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In: Jan Chleboun and Petr Přikryl and Karel Segeth and Jakub Šístek and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 8-13, 2014. Institute of Mathematics AS CR, Prague, 2015. pp. 32–40.

Persistent URL: http://dml.cz/dmlcz/702660

Programs and Algorithms of Numerical Matematics 17
J. Chleboun, P. Přikryl, K. Segeth, J. Šístek, T. Vejchodský (Eds.)
Institute of Mathematics AS CR, Prague 2015

AN A POSTERIORI ERROR ESTIMATE FOR THE STOKES-BRINKMAN PROBLEM IN A POLYGONAL DOMAIN

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Abstract

We derive a residual based a posteriori error estimate for the Stokes-Brinkman problem on a two-dimensional polygonal domain. We use Taylor-Hood triangular elements. The link to the possible information on the regularity of the problem is discussed.

1. Introduction

In the paper we try to contribute to the technique of a posteriori error estimates for the finite element solution of linearized flow problems. In this respect we note that important results have already been obtained: concerning linear elliptic equations let us mention I. Babuška, W. C. Rheinboldt [2], I. Babuška, R. Durán, R. Rodríguez [3], concerning the Stokes problem e.g. M. Ainsworth, J. T. Oden [1], R. E. Bank, D. Welfert [5], C. Carstensen, S. Jansche [7], C. Johnson, R. Rannacher, M. Boman [12], R. Verfürth [15].

The goal of this paper is to link the problem of a posteriori error estimates as much as possible to the information on the regularity of the solution.

Let us illustrate it first on the Dirichlet problem for the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a polygonal domain in R^2. Let u_h be the finite element solution of (1), with linear triangular elements. Let us denote

$$e = u - u_h,$$

the approximation error, and

$$R(u_h) = f + \Delta u_h,$$

the residual. Following the technique of K. Eriksson et al. [10], we first express the error by means of product of residual and solution of the dual problem, then use the Galerkin orthogonality and get the estimate of the error, in the L_2-norm:

$$\|e\|_0^2 \leq \sum_{K\in\mathcal{T}_h} \Big\{ \|R(u_h)\|_{0,K} \|\varphi - \pi_h\varphi\|_{0,K} + \sum_{l\in\partial K} \Big\| \frac{1}{2} \Big[\!\Big[\frac{\partial u_h}{\partial \boldsymbol{n}} \Big]\!\Big]_l \Big\|_{0,l} \|\varphi - \pi_h\varphi\|_{0,l} \Big\}, \tag{2}$$

where φ is the solution of the dual problem

$$\begin{aligned} -\Delta\varphi &= e \quad \text{in} \quad \Omega, \\ \varphi &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3}$$

$\pi_h\varphi$ means the interpolant of φ. The sum in (2) is taken over all triangles in the triangulation $\mathcal{T}_h$, the symbol $[\![\frac{\partial u_h}{\partial \boldsymbol{n}}]\!]_l$ means the jump of the normal derivative $\frac{\partial u_h}{\partial \boldsymbol{n}}$ over the edge l of the triangle K.
Let us now distinguish 3 cases:
A) *General polygonal domain* Ω:
Let h_K be the largest side of the triangle K. The interpolation property together with the (low) regularity of the dual problem (3) yield

$$\|\varphi - \pi_h\varphi\|_{0,K} \leq C_I h_K \|\varphi\|_1 \leq C_I C_R h_K \|e\|_0.$$

Combining this with (2), we come to the a posteriori error estimate

$$\|e\|_0 \leq C_I C_R \sum_{K\in\mathcal{T}_h} h_K \Big\{ \|R(u_h)\|_{0,K} + h_K^{-\frac{1}{2}} \sum_{l\in\partial K} \Big\| \frac{1}{2} \Big[\!\Big[\frac{\partial u_h}{\partial \boldsymbol{n}} \Big]\!\Big]_l \Big\|_{0,l} \Big\}. \tag{4}$$

