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## COUPLED HEAT TRANSPORT AND DARCIAN WATER FLOW IN FREEZING SOILS

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### Abstract

The model of coupled heat transport and Darcian water flow in unsaturated soils and in conditions of freezing and thawing is analyzed. In this contribution, we present results concerning the existence of the numerical solution. Numerical scheme is based on semi-implicit discretization in time. This work illustrates its performance for a problem of freezing processes in vertical soil columns.

### 1. Introduction

Let  $T > 0$  and  $\ell > 0$  be the fixed values,  $\Omega = (0, \ell)$ ,  $I = (0, T)$ ,  $\Omega_T = \Omega \times I$ . We consider a mixed initial-boundary value problem for a general model of the coupled heat and mass flow in freezing soils. The mathematical model consists of the following system (cf. [1]):

$$\frac{\partial \theta_M(z, \vartheta, u)}{\partial t} = \frac{\partial}{\partial z} \left( k(z, \vartheta, u) \frac{\partial u}{\partial z} \right) \quad \text{in } \Omega_T, \quad (1)$$

$$C_a(z, \vartheta, u) \frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial z} \left( \lambda(z, \vartheta, u) \frac{\partial \vartheta}{\partial z} \right) + C_w k(z, \vartheta, u) \frac{\partial \vartheta}{\partial z} \frac{\partial u}{\partial z} \quad \text{in } \Omega_T, \quad (2)$$

$$u(0, t) = u_D(t) \quad \text{and} \quad \vartheta(0, t) = \vartheta_D(t) \quad \text{in } I, \quad (3)$$

$$-k(z, \vartheta, u) \frac{\partial u}{\partial z} = \beta_c(u - u_\infty) \quad \text{and} \quad -\lambda(z, \vartheta, u) \frac{\partial \vartheta}{\partial z} = \alpha_c(\vartheta - \vartheta_\infty) \quad \text{in } I, \quad z = \ell, \quad (4)$$

$$u(z, 0) = u_0(z) \quad \text{and} \quad \vartheta(z, 0) = \vartheta_0(z) \quad \text{in } \Omega. \quad (5)$$

This system describes the one-dimensional coupled water flow and heat transport involving freezing-thawing processes in a vertical soil column. Equations (1) and (2) represent conservation laws for mass and energy, (3) and (4) are prescribed boundary conditions of Dirichlet and Neumann type, respectively, and (5) represents appropriate initial conditions. In (1)–(5)  $u = u(z, t)$  [m] and  $\vartheta = \vartheta(z, t)$  [K] (single-valued

functions of the time  $t$  and the spatial position  $z \in \Omega$  (positive upward) are the primary unknowns for the total pressure head and temperature,  $\theta_M$  [-] is the total water content,  $k$  [ $\text{m s}^{-1}$ ] represents the hydraulic conductivity,  $C_a$  [ $\text{J m}^{-3} \text{K}^{-1}$ ] is the so called apparent heat capacity and  $\lambda$  [ $\text{W m}^{-1} \text{K}^{-1}$ ] is the thermal conductivity of the soil. Material constant parameters in (1)–(5) are the volumetric heat capacity of water  $C_w$  ( $4.181 \times 10^6 \text{ J m}^{-3} \text{K}^{-1}$ ), convective heat and mass transfer coefficients  $\alpha_c$  [ $\text{W m}^{-2} \text{K}^{-1}$ ] and  $\beta_c$  [ $\text{s}^{-1}$ ].

## 2. Freezing and thawing

Define  $\psi$  [m] as the matric potential corresponding to the liquid water content  $\theta_w$  [-] and the matric potential  $h$  [m] corresponding to the total water content  $\theta_V$  [-] (liquid and ice). The amount of water present at a certain matric potential of the porous medium is characterized by the water retention curve  $\theta(\cdot)$ . In particular,  $\theta_V = \theta(h)$ , while  $\theta_w = \theta(\psi)$ . Here we use the relation proposed by van Genuchten [4]  $\theta(h) = \theta_r + (\theta_s - \theta_r)[1 + |\alpha h|^n]^{-m}$ , where  $\theta_s$  is the soil saturated water content [-],  $\theta_r$  is the soil residual water content [-],  $\alpha$  [ $\text{m}^{-1}$ ],  $m$  and  $n$  are parameters.

