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STABILITY ANALYSIS OF THE SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR NONSTATIONARY NONLINEAR CONVECTION-DIFFUSION PROBLEMS

Monika Balázsová, Miloslav Feistauer, Martin Hadrava, Adam Kosík

Faculty of Mathematics and Physics, Charles University in Prague
Sokolovská 83, 186 75 Praha 8, Czech Republic
b.moncsi@gmail.com, feist@karlin.mff.cuni.cz, martin@hadrava.eu, adam.kosik@atlas.cz

Abstract

This paper is concerned with the stability analysis of the space-time discontinuous Galerkin method for the solution of nonstationary, nonlinear, convection-diffusion problems. In the formulation of the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the discretization of diffusion terms and interior and boundary penalty. Then error estimates are briefly characterized. The main attention is paid to the investigation of unconditional stability of the method. Theoretical results are demonstrated by a numerical example.

1. Introduction

One of efficient and robust techniques for the numerical solution of partial differential equations is the discontinuous Galerkin (DG) method. It is based on piecewise polynomial approximations of the sought exact solution over a partition of the computational domain without any requirement of the continuity on interfaces between neighbouring elements. Most of works on the DG method are concerned with space discretization. The numerical simulation of strongly nonstationary transient problems requires the application of numerical schemes of high order of accuracy both in space and in time. For some applications, the standard Euler schemes or θ-schemes are not sufficiently accurate in time. In computational fluid dynamics, Runge-Kutta methods are very popular ([3]). However they are conditionally stable. It appears suitable to use the discontinuous Galerkin discretization with respect to space as well as time for the construction of numerical schemes with high accuracy in space and time for the solution of nonlinear nonstationary problems. The discontinuous Galerkin time discretization was introduced and analyzed e.g. in [4] for the solution of ordinary differential equations. In [6] it was combined with conforming finite elements and applied to parabolic problems. See also the monograph [7].

The papers [2] and [5] are concerned with theoretical analysis of error estimates for the space-time DG method applied to nonlinear nonstationary convection-diffusion problems. However, in a general case the results were obtained under a CFL-like stability condition applied in the vicinity of the boundary. There is a natural question, if this condition is really necessary for guaranteeing the stability. This was the motivation for the investigation of the stability of the space-time DG method. In this paper we present a brief description of the obtained results. The analysis is rather complicated and technical and detailed proofs will be published in [1].

2. Formulation of the continuous problem

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be a bounded domain and $T > 0$. We consider the initial-boundary value problem to find $u : Q_T = \Omega \times (0, T) \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } Q_T, \tag{1}$$

$$u\,|_{\partial\Omega\times(0,T)} = u_D, \tag{2}$$

$$u(x, 0) = u^0(x), \quad x \in \Omega. \tag{3}$$

We assume, that g, u_D, u^0, f_s are given functions and $f_s \in C^1(\mathbb{R}), |f_s'| \le C$, $f_s(0) = 0$, $s = 1, \ldots, d$. Moreover, let the function $\beta : \mathbb{R} \to [\beta_0, \beta_1]$, $0 < \beta_0 < \beta_1 < \infty$, be Lipschitz continuous: $|\beta(u_1) - \beta(u_2)| \le L_\beta |u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$.

3. Space-time discretization

In the time interval $[0, T]$ we introduce a partition formed by time instants $0 = t_0 < t_1 < \ldots < t_M = T$, and denote $I_m = (t_{m-1}, t_m)$, $\tau_m = t_m - t_{m-1}$, $m = 1 \ldots, M$. We set $\tau = \max_{m=1,\ldots,M} \tau_m$. For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote one-sided limits at t_m as $\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t\to t_m\pm} \varphi(t)$ and the jump as $\{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-)$.

For each I_m we consider a system of partitions $\{\mathcal{T}_{h,m}\}_{h\in(0,h_0)}$ with $h_0 > 0$ of $\overline{\Omega}$ into a finite number of closed triangles with mutually disjoint interiors (partitions are in general different for different m). We set $h_K = \operatorname{diam}(K)$ for $K \in \mathcal{T}_{h,m}$, $h_m = \max_{K\in\mathcal{T}_{h,m}} h_K$ and $h = \max_{m=1,\ldots,M} h_m$.