B) *Convex polygon* Ω:
Now the regularity of the dual problem (3) is higher, cf. R. B. Kellogg, J. E. Osborn [13], and together with the interpolation property it gives

$$\|\varphi - \pi_h\varphi\|_{0,K} \leq C_I h_K^2 \|\varphi\|_2 \leq C_I C_R h_K^2 \|e\|_0.$$

Combining this with (2), we come to the more precise a posteriori estimate

$$\|e\|_0 \leq C_I C_R \sum_{K\in\mathcal{T}_h} h_K^2 \Big\{ \|R(u_h)\|_{0,K} + h_K^{-\frac{1}{2}} \sum_{l\in\partial K} \Big\| \frac{1}{2} \Big[\!\Big[\frac{\partial u_h}{\partial \boldsymbol{n}} \Big]\!\Big]_l \Big\|_{0,l} \Big\}. \tag{5}$$

C) *Nonconvex polygon* Ω *with known singularity*:
It is well-known that the solution near the nonconvex corner, in the local spherical coordinates, has the form

$$u(r, \vartheta) = r^\gamma w(\vartheta),$$

where r is the distance from the corner, $\gamma \in (0,1)$. For instance, the case of the L-shaped domain with the interior angle $\omega = \frac{3}{2}\pi$ gives $\gamma = \frac{2}{3}$, cf. also [6]. Now the interpolation together with the above regularity gives

$$\|\varphi - \pi_h\varphi\|_{0,K} \le C_I h_K^{1+\gamma-\varepsilon}\|\varphi\|_{H^{1+\gamma-\varepsilon}} \le C_I C_R h_K^{1+\gamma-\varepsilon}\|e\|_0, \ \forall\varepsilon > 0,$$

which, combined with (2), finally leads to the a posteriori estimate

$$\|e\|_0 \le C_I C_R \sum_{K\in\mathcal{T}_h} h_K^{1+\gamma-\varepsilon}\left\{\|R(u_h)\|_{0,K} + h_K^{-\frac{1}{2}}\sum_{l\in\partial K}\left\|\frac{1}{2}\left[\left[\frac{\partial u_h}{\partial \boldsymbol{n}}\right]\right]_l\right\|_{0,l}\right\}, \tag{6}$$

valid $\forall\varepsilon > 0$. Of course, in (6) the parameter γ applies only in the nearest neighborhood of the corner.

Comparing the estimates (4), (5), (6) we see that the a posteriori error estimate depends significantly on the regularity of the problem. Having this in mind, we try to derive the a posteriori error estimate for the Stokes-Brinkman problem.

2. The Stokes-Brinkman model

Let Ω be a bounded Lipschitzian domain, $\Omega \subset R^2$, which consists of two parts: porous part Ω_p and fluid part Ω_f, $\bar{\Omega} = \bar{\Omega}_p \cup \bar{\Omega}_f$. The Stokes-Brinkman equation representing a mathematical model of a single phase flow in a porous/free flow media has the following form

$$\nu\mathbf{K}^{-1}\mathbf{v} + \nabla p - \nu^*\Delta\mathbf{v} = \mathbf{f} \quad \text{in } \Omega, \tag{7}$$

$$\nabla\cdot\mathbf{v} = 0 \quad \text{in } \Omega, \tag{8}$$

$$\mathbf{v} = \mathbf{w} \quad \text{on } \partial\Omega_D, \qquad \frac{\partial\mathbf{v}}{\partial n} - \mathbf{n}p = \mathbf{s} \quad \text{on } \partial\Omega_N, \tag{9}$$