Water in soil pores does not freeze at 273.15 K, but is subject to a freezing-point depression caused by interaction between water, soil particles and solutes. The generalized Clapeyron equation is used to describe the condition for the co-existence of water and ice. The local freezing point of pore fluid can be obtained from the generalized Clapeyron equation [1, 2]

$$dp = \frac{\rho_w L_f}{\vartheta} d\vartheta, \quad (6)$$

where  $p$  [Pa] is the water pressure,  $p = \rho_w g h$ ,  $h = u - z$ ,  $g$  is the acceleration due to gravity ( $9.81 \text{ m s}^{-2}$ ),  $h$  [m] the pressure head (matric potential),  $\rho_w$  the density of liquid water (approximately  $1000.0 \text{ kg m}^{-3}$ ) and  $L_f$  is the latent heat of fusion ( $3.34 \times 10^5 \text{ J kg}^{-1}$ ). Let  $\vartheta_0 = 273.15$  be the freezing temperature at zero pressure head. If the soil is unsaturated, the surface tension at the water/air interface decreases the water freezing temperature to  $\vartheta_f < 273.15 \text{ K}$ . To obtain the value  $\vartheta_f$  at the given pressure  $P$  integrate (6) in temperature from 273.15 to  $\vartheta_f$  and from 0 to  $P$  in pressure to obtain

$$\int_0^P dp = \int_{273.15}^{\vartheta_f} \frac{\rho_w L_f}{\vartheta} d\vartheta, \quad \text{which yields } \vartheta_f = 273.15 e^{hg/L_f} = 273.15 e^{(u-z)g/L_f}. \quad (7)$$

Similarly, integrating (6) in temperature from  $\vartheta_f$  to  $\vartheta$  and from  $P(= h\rho_w g)$  to  $P_\psi(= \psi\rho_w g)$  in pressure yields (recall  $u = h + z$ )

$$\psi(z, \vartheta, u) = \psi(\vartheta, u - z) \equiv \psi(\vartheta, h) = h + \frac{L_f}{g} \ln \frac{\vartheta}{\vartheta_f} = u - z + \frac{L_f}{g} \ln \frac{\vartheta}{\vartheta_f}. \quad (8)$$

The above equation is valid for  $\vartheta < \vartheta_f$ . If  $\vartheta \geq \vartheta_f$  all water is unfrozen and  $h = \psi$  and  $\theta_w = \theta_V$ . Consequently, whenever  $\vartheta < \vartheta_f$ , the ice fraction  $\theta_i$  [-] can be expressed

as  $\theta_i = \theta_V - \theta_w$  [-]. In addition, the total water content  $\theta_M$  (present in (1)) as derived by the fraction of total mass of liquid water and ice (see [1, Appendix A]) reads

$$\theta_M(z, \vartheta, u) = \theta_w(\psi(z, \vartheta, u)) + \frac{\rho_i}{\rho_w} \theta_i(z, \vartheta, u) = \frac{\rho_i}{\rho_w} \theta_V(z, u) + \left(1 - \frac{\rho_i}{\rho_w}\right) \theta_w(\psi(z, \vartheta, u)),$$

where  $\rho_i$  is the density of ice (918 kg m<sup>-3</sup>).

### 2.1. Structural conditions and assumptions on physical parameters

Let us present some properties and additional assumptions on physical parameters introduced in the model.

$\mathbb{A}_1$  The parameters  $\rho_w$ ,  $\rho_i$ ,  $\theta_s$ ,  $\theta_r$ ,  $C_w$ ,  $L_f$ ,  $\alpha_c$  and  $\beta_c$  are real positive constants and  $\rho_i < \rho_w$ .