By $\mathcal{F}_{h,m}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,m}$. It consists of the set of all inner faces $\mathcal{F}_{h,m}^I$ and the set of all boundary faces $\mathcal{F}_{h,m}^B$. Each $\Gamma \in \mathcal{F}_{h,m}$ will be associated with a unit normal vector $\mathbf{n}_\Gamma$. By $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)} \in \mathcal{T}_{h,m}$ we denote the elements adjacent to the face $\Gamma \in \mathcal{F}_{h,m}$. We shall use the convention, that $\mathbf{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$. Over a triangulation $\mathcal{T}_{h,m}$, for each positive integer k, we define the broken Sobolev space $H^k(\Omega, \mathcal{T}_{h,m}) = \{v; v|_K \in H^k(K)\, \forall K \in \mathcal{T}_{h,m}\}$.

If $v \in H^1(\Omega, \mathcal{T}_{h,m})$ and $\Gamma \in \mathcal{F}_{h,m}$, then $v|_\Gamma^{(L)}, v|_\Gamma^{(R)}$ will denote the traces of v on Γ from the side of the elements $K_\Gamma^{(L)}, K_\Gamma^{(R)}$ adjacent to Γ. For $\Gamma \in \mathcal{F}_{h,m}^I$ we set

$$\langle v\rangle_\Gamma \;=\; \frac{1}{2}\left(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\right), \quad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}.$$

We use the notation

$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2} \quad \text{for } \Gamma \in \mathcal{F}_{h,m}^I, \quad h(\Gamma) = h_{K_\Gamma^{(L)}} \quad \text{for } \Gamma \in \mathcal{F}_{h,m}^B.$$

If $u, \varphi \in H^2(\Omega, \mathcal{T}_{h,m})$ and $c_W > 0$, we introduce the forms

$$\begin{aligned}
&a_{h,m}(u,\varphi) = \sum_{K\in\mathcal{T}_{h,m}} \int_K \beta(u)\nabla u \cdot \nabla\varphi \, dx \\
&- \sum_{\Gamma\in\mathcal{F}_{h,m}^I} \int_\Gamma \left(\langle \beta(u)\nabla u\rangle \cdot \mathbf{n}_\Gamma \, [\varphi] + \theta \, \langle \beta(u)\nabla\varphi\rangle \cdot \mathbf{n}_\Gamma \, [u] \right) dS \\
&- \sum_{\Gamma\in\mathcal{F}_{h,m}^B} \int_\Gamma \left(\beta(u)\nabla u \cdot \mathbf{n}_\Gamma \, \varphi + \theta\beta(u)\nabla\varphi \cdot \mathbf{n}_\Gamma \, u - \theta\beta(u)\nabla\varphi\cdot \mathbf{n}_\Gamma \, u_D \right) dS, \\
&J_{h,m}(u,\varphi) = c_W \sum_{\Gamma\in\mathcal{F}_{h,m}^I} h(\Gamma)^{-1} \int_\Gamma [u]\,[\varphi] dS + c_W \sum_{\Gamma\in\mathcal{F}_{h,m}^B} h(\Gamma)^{-1} \int_\Gamma u\,\varphi\, dS, \\
&b_{h,m}(u,\varphi) = -\sum_{K\in\mathcal{T}_{h,m}} \int_K \sum_{s=1}^d f_s(u) \frac{\partial\varphi}{\partial x_s} \, dx \\
&+ \sum_{\Gamma\in\mathcal{F}_{h,m}^I} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \mathbf{n}_\Gamma) \, [\varphi] \, dS + \sum_{\Gamma\in\mathcal{F}_{h,m}^B} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \mathbf{n}_\Gamma) \, \varphi \, dS, \qquad (4) \\
&l_{h,m}(\varphi) = \sum_{K\in\mathcal{T}_{h,m}} \int_K g\varphi \, dx + \beta_0 \, c_W \sum_{\Gamma\in\mathcal{F}_{h,m}^B} h(\Gamma)^{-1} \int_\Gamma u_D \, \varphi \, dS.
\end{aligned}$$

Let us note that in integrals over faces we omit the subscript Γ. We consider $\theta = 1$, $\theta = 0$ and $\theta = -1$ and get the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively. In (4), H is a numerical flux, which is Lipschitz-continuous, consistent and conservative.