where $\mathbf{v}$ is the vector of velocity, P is the pressure, $\mathbf{f}$ is the vector of external force, $\mathbf{n}$ is the outward-pointing normal to the boundary, ν^* is the effective viscosity and ν - the physical viscosity - is a uniform constant in the entire domain Ω. $\mathbf{K}$ is a symmetric permeability tensor, which in Ω_p is equal to the Darcy permeability of the porous media. Note that with the choice $\nu^* = 0$ in the vugular region Ω_p, the equation (7) reduces to the problem of Darcy's law. On the other hand by choosing $k_{ij} \to \infty$ (or very large) in fluid domain Ω_f, the equation (7) reduces to the problem of Stokes flow (here ν^* is taken equal to the physical fluid viscosity ν). Thus, the Stokes or Darcy's equations can be obtained by suitable choices of the parameters ν^* and $\mathbf{K}$ by defining them in vugular and rock matrix regions, respectively.

In the porous region ($\mathbf{K} < \infty$) it is known [14], that for moderately small permeabilities and pore fractions, the diffusive term $\nu^*\Delta\mathbf{v}$, where ν^* takes values close to the fluid viscosity ν, intoroduces only a small perturbation of the velocity and pressure fields in comparison with a pure Darcy law with $\nu^* = 0$. In [14] it is shown that Stokes-Brinkman equation with the choice $\nu^* = \nu$ in the porous region is very close to the solution of coupled Stokes and Darcy's equations.

The advantage of Stokes-Brinkman model is usage of uniform equations for porous and free flow domains. Boundary conditions between these two domains are represented by $\mathbf{K}$. This approach makes it possible to model heterogeneous material. Moreover, by a numerical point of view, it is easier to solve a monolithic system such as Stokes-Brinkman, in contrast to a coupled Darcy-Stokes system which requires an additional iterative scheme. Also, near the interface, Stokes-Brinkman equations allow us to avoid the typical grid refinement issues necessary for solving the interface between Darcy and Stokes region. On the other hand usage of Taylor-Hood elements for the whole domain requires big load of memory.

3. Weak formulation of Stokes-Brinkman equations

In what follows we denote $G = \mathbf{K}^{-1}$ and assume G is symmetric.

For the weak formulation we denote

$$\mathbf{H}^1_E := \{\mathbf{u} \in H^1(\Omega)^2 | \mathbf{u} = \mathbf{w} \text{ na } \partial\Omega_D\}, \tag{10}$$

$$\mathbf{H}^1_{E_0} := \{\mathbf{v} \in H^1(\Omega)^2 | \mathbf{v} = \mathbf{0} \text{ na } \partial\Omega_D\}. \tag{11}$$

Now the weak form of the Stokes-Brinkman problem reads:
Find $\boldsymbol{v} \in \mathbf{H}^1_{E_0}$ and $p \in L^2_0(\Omega)$ such that

$$\nu^* \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v^*} + \nu \int_\Omega \boldsymbol{v^T} G \boldsymbol{v^*} - \int_\Omega p \nabla \cdot \boldsymbol{v^*} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v^*} \qquad \forall \boldsymbol{v^*} \in \mathbf{H}^1_{E_0}, \tag{12}$$

$$\int_\Omega q \nabla \cdot \boldsymbol{v} = 0 \qquad \forall q \in L^2_0(\Omega). \tag{13}$$

Here $L^2_0(\Omega)$ is the space of L^2 functions having mean value zero.

On the space $V = \Big(H^1_0(\Omega)^2 \times L^2_0(\Omega) \Big)$ we define the bilinear form

$$\mathcal{A}\Big(\{\boldsymbol{v}, p\}, \{\boldsymbol{v}^*, p^*\} \Big) = \nu^* \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{v}^* + \nu \int_\Omega \boldsymbol{v^T} G \boldsymbol{v}^* - \int_\Omega p \nabla \cdot \boldsymbol{v}^* - \int_\Omega p^* \nabla \cdot \boldsymbol{v} \tag{14}$$

where $(.,.)_0$ means the scalar product in L^2.