$\mathbb{A}_2$  The thermal conductivity  $\lambda$ , apparent thermal capacity  $C_a$  and hydraulic conductivity  $k$  are assumed to be positive continuous functions of their arguments (see [2] for specific examples). In addition,

$$0 < C_a(z, \xi, \zeta) \leq C_a^\# < +\infty \quad \forall \xi, \zeta \in \mathbb{R} \quad (C_a^\# = \text{const} > 0). \quad (9)$$

$\mathbb{A}_3$  Functions  $\theta_w = \theta_w(z, \cdot)$  and  $\theta_V = \theta_V(z, \cdot)$  (for  $z \in \Omega$ ) are positive, nondecreasing, continuous and bounded functions such that

$$\theta_r \leq \theta_w(z, \xi) \leq \theta_s, \quad \theta_w(z, \xi) \leq \theta_V(z, \xi) \leq \theta_s \quad \forall \xi \in \mathbb{R}. \quad (10)$$

Consequently,  $\theta_M$  is a positive continuous function such that

$$0 < \theta_M(z, \xi, \zeta) = \frac{\rho_i}{\rho_w} \theta_V(z, \xi) + \left(1 - \frac{\rho_i}{\rho_w}\right) \theta_w(z, \zeta) \leq \theta_s \quad \text{for all } \xi, \zeta \in \mathbb{R}.$$

$\mathbb{A}_4$  Functions  $u_D$ ,  $\vartheta_D$ ,  $u_\infty$ ,  $\vartheta_\infty$  are continuous on  $[0, T]$ ,  $u_0, \vartheta_0 \in W^{1,2}(\Omega)^2$  and the compatibility conditions  $u_0(0) = u_D(0)$  and  $\vartheta_0(0) = \vartheta_D(0)$  hold.

### 3. The approximate solution

Albeit the coupled problem (1)–(5) is essentially non-stationary in their nature, we shall formulate and analyze a weak form of the stationary problem. It has a significant mathematical interest because the time discretization of the evolution problem leads, in each time step, to a coupled system of stationary equations.

Let  $0 = t_0 < t_1 < \dots < t_N = T$  be an equidistant partitioning of time interval  $[0; T]$  with step  $\Delta t$ . Set a fixed integer  $n$  such that  $0 \leq n \leq N - 1$ . In what follows we abbreviate  $\phi(z, t_n)$  by  $\phi_n$  ( $\equiv \phi(z)_n$ ) for any function  $\phi$ . The time discretization of the continuous model is accomplished through a semi-implicit difference scheme.

Consequently, we have to solve, successively for  $n = 0, \dots, N - 1$ , the following semilinear system with primary unknowns  $[\vartheta_{n+1}, u_{n+1}]$

$$\frac{\theta_M(z, \vartheta_{n+1}, u_{n+1}) - \theta_M(z, \vartheta_n, u_n)}{\Delta t} = \frac{\partial}{\partial z} \left( k_n \frac{\partial u_{n+1}}{\partial z} \right), \quad (11)$$

$$(C_a)_{n+1} \frac{\vartheta_{n+1} - \vartheta_n}{\Delta t} = \frac{\partial}{\partial z} \left( \lambda_n \frac{\partial \vartheta_{n+1}}{\partial z} \right) + C_w k_n \frac{\partial u_n}{\partial z} \frac{\partial \vartheta_n}{\partial z}, \quad (12)$$

$$u(0)_{n+1} = (u_D)_{n+1} \quad \text{and} \quad \vartheta(0)_{n+1} = (\vartheta_D)_{n+1}, \quad (13)$$

$$-k_n \frac{\partial u_{n+1}}{\partial z} \Big|_{z=\ell} = \beta_c (u(\ell)_{n+1} - u_\infty(\ell)_{n+1}), \quad (14)$$

$$-\lambda_n \frac{\partial \vartheta_{n+1}}{\partial z} \Big|_{z=\ell} = \alpha_c (\vartheta(\ell)_{n+1} - \vartheta_\infty(\ell)_{n+1}). \quad (15)$$

Here, we assume that the functions  $u_n$  and  $\vartheta_n$  are known and (for the sake of simplicity) we put  $k_n = k(z, \vartheta_n, u_n)$ ,  $\lambda_n = \lambda(z, \vartheta_n, u_n)$ ,  $(C_a)_{n+1} = C_a(z, \vartheta_{n+1}, u_{n+1})$ . In what follows we study the problem of the existence of the solution  $u_{n+1}$  and  $\vartheta_{n+1}$ .