Let $p, q \geq 1$ be integers. For each $m = 1, \ldots, M$ we define the spaces

$$\begin{aligned}
S_{h,m}^p &= \{\varphi \in L^2(\Omega);\, \varphi|_K \in P^p(K) \quad \forall K \in \mathcal{T}_{h,m}\}, \\
S_{h,\tau}^{p,q} &= \{\varphi \in L^2(Q_T);\, \varphi|_{I_m} = \sum_{i=0}^q t^i \varphi_i \quad \text{with } \varphi_i \in S_{h,m}^p,\, m = 1, \ldots, M\}.
\end{aligned}$$

By $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the scalar product and the norm in $L^2(\Omega)$. The symbol $|\cdot|_{H^1(K)}$ denotes the seminorm in the space $H^1(K)$. The space $H^1(\Omega, \mathcal{T}_{h,m})$ will be equipped with the norm

$$\|\varphi\|_{DG,m} = \left(\sum_{K\in\mathcal{T}_{h,m}} |\varphi|_{H^1(K)}^2 + J_{h,m}(\varphi,\varphi) \right)^{1/2}.$$

Definition. *We say that U is an approximate solution of (1)-(3), if $U \in S_{h,\tau}^{p,q}$ and*

$$\int_{I_m} \left(\left(\frac{\partial U}{\partial t}, \varphi \right) + a_{h,m}(u,\varphi) + \beta_0\, J_{h,m}(u,\varphi) + b_{h,m}(U,\varphi) \right) dt + (\{U\}_{m-1}, \varphi_{m-1}^{+})$$
$$= \int_{I_m} l_{h,m}(\varphi)\, dt, \quad \forall \varphi \in S_{h,\tau}^{p,q}, \quad m = 1, \ldots, M, \tag{5}$$
$$U_0^- := L^2(\Omega)\text{-projection of } u^0 \text{ on } S_{h,1}^p.$$

4. Summary of results on error estimates

The papers [5] and [2] were devoted to the analysis of the STDG method applied to problem in the case of linear diffusion and nonlinear diffusion, respectively. Under the assumptions on the regularity of the exact solution

$$u \in H^{q+1}(0,T; H^1(\Omega)) \cap C([0,T]; H^{p+1}(\Omega)),$$
$$\|\nabla u\|_{L^\infty(\Omega)} \le c_R \quad \text{for a. e.} \quad t \in (0,T),$$

using approximation properties of the $S_{h,m}^p$- and $S_{h,\tau}^{p,q}$- interpolation operators, assumptions on the properties of the meshes, namely the shape regularity and local quasiuniformity, and the condition $\tau_m \ge c\, h_m^2$, $m = 1, \ldots, M$, error estimates in terms of h and τ were proven.

Theorem 1. *There exists a constant $c > 0$ such that*

$$\|e_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|e\|_{DG,j}^2 dt \le c \left(h^{2p} |u|_{C([0,T];H^{p+1}(\Omega))}^2 + \tau^{2q+\alpha} |u|_{H^{q+1}(0,T;H^1(\Omega))}^2 \right),$$
$$m = 1, \ldots, M, \quad h \in (0, h_0). \tag{6}$$

Here $\alpha = 2$, if u_D is a polynomial of degree $\le q$ in t. Otherwise, under the assumption that the condition

$$\tau_m \le C_{CFL} h_{K_\Gamma^{(L)}} \tag{7}$$

with a constant C_{CFL} independent of h_K, τ_m and M is satisfied for all elements K adjacent to the boundary $\partial\Omega$, estimate (6) *holds with $\alpha = 0$.*

5. Analysis of stability

There is a natural question, if condition (7) reminding the CFL stability condition is necessary for the derivation of the error estimate (6), or it is also important for guaranteeing the stability of the STDG method (5). In what follows, we shall show that method (5) is unconditionally stable. This means that our goal is to prove that the approximate solution U of problem (1)-(3) is bounded by the L^2-norm of g, u^0 and by the $\|\cdot\|_{DGB,m}$-norm of u_D, which is defined as

$$\|u_D\|_{DGB,m} := \left(J^B_{h,m}(u_D,u_D)\right)^{1/2} = \left(c_W \sum_{\Gamma\in\mathcal{F}^B_{h,m}} h^{-1}(\Gamma)\int_\Gamma |u_D|^2 dS\right)^{1/2}.$$

