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Martin Horák; Mathieu Charlebois; Milan Jirásek; Philippe K. Zysset
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NONLOCAL TANGENT OPERATOR FOR DAMAGE PLASTICITY MODEL*

Martin Horák, Mathieu Charlebois, Milan Jirásek, Philippe K. Zysset

1 Introduction

Realistic description of the mechanical behaviour of quasi-brittle materials requires a constitutive law with softening. Softening is one of the destabilising factors that may lead to localisation of inelastic processes into narrow bands. Standard "local" models fail to describe this phenomenon in an objective way. The boundary value problem becomes ill-posed due to the loss of ellipticity of the governing differential equation and results obtained numerically are not objective with respect to the discretisation. To avoid pathological sensitivity of the numerical results to the finite element mesh, the model is regularised by a nonlocal formulation based on a spatial averaging procedure, which acts as a localisation limiter. The return mapping algorithm based on the closest-point projection is developed and the corresponding consistent algorithmic stiffness is derived using an extension of the approach proposed in [2] for nonlocal damage models.

2 Constitutive model

In this section a model combining anisotropic elasticity and anisotropic plasticity coupled with isotropic damage is described. This model was first presented in [4]. The main feature of plasticity models is irreversibility of plastic strain while irreversible processes related to damage lead to degradation of stiffness. The basic equations include an additive decomposition of total strain into elastic (reversible) part and plastic (irreversible) part,

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p},$$

the stress strain law,

$$\sigma_{ij} = (1 - \omega(\kappa))\, \bar{\sigma}_{ij} = (1 - \omega(\kappa))\, D_{ijkl}^{e} \varepsilon_{kl}^{e},$$

loading-unloading conditions in Kuhn-Tucker form,

$$f(\bar{\sigma}_{ij}, \kappa) \leq 0 \qquad \dot{\lambda} \geq 0 \qquad \dot{\lambda} f(\bar{\sigma}_{ij}, \kappa) = 0,$$

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evolution laws for plastic strain,

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \bar{\sigma}_{ij}},$$

and for cumulated plastic strain,

$$\dot{\kappa} = \sqrt{\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}},$$

the law governing the evolution of the damage variable,

$$\omega(\kappa) = \omega_c(1 - \mathrm{e}^{-a\kappa}),$$

and the hardening law,

$$\sigma_Y(\kappa) = 1 + \sigma_H(1 - \mathrm{e}^{-s\kappa}).$$

In the equations above, $\bar{\sigma}_{ij}$ is the effective stress tensor, D^e_{ijkl} is the elastic stiffness tensor, f is the yield function, λ is the plastic multiplier, ω is the damage variable, κ is the cumulated plastic strain, σ_Y is the yield stress and s, a, σ_H and ω_c are positive material parameters, to be identified from experiments. Superior dot marks the derivative with respect to time. To complete the formulation, the specific form of yield function needs to be introduced:

$$f(\bar{\sigma}_{ij}, \kappa) = \sqrt{\bar{\sigma}_{ij}F_{ijkl}\bar{\sigma}_{kl}} - \sigma_Y(\kappa).$$

Material anisotropy is characterised by the second-order fabric tensor. The eigenvectors of the fabric tensor determine the directions of material orthotropy and the components of the elastic stiffness tensor D^e_{ijkl} are linked to eigenvalues of the fabric tensor. Similar relations are postulated for the components of the fourth-order tensor F_{ijkl} used in the yield condition.

3 Nonlocal formulation

The standard elasto-plasto-damage model based on continuum approach was described in the previous section. However, such a model fails after the loss of ellipticity, which leads to an ill-posed boundary value problem. From the numerical point of view, ill-posedness is manifested by a pathological sensitivity of the numerical results to the size of finite elements. One possible regularisation technique is a nonlocal formulation based on spatial averaging. The model is regularised by the over-nonlocal formulation with damage driven by a combination of local and nonlocal cumulated plastic strain,

$$\hat{\kappa} = (1 - m)\kappa + m\bar{\kappa},$$

where

$$\bar{\kappa}(x) = \int_V \alpha(x, s)\kappa(s)\,\mathrm{d}s \tag{1}$$

is the nonlocal cumulated plastic strain and m is a model parameter that should exceed unity to suppress the sensitivity of the numerical solution to the mesh shape. The nonlocal weight function is usually defined as

$$\alpha(x,s) = \frac{\alpha_0(\|x-s\|)}{\int_V \alpha_0(\|x-t\|)\,\mathrm{d}t}$$

where

$$\alpha_0(r) = \begin{cases} (1 - r^2/R^2)^2 & \text{if } r < R \\ 0 & \text{if } r \geq R \end{cases}$$

is a nonnegative function, for $r < R$ monotonically decreasing with increasing distance $r = \|x - s\|$, and V denotes the domain occupied by the investigated material body. The key idea is that the damage evolution at a certain point depends not only on the cumulated plastic strain at that point, but also on points at distances smaller than the interaction radius R, considered as a new material parameter. Note that the over-nonlocal cumulated plastic strain affects only damage evolution while the yield condition remains local.