Let  $\mathbb{V}$  be a closure of the space  $\{\mathbf{v} \in C^\infty(\overline{\Omega})^2; \mathbf{v}(0) = \mathbf{0}\}$  in the norm of  $W^{1,2}(\Omega)^2$ . By  $\langle \cdot, \cdot \rangle$  we denote the duality between  $\mathbb{V}$  and  $\mathbb{V}^*$ , where  $\mathbb{V}^*$  represents the dual space corresponding to  $\mathbb{V}$ . Define an operator  $\mathcal{A} : W^{1,2}(\Omega)^2 \rightarrow \mathbb{V}^*$  given by the equation

$$\begin{aligned} \langle \mathcal{A}([\vartheta_{n+1}, u_{n+1}]), \boldsymbol{\varphi} \rangle &:= \int_{\Omega} k_n \frac{\partial u_{n+1}}{\partial z} \frac{\partial \varphi_1}{\partial z} + \lambda_n \frac{\partial \vartheta_{n+1}}{\partial z} \frac{\partial \varphi_2}{\partial z} \, dz \\ &+ \frac{1}{\Delta t} \int_{\Omega} \theta_M(z, \vartheta_{n+1}, u_{n+1}) \varphi_1 + (C_a)_{n+1} (\vartheta_{n+1} - \vartheta_n) \varphi_2 \, dz \\ &+ \beta_c u(\ell)_{n+1} \varphi_1(\ell) + \alpha_c \vartheta(\ell)_{n+1} \varphi_2(\ell) \end{aligned} \quad (16)$$

for every  $\boldsymbol{\varphi} = [\varphi_1, \varphi_2] \in \mathbb{V}$  and the functional  $\mathbf{f} \in \mathbb{V}^*$  by the equation

$$\begin{aligned} \langle \mathbf{f}, \boldsymbol{\varphi} \rangle &:= \frac{1}{\Delta t} \int_{\Omega} \theta_M(z, \vartheta_n, u_n) \varphi_1 \, dz + \int_{\Omega} C_w k_n \frac{\partial u_n}{\partial z} \frac{\partial \vartheta_n}{\partial z} \varphi_2 \, dz \\ &+ \beta_c u_\infty(\ell)_{n+1} \varphi_1(\ell) + \alpha_c \vartheta_\infty(\ell)_{n+1} \varphi_2(\ell) \end{aligned} \quad (17)$$

for all  $\boldsymbol{\varphi} = [\varphi_1, \varphi_2] \in \mathbb{V}$ . It can be shown that the operator  $\mathcal{A}$  and the functional  $\mathbf{f}$  are well defined. Let  $[\vartheta_n, u_n] \in [(\vartheta_D)_n, (u_D)_n] + \mathbb{V}$ . We say that the couple  $[\vartheta_{n+1}, u_{n+1}] \in [(\vartheta_D)_{n+1}, (u_D)_{n+1}] + \mathbb{V}$  is the weak solution of the problem (11)–(15) whenever  $\langle \mathcal{A}([\vartheta_{n+1}, u_{n+1}]), \boldsymbol{\varphi} \rangle = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle$  for all  $\boldsymbol{\varphi} = [\varphi_1, \varphi_2] \in \mathbb{V}$ .