In what follows we assume $\mathbf{w} = 0$, i. e. only zero Dirichlet condition on the whole boundary $\partial\Omega$. Problem (12), (13) can be written as follows: find $\{\boldsymbol{v}, p\} \in V$, such that

$$\mathcal{A}\Big(\{\boldsymbol{v}, p\}, \{\boldsymbol{v}^*, p^*\} \Big) = (\boldsymbol{f}, \boldsymbol{v}^*)_0, \quad \forall \{\boldsymbol{v}^*, p^*\} \in V. \tag{15}$$

4. Finite element approximation

We suppose Ω to be a polygon, for simplicity. Let $\mathcal{T}_h$ be regular [11] triangulations of Ω. Let X^h, M^h be the finite element spaces of Taylor-Hood elements (cf. e.g. F. Brezzi, M. Fortin [4]), i.e.

$$X^h = \{\boldsymbol{v} \in H^1_0(\Omega)^2, \boldsymbol{v}/_T \in P^2(T)^2, T \in \mathcal{T}_h\},$$
$$M^h = \{p \in L^2_0(\Omega), p/_T \in P^1(T), T \in \mathcal{T}_h\}.$$

These satisfy the Babuška-Brezzi condition [4]. The finite element approximation of the Stokes-Brinkman problem consists in finding $\{\boldsymbol{v}_h, p_h\} \in\ X^h \times M^h$ such that

$$\mathcal{A}\Big(\{\boldsymbol{v}_h, p_h\}, \{\boldsymbol{v}_h^*, p_h^*\}\Big) = (\boldsymbol{f}, \boldsymbol{v}_h^*)_0\ ,\ \ \forall\{\boldsymbol{v}_h^*, p_h^*\} \in X^h \times M^h. \tag{16}$$

5. A posteriori error estimate

We follow the idea of K. Eriksson et al. [10] who proved the a posteriori error estimate for the Poisson equation. We define the residual components by the relations

$$\boldsymbol{R_1}\{\boldsymbol{v}_h, p_h\} = \boldsymbol{f} + \nu^*\Delta\boldsymbol{v}_h - \nu G\boldsymbol{v}_h - \nabla p_h, \quad R_2\{\boldsymbol{v}_h, p_h\} = \text{div}\ \boldsymbol{v}_h. \tag{17}$$

Next we study the properties of the errors

$$\boldsymbol{e}_v = \boldsymbol{v} - \boldsymbol{v}_h\ ,\ e_p = p - p_h\ ,$$

where $\{\boldsymbol{v}, p\}$ is the exact solution of (15), $\{\boldsymbol{v}_h, p_h\}$ is the approximate solution defined in (16). The V norm of $\{\boldsymbol{e}_v, e_p\}$ is

$$\|\{\boldsymbol{e}_v, e_p\}\|_V^2 = (\boldsymbol{e}_v, \boldsymbol{e}_v)_1 + (e_p, e_p)_0 = \int_\Omega (\boldsymbol{e}_v \cdot \boldsymbol{e}_v + \nabla\boldsymbol{e}_v : \nabla\boldsymbol{e}_v) + \int_\Omega e_p e_p.$$

By the Poincaré-Friedrichs inequality, cf. [9], as $\boldsymbol{e}_v \in H_0^1(\Omega)^2$

$$(\boldsymbol{e}_v, \boldsymbol{e}_v)_1 \le C_P \int_\Omega \nabla\boldsymbol{e}_v : \nabla\boldsymbol{e}_v \tag{18}$$