4 Numerical algorithm

To implement the constitutive model into a displacement-driven finite element code, the governing equations need to be expressed in an incremental form, and an algorithm for the evaluation of the stress increment from a given strain increment must be developed. In plasticity, this procedure is often called the stress-return algorithm. Within a computational increment number $n+1$, the mapping of strain $\boldsymbol{\varepsilon}^{n+1}$ at the end of the step onto the effective stress $\bar{\boldsymbol{\sigma}}^{n+1}$ at the end of the step, provided by the stress-return algorithm, is denoted as function $\boldsymbol{\theta}$, and the mapping of strain $\boldsymbol{\varepsilon}^{n+1}$ onto the cumulated plastic strain κ^{n+1} at the end of the step is denoted as function η. The Jacobi matrix of $\boldsymbol{\theta}$, denoted as $\partial\boldsymbol{\theta}/\partial\boldsymbol{\varepsilon}$, is the consistent elasto-plastic material stiffness. Using the standard finite element assembly procedure, the consistent structural tangent stiffness can be constructed. However, for an elasto-plastic model with damage, it is necessary to take into account additional terms that result from damage growth, and if damage is driven by the over-nonlocal cumulated plastic strain $\hat{\kappa}$, such terms have a more complicated structure than usual, but are still manageable. The resulting nonlocal tangent stiffness matrix of the structural (finite element) model is used in equilibrium iterations of the Newton-Raphson type and leads to quadratic convergence, provided that the linearisation is done in a fully consistent manner.

4.1 Predictor-corrector scheme

The stress return algorithm is based on elastic-plastic operator split, which consist of a trial elastic predictor followed by the return mapping algorithm. The over-nonlocal formulation described in the previous section has computational advantages

Algorithm 1 Return mapping algorithm

given ε^{n+1}, $\varepsilon^{p,n}$, κ^n, ω^n

compute elastic predictor

$\sigma_{ij}^{tr} = D_{ijkl}^e(\varepsilon_{kl}^{e,n+1} - \varepsilon_{kl}^{p,n})$ **and** $f^{tr} = f(\sigma_{ij}^{tr}, \kappa^n)$

check for plastic process

if $f \leq 0$ **then**

elastic step: *set* $\bar{\boldsymbol{\sigma}}^{n+1} = \bar{\boldsymbol{\sigma}}^{tr}$, $\boldsymbol{\varepsilon}^{p,n+1} = \boldsymbol{\varepsilon}^{p,n}$, $\kappa^{n+1} = \kappa^n$, $\omega^{n+1} = \omega^n$

else

return mapping algorithm

1. solve system of nonlinear equations

$\bar{\sigma}_{ij}^{n+1} = \bar{\sigma}_{ij}^{tr} - \Delta\kappa \dfrac{D_{ijkl}^e F_{klmn} \bar{\sigma}_{mn}^{n+1}}{\|F_{ijkl}\bar{\sigma}_{kl}^{n+1}\|}$ $\quad\quad \sqrt{\bar{\sigma}_{ij}^{n+1} F_{ijkl} \bar{\sigma}_{kl}^{n+1}} - \sigma_Y(\kappa + \Delta\kappa) = 0$

$\implies \bar{\boldsymbol{\sigma}}^{n+1} = \theta(\boldsymbol{\varepsilon}^{n+1})$; $\Delta\kappa = \eta(\boldsymbol{\varepsilon}^{n+1})$

2. update state variables

$\varepsilon_{ij}^{p,n+1} = \varepsilon_{ij}^{p,n} + \Delta\kappa \frac{F_{ijkl}\bar{\sigma}_{kl}^{n+1}}{\|F_{ijkl}\bar{\sigma}_{kl}^{n+1}\|}, \quad \kappa^{n+1} = \kappa^n + \Delta\kappa, \quad \hat{\kappa}^{n+1} = (1-m)\kappa^{n+1} + m\bar{\kappa}^{n+1}$

$\omega^{n+1} = \omega(\hat{\kappa}^{n+1}), \quad \sigma_{ij}^{n+1} = (1 - \omega^{n+1})\bar{\sigma}_{ij}^{n+1}$

end if

because the plastic part of the model remains local and the standard return mapping algorithm can be applied at each Gauss point separately. After that, the nonlocal cumulated plastic strain and damage are evaluated in a fully explicit manner. This procedure is summarised in Algorithm 1.

4.2 Consistent tangent operator

The concept of a consistent tangent operator was first presented in [1] for the case of a local elastoplastic problem. As shown in [2], the quadratic convergence is preserved also for a nonlocal damage problem, but only with a consistent nonlocal tangent operator. The consistent stiffness operator is obtained by differentiating the internal force vector with respect to the nodal displacements:

$$\boldsymbol{K} = \frac{\partial \boldsymbol{f}_{int}}{\partial \boldsymbol{d}}.$$

The internal force vector is defined as

$$\int_V \boldsymbol{B}^T \boldsymbol{\sigma} \, \mathrm{d}x \approx \sum_r w_r \boldsymbol{B}_r^T \boldsymbol{\sigma}_r = \boldsymbol{f}_{int} \tag{2}$$

In the above, subscript r refers to the integration points of the finite element model, w_r are the corresponding integration weights and $\boldsymbol{B}$ is the usual strain-displacement matrix.