**Theorem 1.** *For a given couple  $[\vartheta_n, u_n] \in [(\vartheta_D)_n, (u_D)_n] + \mathbb{V}$  there exists a weak solution  $[\vartheta_{n+1}, u_{n+1}] \in [(\vartheta_D)_{n+1}, (u_D)_{n+1}] + \mathbb{V}$  of the problem (11)–(15).*

*Sketch of the proof.* Note that the couple  $[\vartheta_{n+1}, u_{n+1}] \in [(\vartheta_D)_{n+1}, (u_D)_{n+1}] + \mathbb{V}$  is the weak solution of the problem (11)–(15) iff it is a solution of the operator equation

$$\mathcal{A}([\vartheta_{n+1}, u_{n+1}]) = \mathbf{f}.$$

Let us define  $\bar{\mathcal{A}} : \mathbb{V} \rightarrow \mathbb{V}^*$  by  $\bar{\mathcal{A}}([\bar{\vartheta}_{n+1}, \bar{u}_{n+1}]) := \mathcal{A}([\bar{\vartheta}_{n+1}, \bar{u}_{n+1}] + [(\vartheta_D)_{n+1}, (u_D)_{n+1}])$ . The abstract equation  $\bar{\mathcal{A}}([\bar{\vartheta}_{n+1}, \bar{u}_{n+1}]) = \mathbf{f}$  has a solution  $[\bar{\vartheta}_{n+1}, \bar{u}_{n+1}] \in \mathbb{V}$  if and only if  $[\vartheta_{n+1}, u_{n+1}] = [\bar{\vartheta}_{n+1}, \bar{u}_{n+1}] + [(\vartheta_D)_{n+1}, (u_D)_{n+1}] \in W^{1,2}(\Omega)^2$  is the solution of the equation  $\mathcal{A}([\vartheta_{n+1}, u_{n+1}]) = \mathbf{f}$ . Note that the equation  $\bar{\mathcal{A}}([\bar{\vartheta}_{n+1}, \bar{u}_{n+1}]) = \mathbf{f}$  represents a variational formulation corresponding to the system of coupled semilinear equations. It can be shown that the operator  $\bar{\mathcal{A}} : \mathbb{V} \rightarrow \mathbb{V}^*$  is pseudomonotone and coercive. Now [3, Theorem 3.3.42] yields the existence of the solution  $[\bar{\vartheta}_{n+1}, \bar{u}_{n+1}] \in \mathbb{V}$  to the equation  $\bar{\mathcal{A}}([\bar{\vartheta}_{n+1}, \bar{u}_{n+1}]) = \mathbf{f}$ .  $\square$

#### 4. Numerical solution and results

By means of the model described above, we briefly present numerical results for coupled water flow and heat transport involving freezing-thawing cycle in a vertical soil column. The soil thickness in the numerical simulation for the one-dimensional vertical transport is 1 m, see Fig. 1. The spatial discretization of the system (11)–(15) is carried out by means of the FE-method with piecewise linear elements with spatial discretization as indicated in Fig. 1. This resulting system is solved using the well-known Newton method at each time step with  $\Delta t = 30$  s. Physical properties of soil are taken from [1, 2, 4]. The initial and boundary conditions are set as follows:  $\vartheta_0 = \vartheta_D = 277.15$  K,  $u_0 = -0.1241 + z$  m,  $u_\infty$  decreases from the value  $u_0 + 1$  m to  $-100$  m during the first two days and then taken constant, the distribution of  $\vartheta_\infty$  is shown in Fig. 2. The progress of freezing and thawing in a soil column based on numerical simulation is clearly visible in Figures 3 and 4 which show the vertical distributions of the temperature, water content and ice during the 8-days period.

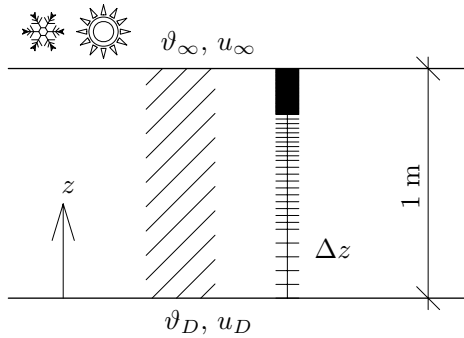


Figure 1: Analyzed soil profile.