The stability analysis starts by setting $\varphi := U$ in the basic relation (5). We get

$$\begin{aligned}&\int_{I_m}\left(\left(\frac{\partial U}{\partial t},U\right) + a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U) + b_{h,m}(U,U)\right)dt \\ &+ (\{U\}_{m-1},\varphi^+_{m-1}) = \int_{I_m} l_{h,m}(U)\,dt.\end{aligned} \tag{8}$$

After some manipulations we can derive the following identity

$$\int_{I_m}\left(\frac{\partial U}{\partial t},U\right)dt + (\{U\}_{m-1},U^+_{m-1}) = \frac{1}{2}\left(\|U^-_m\|^2 - \|U^-_{m-1}\|^2 + \|\{U\}_{m-1}\|^2\right). \tag{9}$$

For a sufficiently large constant c_W, whose lower bound is determined by β_0 and the constants from the multiplicative trace inequality, inverse inequality, local quasiuniformity of the meshes, we can prove the coercivity of the diffusion term:

$$\int_{I_m}\left(a_{h,m}(U,U) + \beta_0\, J_{h,m}(U,U)\right)dt \ge \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,m}\,dt - \frac{\beta_0}{2}\int_{I_m}\|u_D\|^2_{DGB,m}\,dt. \tag{10}$$

Furthermore, if $k_1, k_2 > 0$ then there exists a constant $c_b = c_b(k_1)$ such that the following inequalities for the convection term and for the right-hand side form hold:

$$\int_{I_m}|b_{h,m}(U,U)|\,dt \le \frac{\beta_0}{k_1}\int_{I_m}\|U\|^2_{DG,m}\,dt + c_b\int_{I_m}\|U\|^2\,dt. \tag{11}$$

$$\begin{aligned}\int_{I_m}|l_{h,m}(U)|\,dt \le{}& \frac{1}{2}\int_{I_m}\left(\|g\|^2 + \|U\|^2\right)dt + \beta_0 k_2\int_{I_m}\|u_D\|^2_{DGB,m}\,dt \\ &+ \frac{\beta_0}{k_2}\int_{I_m}\|U\|^2_{DG,m}\,dt.\end{aligned} \tag{12}$$

If we substitute estimates (9)-(12) into our basic identity (8) and set $k_1 = k_2 = 8$, $c = \max\{2c_b + 1, 17\beta_0\}$, after some manipulation we get

$$\begin{aligned}&\|U^-_m\|^2 - \|U^-_{m-1}\|^2 + \frac{\beta_0}{2}\int_{I_m}\|U\|^2_{DG,m}dt \\ &\le c\left(\int_{I_m}\|g\|^2\,dt + \int_{I_m}\|U\|^2\,dt + \int_{I_m}\|u_D\|^2_{DGB,m}dt\right).\end{aligned} \tag{13}$$

Now our further task is to estimate the expression $\int_{I_m} \|U\|^2\, dt$ in terms of g and u_D. The main tool is the concept of the discrete characteristic function $\zeta_y \in S_{h,\tau}^{p,q}$ to U for $y \in I_m = (t_{m-1}, t_m)$ defined by

$$\int_{I_m} (\zeta_y, \varphi)\, dt = \int_{t_{m-1}}^{y} (U, \varphi)\, dt \quad \forall \varphi \in S_{h,\tau}^{p,q-1}, \quad \zeta_y(t_{m-1}^+) = U(t_{m-1}^+).$$

The operator assigning ζ_y to U is continuous, i.e, there exists $c_q > 0$, depending on q only, such that

$$\int_{I_m} \|\zeta_y\|_{DG,m}^2\, dt \le c_q \int_{I_m} \|U\|_{DG,m}^2\, dt, \qquad \int_{I_m} \|\zeta_y\|^2\, dt \le c_q \int_{I_m} \|U\|^2\, dt.$$

Then, after a technical and complicated analysis, it is possible to prove this important estimate: there exists a constant $c > 0$ such that

$$\int_{I_m} \|U\|^2\, dt \le c\, \tau_m \left(\|U_{m-1}^-\|^2 + \int_{I_m} \|g\|^2 + \|u_D\|_{DGB,m}^2\, dt \right). \tag{14}$$

Now we come to the formulation of our main result, which demonstrates the unconditional stability of the STDG method in the discrete $L^2(L^\infty)$-norm, energy DG-norm and $L^2(L^2)$-norm. (A detailed proof can be found in [1].)