5.1. Dual Stokes-Brinkman problem

To study the above norms we introduce the dual Brinkman-Stokes problem by

$$\begin{aligned} -\nu^*\Delta\boldsymbol{\varphi_v} + \nu G\boldsymbol{\varphi_v} + \nabla\varphi_p &= -\Delta\boldsymbol{e}_v \quad \text{in } \Omega,\ \text{here } \Delta\boldsymbol{e}_v \in H^{-1}(\Omega) \\ -\text{div}\ \boldsymbol{\varphi_v} &= e_p \quad \text{in } \Omega, \\ \boldsymbol{\varphi_v} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{19}$$

which in a weak form is: find $\boldsymbol{\varphi_v} \in H^1(\Omega)^2$ and $\varphi_p \in L_0^2(\Omega)$ such that

$$\begin{aligned} (\nu^*\nabla\boldsymbol{\varphi_v}, \nabla\boldsymbol{v}^*)_0 + \nu((G\boldsymbol{\varphi_v}), \boldsymbol{v}^*) - (\varphi_p, \nabla\boldsymbol{v}^*)_0 &= (\nabla\boldsymbol{e}_v, \nabla\boldsymbol{v}^*)_0,\ \forall\boldsymbol{v}^* \in H_0^1(\Omega)^2, \\ (-\text{div}\ \boldsymbol{\varphi_v}, p^*)_0 &= (e_p, p^*)_0,\ \forall p^* \in L_0^2(\Omega), \end{aligned} \tag{20}$$

or, using the notation (14)

$$\mathcal{A}(\{\boldsymbol{\varphi_v}, \varphi_p\}, \{\boldsymbol{v}^*, p^*\}) = (\nabla\boldsymbol{e}_v, \nabla\boldsymbol{v}^*)_0 + (e_p, p^*)_0\ ,\ \forall\{\boldsymbol{v}^*, p^*\} \in V\ . \tag{21}$$

By (18) and (20) where we put $\boldsymbol{v}^* = \boldsymbol{e}_v,\ p^* = e_p$, we get

$$\begin{aligned} \frac{1}{C_P}(\boldsymbol{e}_v, \boldsymbol{e_v})_1 \le\ (\nabla\boldsymbol{e_v}, \nabla\boldsymbol{e_v})_0 &= \nu^*(\nabla\boldsymbol{\varphi_v}, \nabla\boldsymbol{e_v})_0 + \nu((G\boldsymbol{\varphi_v})\boldsymbol{,}\boldsymbol{e_v}) - (\varphi_p\nabla, \boldsymbol{e_v})_0 \\ &= \nu^*(\nabla\boldsymbol{\varphi_v}, \nabla\boldsymbol{v})_0 + \nu((G\boldsymbol{\varphi_v})\boldsymbol{v}) - (\varphi_p\nabla, \boldsymbol{v})_0 - \nu^*(\nabla\boldsymbol{\varphi_v}, \nabla\boldsymbol{v}_h)_0 \\ &\quad - \nu((G\boldsymbol{\varphi_v})\boldsymbol{v_h}) + (\varphi_p\nabla, \boldsymbol{v_h})_0, \end{aligned} \tag{22}$$

$$(e_p, e_p)_0 = (e_p, -\text{div}\ \boldsymbol{\varphi_v})_0 = -(p\nabla, \boldsymbol{\varphi_v})_0 + (p_h\nabla, \boldsymbol{\varphi_v})_0. \tag{23}$$

5.2. Estimation of the error by means of the residual and solution of the dual problem

Combining (22), (23), and (19) we get (as $C_P \geq 1$)