Using the expression for stress at Gauss point r,

$$\boldsymbol{\sigma}_r = (1 - \omega_r)\bar{\boldsymbol{\sigma}}_r,$$

we can expand (2) as

$$\boldsymbol{f}_{int} = \sum_r w_r \boldsymbol{B}_r^T (1-\omega_r)\bar{\boldsymbol{\sigma}}_r.$$

The effective stress at Gauss point r is given by the return mapping evaluated for strain at Gauss point r:

$$\bar{\boldsymbol{\sigma}}_r = \theta_r(\boldsymbol{\varepsilon}_r).$$

One can then express damage as

$$\omega_r = \omega(\hat{\kappa}_r) = \omega\left(m\bar{\kappa}_r + (1-m)\,\kappa_r\right)$$

and after numerical approximation of integral (1) by

$$\bar{\kappa} \approx \sum_s \alpha_{rs}\kappa_s$$

one gets

$$\omega_r = \omega\left(m\sum_s \alpha_{rs}\kappa_s + (1-m)\,\kappa_r\right),$$

where

$$\kappa_s = \eta(\boldsymbol{\varepsilon}_s)$$

is also supplied by the return mapping algorithm. Combining all this with the standard relation $\boldsymbol{\varepsilon}_s = \boldsymbol{B}_s\boldsymbol{d}$, one can evaluate the consistent nonlocal tangent stiffness operator as

$$K = \sum_r w_r(1-\omega_r)\boldsymbol{B}_r^T \frac{\partial\theta(\boldsymbol{\varepsilon}_r)}{\partial\boldsymbol{\varepsilon}}\boldsymbol{B}_r - (1-m)\sum_r w_r\omega_r'\boldsymbol{B}_r^T\bar{\boldsymbol{\sigma}}_r\left(\frac{\partial\eta(\boldsymbol{\varepsilon}_r)}{\partial\boldsymbol{\varepsilon}}\right)^T\boldsymbol{B}_r$$
$$-m\sum_r\sum_s w_r\omega_r'\alpha_{rs}\boldsymbol{B}_r^T\bar{\boldsymbol{\sigma}}_r\left(\frac{\partial\eta(\boldsymbol{\varepsilon}_s)}{\partial\boldsymbol{\varepsilon}}\right)^T\boldsymbol{B}_s \qquad (3)$$

5 Numerical example

The algorithm described in section 4 has been implemented into the open-source finite element code OOFEM [5, 6]. Properties of the model have been explored for several examples in [4], but with the secant stiffness matrix, which provides only linear convergence rate. The compression of a cylinder is simulated in 100 incremental steps, using a three-dimensional model containing 915 nodes and 609 linear brick elements. As follows from (3), the nonlocal tangent operator is nonsymmetric. One important consequence of nonlocality is a growing profile of the stiffness matrix, caused by the stepwise activation of interaction between pairs of Gauss points belonging to different elements. The evolution of error versus the number of iteration for three steps, corresponding to a pre-peak, peak and post-peak state as indicated in the load-displacement curve in Figure 1(a), is depicted in Figure 1(b) in a semilogarithmic scale. The convergence curves are approximately parabolic, i.e., the convergence rate is quadratic and the equilibrium is reached in a few iterations.

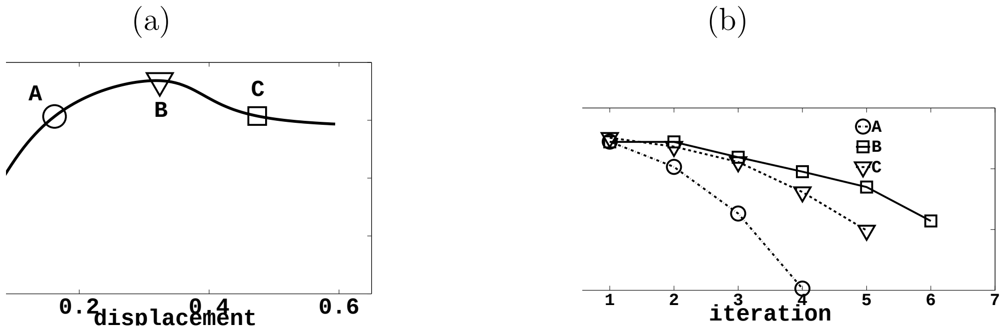

Fig. 1: *(a) Load-displacement curve. (b) Evolution of error during the equilibrium iteration process.*

6 Conclusions

The constitutive law combining anisotropic elasticity, anisotropic plasticity and isotropic damage with the over-nonlocal regularisation is presented. The stress-return algorithm is described and the nonlocal consistent tangent operator is derived. It is shown by a numerical example that the nonlocal consistent tangent operator leads to a quadratic rate of convergence, even if the tangent operator is nonsymmetric and the profile of nonzero elements is growing during the simulation.

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