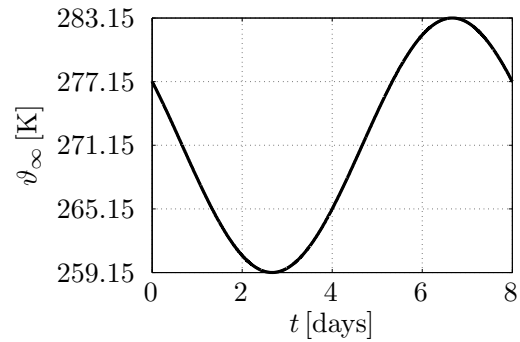


Figure 2: Temperature  $\vartheta_\infty$ .

#### Acknowledgements

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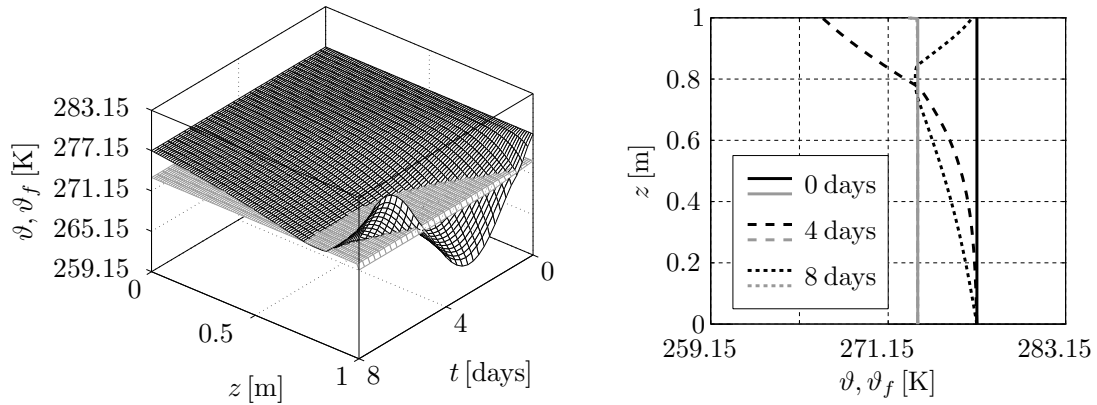


Figure 3: Spatial and time distribution of temperature (black lines) and freezing temperature (gray lines),  $\vartheta$  and  $\vartheta_f$ , respectively, for the analyzed soil profile.

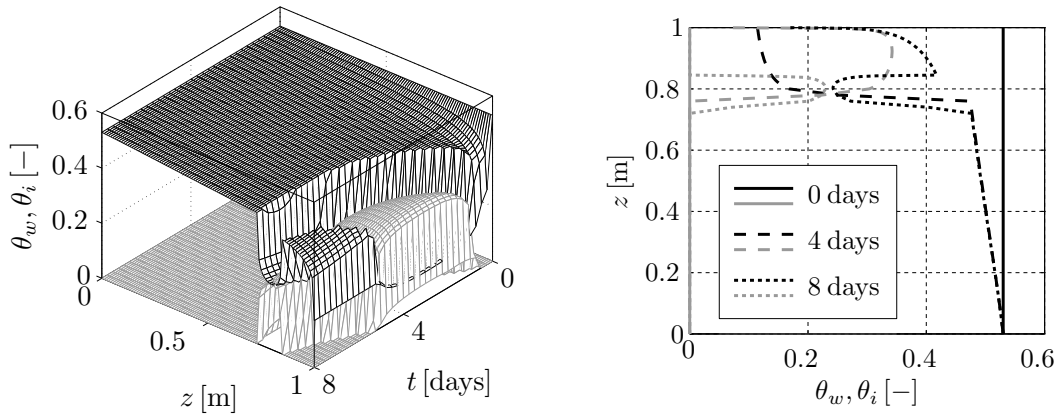


Figure 4: Spatial and time distribution of water (black lines) and ice (gray lines) content,  $\theta_w$  and  $\theta_i$ , respectively, for the analyzed soil profile.

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