$$
\begin{aligned}
&\frac{1}{C_P}\Big\{(\boldsymbol{e}_v, \boldsymbol{e}_v)_1 + (e_p, e_p)_0\Big\} \\
&\leq \nu^*(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi_v})_0 + \nu((G\boldsymbol{v}\varphi_v)) - (p, \nabla \boldsymbol{\varphi_v})_0 - (\nabla \boldsymbol{v}, \varphi_p)_0 \\
&\quad + \sum_{K\in\mathcal{T}_h} \{-\nu^*(\nabla \boldsymbol{\varphi_v}, \nabla \boldsymbol{v}_h)_{0,K} - \nu((G\boldsymbol{v_h}\varphi_v)) + (p_h, \nabla\boldsymbol{\varphi_v})_{0,K} + (\varphi_p, \nabla \boldsymbol{v}_h)_{0,K}\} \\
&= (\boldsymbol{f}, \boldsymbol{\varphi_v})_0 + \sum_{K\in\mathcal{T}_h} \Big\{(\nu^*\Delta\boldsymbol{v}_h, \boldsymbol{\varphi_v})_{0,K} - \int_{\partial K} \nu^* \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \boldsymbol{\varphi_v} ds\Big\} - \nu((G\boldsymbol{v_h}\varphi_v)) \qquad (24) \\
&\quad - \sum_{K\in\mathcal{T}_h} \Big\{(\nabla p_h, \boldsymbol{\varphi_v})_{0,K} + \int_{\partial K} p_h \boldsymbol{\varphi_v}\cdot \boldsymbol{n} ds \ + (\text{div } \boldsymbol{v}_h, \varphi_p)_{0,K}\Big\} \\
&= \sum_{K\in\mathcal{T}_h} (\boldsymbol{f} + \nu^*\Delta\boldsymbol{v}_h - \nu((G\boldsymbol{v_h}\varphi_v)) - \nabla p_h, \boldsymbol{\varphi_v})_{0,K} + \sum_{K\in\mathcal{T}_h} (\text{div } \boldsymbol{v}_h, \varphi_p)_{0,K} \\
&\quad - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nu^* \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \boldsymbol{\varphi_v} ds + \sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h \boldsymbol{\varphi_v}\cdot\boldsymbol{n} ds
\end{aligned}
$$

In view of (16) we also have

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} (\boldsymbol{f} + \nu^*\Delta\boldsymbol{v}_h - \nu G\boldsymbol{v}_h - \nabla p_h, \boldsymbol{v}_h^*)_{0,K} + \ (\text{div } \boldsymbol{v}_h, p_h^*)_0 \\
&= (\boldsymbol{f}, \boldsymbol{v}_h^*)_0 + \sum_{K\in\mathcal{T}_h} \Big\{(-\nu^*\nabla\boldsymbol{v}_h, \nabla\boldsymbol{v}_h^*)_{0,K} - \nu(G\boldsymbol{v}_h, \boldsymbol{v}_h^*) + \int_{\partial K} \nu^* \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \boldsymbol{v}_h^* ds\Big\} \\
&\quad + (\nabla p_h, \boldsymbol{v}_h^*)_0 - \sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h \boldsymbol{v}_h^* \cdot \boldsymbol{n} ds + (\text{div } \boldsymbol{v}_h, p_h^*)_0 \qquad (25) \\
&= 0 + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nu \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \boldsymbol{v}_h^* ds - \sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h \boldsymbol{v}_h^* \cdot \boldsymbol{n} ds, \ \forall \{\boldsymbol{v}_h^*, p_h^*\} \in X^h \times M^h.
\end{aligned}
$$

This implies, taking $\boldsymbol{v}_h^* = \pi_h \boldsymbol{\varphi_v}$, $p_h^* = \pi_h \varphi_p$, the Clement interpolants, (cf. e.g. [8], p. 146) that

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} (\boldsymbol{f} + \nu^*\Delta\boldsymbol{v}_h - \nu G\boldsymbol{v}_h - \nabla p_h, \pi_h\boldsymbol{\varphi_v}) + \ (\text{div } \boldsymbol{v}_h, \pi_h\varphi_p)_0 \\
&\qquad - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nu^* \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} \pi_h \boldsymbol{\varphi_v} ds - \sum_{K\in\mathcal{T}_h} \int_{\partial K} p_h \pi_h \boldsymbol{\varphi_v} \cdot \boldsymbol{n} ds = 0 \qquad (26)
\end{aligned}
$$

