) we introduce the *penalty*

$$\boldsymbol{J}_h(\boldsymbol{w}_h^k, \boldsymbol{w}_h^{k+1}, \boldsymbol{\varphi}_h) = \sum_{\Gamma\in\mathcal{F}_h^I} \int_\Gamma \mu_{art}(\boldsymbol{w}_h^k) C_W |\Gamma|^{-1} [\boldsymbol{w}_h^{k+1}]_\Gamma \cdot [\boldsymbol{\varphi}_h]_\Gamma \, \mathrm{d}S, \qquad (11)$$

where $C_W > 0$ is a suitable constant depending on the used variant of stabilization and on the degree of polynomial approximation. The scheme (9) requires a solution of linear algebraic problem at each time level and gives practically unconditionally stable scheme, see [4].

The key ingredient of the scheme (SC) is the nonlinear viscosity μ_{art} which is chosen proportionally to the residual of the entropy equation in the spirit of [6]. It is known from thermodynamics that $S = \frac{1}{\gamma-1} \ln(p/\rho^\gamma)$ is an entropy functional for perfect gas which satisfies the following energy equation (see [5]) written in the entropy form

$$\frac{\partial \rho S}{\partial t} + \mathrm{div}(\rho S \boldsymbol{v}) = \frac{D(\boldsymbol{v})}{\theta} + \mu_{art} \frac{\gamma}{Pr} \frac{\mathrm{div}(\nabla\theta)}{\theta}, \qquad (12)$$

where $D(\boldsymbol{v}) = -\frac{2}{3}\mu_{art}(\mathrm{div}(\boldsymbol{v}))^2 + 2\mu_{art}\boldsymbol{D}(\boldsymbol{v})\cdot\boldsymbol{D}(\boldsymbol{v})$ is a dissipation and $\boldsymbol{D}(\boldsymbol{v})$ denotes symmetric part of the velocity gradient.

To construct μ_{art}, we first evaluate the discrete entropy residual $r_S = r_S(\boldsymbol{w}_h)$, which is considered in the following weak formulation as $r_S \in S_{h\mathsf{p}}$ such that

$$\int_\Omega r_S \cdot \varphi_h \, \mathrm{d}x = \int_\Omega \left(\frac{\partial \rho S}{\partial t} + \mathrm{div}(\rho S \boldsymbol{v}) - \frac{D(\boldsymbol{v})}{\theta} - \mu_{art} \frac{\gamma}{Pr} \frac{\mathrm{div}(\nabla\theta)}{\theta} \right) \varphi_h \, \mathrm{d}x \quad \forall\, \varphi_h \in S_{h\mathsf{p}}. \qquad (13)$$

In view of (13), the function r_S is L^2-projection onto $S_{h\mathsf{p}}$, i.e. $r_S|_K \in P_{p_K}(K)$, $K \in \mathcal{T}_h$. Further, we construct a piecewise constant limiting viscosity as follows

$$\mu_{max}^K = \frac{\mathrm{diam}(K)}{p_K} \max_K \left(\rho|\boldsymbol{v}| + \rho\sqrt{\gamma\theta} \right) \Big|_K, \quad K \in \mathcal{T}_h \qquad (14)$$

and finally set

$$\mu_{art}(\boldsymbol{w}_h)|_K = \min\left(\mu^K_{max}, \beta L\,\mathrm{diam}(K)\Big|r_S|_K(\boldsymbol{w}_h)\Big|\right), \quad K \in \mathcal{T}_h, \tag{15}$$

where L denotes the characteristic length (e.g. length of channel or airfoil) and β is a user-dependent parameter, typically β can reasonably be chosen in the range $[0.05, 5]$ without that choice dramatically affecting the results.

5 Numerical example

We consider inviscid steady transonic flow past a single NACA0012 airfoil of unit length at free stream Mach number $M_\infty = 0.8$ with angle of attack $\alpha = 1.25°$. The computation domain is a circle with radius of 50. We use a fixed relatively coarse triangular mesh having 4544 elements which was adaptively refined and uses curved elements along the airfoil. The characteristic feature of this flow is a relatively strong shock at the suction side and a very weak shock at the pressure side.

We carried out computations with the shock capturing scheme (SC) by P_1, P_2 and P_3 approximations and set $\Theta = 1$ (non-symmetric variant) with $C_W = 1$. These values guarantee the stability of the scheme (SC), for more details see [4]. The initial condition was set as a constant vector taken from the prescribed boundary conditions at infinity: $\rho = 1$, $v_1 = 0.999762027$, $v_2 = 0.021814885$ and the Mach number $M_\infty = 0.8$. This test case represents a stationary problem. Therefore, the computational process was stopped, after the residue of the solution had reached the prescribed tolerance.

Table 1 illustrates the asymptotic convergence of drag (c_D) and lift (c_L) coefficients and comparison with reference values from [1]. Figure 1 shows the pressure coefficient c_p along the airfoil with resolved shocks. We obtained satisfactory results and quite good agreement was already achieved for piecewise cubic approximation with reference results from [1] using P_5 approximation.

method	c_D	c_L	$\#DOF$
SC-DGM – P_1	0.02426	0.33684	54 528
SC-DGM – P_2	0.02300	0.34065	109 056
SC-DGM – P_3	0.02277	0.35587	181 760
ref. value [1] – P_5	0.02276	0.35366	381 696

Tab. 1: *Computed values of force coefficients in comparison with [1].*

6 Conclusion

We dealt with the numerical solution of the compressible Euler equations via discontinuous Galerkin method. We presented the shock-capturing technique avoiding a failure of computational processes and most of Gibbs phenomena. Preliminary numerical example gives promising results.

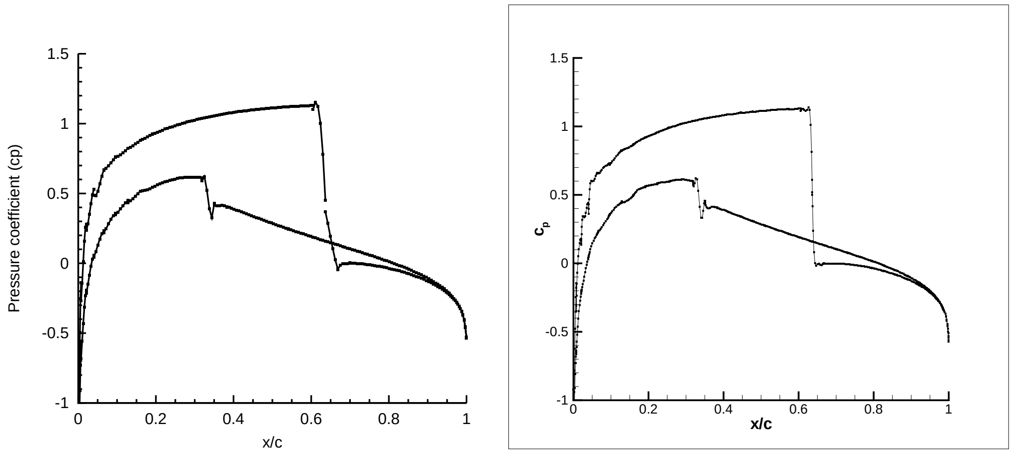

Fig. 1: *Pressure coefficient comparison, SC-DGM scheme with P_3 (left), DG scheme with P_5 described in [1] (right).*

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