Now subtracting zero in (26) from (24) we get

$$
\begin{aligned}
&\frac{1}{C_P}\Big\{(\boldsymbol{e}_v,\boldsymbol{e}_v)_1+(e_p,e_p)_0\Big\}\\
&\le \sum_{K\in\mathcal{T}_h}(\boldsymbol{f}+\nu^*\Delta\boldsymbol{v}_h-\nu G\boldsymbol{v}_h-\nabla p_h,\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})_{0,K}+\ (\text{div } \boldsymbol{v}_h,\varphi_p-\pi_h\varphi_p)_0\\
&\quad-\sum_{K\in\mathcal{T}_h}\int_{\partial K}\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}(\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})ds+\sum_{K\in\mathcal{T}_h}\int_{\partial K}p_h(\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})\cdot\boldsymbol{n}ds \qquad (27)\\
&=\sum_{K\in\mathcal{T}_h}(\boldsymbol{f}+\nu^*\Delta\boldsymbol{v}_h-\nu G\boldsymbol{v}_h-\nabla p_h,\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})_{0,K}+\ (\text{div } \boldsymbol{v}_h,\varphi_p-\pi_h\varphi_p)_0\\
&\quad-\sum_{K\in\mathcal{T}_h}\sum_{l\in\partial K}\int_l\left(\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right)(\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v})ds,
\end{aligned}
$$

where we denoted

$$
\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l=\left(\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right)\Big/_{l+}-\left(\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right)\Big/_{l_-}
$$

the jump along the common side l of two adjacent triangles. Then, using in turn the Schwarz inequality, the interpolation properties of $X^h,\ M^h$ (cf. e.g. [4]), and the estimate of the solution of the dual problem (19) (cf. [4]), we get the inequalities

$$
\begin{aligned}
&\|\boldsymbol{e}_v\|_1^2+\|e_p\|_0^2\\
&\le C_P\sum_{K\in\mathcal{T}_h}\Big\{\|\boldsymbol{R_1}\{\boldsymbol{v}_h,p_h\}\|_{0,K}\|\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v}\|_{0,K}+\|R_2\{\boldsymbol{v}_h,p_h\}\|_{0,K}\|\varphi_p-\pi_h\varphi_p\|_{0,K}\Big\}\\
&\quad+C_P\sum_{K\in\mathcal{T}_h}\sum_{l\in\partial K}\left\|\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right\|_{0,l}\|\boldsymbol{\varphi_v}-\pi_h\boldsymbol{\varphi_v}\|_{0,l} \qquad (28)\\
&\le C_PC_I\sum_{K\in\mathcal{T}_h}\Big\{h_K\ \|\boldsymbol{R_1}\{\boldsymbol{v}_h,p_h\}\|_{0,K}\ \|\boldsymbol{\varphi_v}\|_1+\|R_2\{\boldsymbol{v}_h,p_h\}\|_{0,K}\ \|\varphi_p\|_0\Big\}\\
&\quad+C_PC_I\sum_{K\in\mathcal{T}_h}(h_K)^{\frac{1}{2}}\sum_{l\in\partial K}\left\|\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right\|_{0,l}\|\boldsymbol{\varphi_v}\|_1\\
&\le C_PC_IC_R\sum_{K\in\mathcal{T}_h}\left\{h_K\ \|\boldsymbol{R}_1\{\boldsymbol{v}_h,p_h\}\|_{0,K}\ +\|R_2\{\boldsymbol{v}_h,p_h\}\|_{0,K}\right.\\
&\quad\left.+\sum_{l\in\partial K}(h_K)^{\frac{1}{2}}\ \left\|\frac{1}{2}\left[\!\!\left[\nu\frac{\partial\boldsymbol{v}_h}{\partial\boldsymbol{n}}-p_h\boldsymbol{n}\right]\!\!\right]_l\right\|_{0,l}\right\}\cdot\big\{\|\Delta\boldsymbol{e}_v\|_{-1}+\|e_p\|_0\big\}.
\end{aligned}
$$

Using then the relations

$$
\begin{aligned}
\|\Delta\boldsymbol{e}_v\|_{-1}\equiv&\sup_{\boldsymbol{v}^*\in H_0^1,\boldsymbol{v}^*\neq0}\frac{|(\Delta\boldsymbol{e}_v,\boldsymbol{v}^*)_0|}{\|\boldsymbol{v}^*\|_1}=\sup_{\boldsymbol{v}^*\in H_0^1,\boldsymbol{v}^*\neq0}\frac{|(\nabla\boldsymbol{e}_v,\nabla\boldsymbol{v}^*)_0|}{\|\boldsymbol{v}^*\|_1}\\
&\le\sup_{\boldsymbol{v}^*\in H_0^1,\boldsymbol{v}^*\neq0}\frac{\|\nabla\boldsymbol{e}_v\|_0\ \|\nabla\boldsymbol{v}^*\|_0}{\|\boldsymbol{v}^*\|_1}\le\|\nabla\boldsymbol{e}_v\|_0\le\|\boldsymbol{e}_v\|_1
\end{aligned}
$$

we get, by (28)

$$\{\|\boldsymbol{e}_v\|_1 + \|e_p\|_0\}^2 \le 2\{\|\boldsymbol{e}_v\|_1^2 + \|e_p\|_0^2\} \le 2C_P C_I C_R \sum_{K\in\mathcal{T}_h} \Big\{ h_K \|\boldsymbol{R_1}\{\boldsymbol{v}_h, p_h\}\|_{0,K}$$

$$+ \|R_2\{\boldsymbol{v}_h, p_h\}\|_{0,K} + h_K^{\frac{1}{2}} \sum_{l\in\partial K} \left\| \frac{1}{2} \left[\!\!\left[\nu \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} - p_h \boldsymbol{n} \right]\!\!\right]_l \right\|_{0,l} \Big\} \cdot \{\|\boldsymbol{e}_v\|_1 + \|e_p\|_0\}. \quad (29)$$

Upon cancelling $\{\|\boldsymbol{e}_v\|_1 + \|e_p\|_0\}$ in (29) we finally get the following theorem:

Theorem 1. *Let Ω be a polygon in R^2. Let $\mathcal{T}_h$ be a family of regular triangulations of Ω. Let $\{\boldsymbol{v_h}, p_h\}$ be the Taylor-Hood approximation of the solution $\{\boldsymbol{v}, p\}$ of the Stokes-Brinkman problem. Then the error $\{\boldsymbol{e}_v, e_p\}$ satisfies the following a posteriori estimate*

$$\|\boldsymbol{e}_v\|_1 + \|e_p\|_0 \le 2C_P C_I C_R \sum_{K\in\mathcal{T}_h} \Big\{ h_K \|\boldsymbol{R}_1\{\boldsymbol{v}_h, p_h\}\|_{0,K} + \|R_2\{\boldsymbol{v}_h, p_h\}\|_{0,K}$$

$$+ h_K^{\frac{1}{2}} \sum_{l\in\partial K} \left\| \frac{1}{2} \left[\!\!\left[\nu \frac{\partial \boldsymbol{v}_h}{\partial \boldsymbol{n}} - p_h \boldsymbol{n} \right]\!\!\right]_l \right\|_{0,l} \Big\}. \quad (30)$$

where C_P, C_I, C_R are positive constants, residuals $\boldsymbol{R}_1$ and R_2 are defined in (17) .

Conclusions

The estimate in Theorem 1 applies to more general class of elements. Of course, for Taylor-Hood elements with continuous pressure the jumps of p_h along the common sides disappear.

Let us note that for convex domains stronger regularity applies to the Stokes problem, cf. [13], and better a posteriori error estimate may be expected.

For nonconvex domains with corners we do not obtain so strong regularity as in [13], cf. e.g. [6], but still the a posteriori error estimate should be better than in (30), as it was for the Poisson equation in (2).

Acknowledgments

This work was supported by the IT4Innovations Centre of Excellence project, reg. no. CZ.1.05/1.1.00/02.0070, and by the grant Kontakt II number LH11004.

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