Theorem 2. *There exists a constant $c > 0$ such that*

$$\|U_m^-\|^2 + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|_{DG,j}^2\, dt \le c \left(\|U_0^-\|^2 + \sum_{j=1}^{m} \int_{I_j} (\|g\|^2 + \|u_D\|_{DGB,j}^2)\, dt \right),$$
$$m = 1, \ldots, M,\ h \in (0, h_0),$$

$$\|U\|_{L^2(Q_T)}^2 \le c \left(\|U_0^-\|^2 + \sum_{m=1}^{M} \int_{I_m} (\|g\|^2 + \|u_D\|_{DGB,m}^2)\, dt \right), \quad h \in (0, h_0).$$

6. Numerical experiment

We consider the problem

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + u \frac{\partial u}{\partial x_2} = \epsilon \Delta u + g \quad \text{in} \quad (0,1)^2 \times (0, 10),$$

with $\epsilon = 0.1$ and such initial and Dirichlet boundary conditions that the exact solution has the form

$$u(x_1, x_2, t) = (1 - e^{-10t})\, \hat{u}(x_1, x_2),$$

where $\hat{u}(x_1, x_2) = 2r^\alpha x_1 x_2 (1 - x_1)(1 - x_2)$, $r = (x_1 + x_2)^{1/2}$ and $\alpha \in \mathbb{R}$ is a constant. It is possible to prove that $u \in H^{q+1}(0, T; H^\beta(\Omega))$ for all $\beta \in (0, \alpha + 3)$. (Here $H^\beta(\Omega)$ denotes the Sobolev-Slobodetskii space of functions with "noninteger derivatives".)

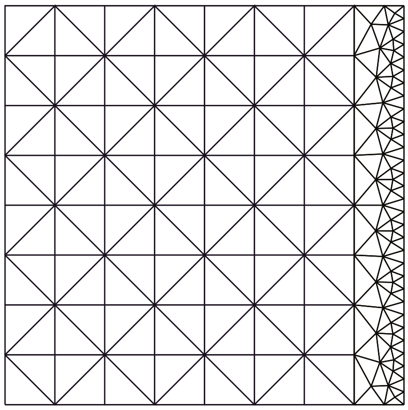

Figure 1: Coarse mesh with 235 elements

We used five special triangular meshes having $235, 333, 749$, 1622 and 2521 elements. All these meshes have refined elements along the right-hand side of the boundary. Figure 1 shows the coarsest mesh. In numerical experiments space polynomial degree $p = 1, 2, 3$ and time polynomial degree $q = 2$ were used. We choose fixed time step $\tau = 0.025$ and set $c_W = 100$ for SIPG. Tables show the computational errors in the $L^\infty(L^2(\Omega))$-norm along the time interval $[0, 10]$, and the corresponding orders of convergence (EOC). It is seen, that for a sufficiently regular exact solution (case $\alpha = 4$), for the SIPG method we have optimal order of convergence $O(h^{p+1})$ for $p = 1, 2, 3$, whereas in the case with irregular solution ($\alpha = -3/2$) the error estimates are of order $O(h^{3/2})$ for $p = 1, 2, 3$ (this result can be proven with the aid of estimates in Sobolev-Slobodetskii spaces). The presented numerical experiments demonstrate the unconditional stability of the numerical process without the CFL-like condition (7). Further numerical experiments including also the NIPG case can be found in [1].

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC	$\|e_h\|$	EOC	$\|e_h\|$	EOC
1	1.768E-01	2.167E-03	-	1.305E-04	-	6.681E-06	-
2	1.414E-01	1.488E-03	1.685	7.218E-05	2.654	2.948E-06	3.667
3	8.839E-02	6.549E-04	1.746	1.984E-05	2.748	5.019E-07	3.767
4	5.657E-02	2.914E-04	1.814	5.615E-06	2.828	9.011E-08	3.848
5	4.419E-02	1.842E-04	1.858	2.764E-06	2.872	3.440E-08	3.901

Table 1: Computational errors and the corresponding experimental orders (EOC) of convergence of the SIPG method for $\alpha = 4$

		p=1		p=2		p=3	
Mesh	h	$\|e_h\|$	EOC	$\|e_h\|$	EOC	$\|e_h\|$	EOC
1	1.768E-01	2.668E-02	-	6.038E-03	-	2.784E-03	-
2	1.414E-01	1.946E-02	1.415	4.330E-03	1.490	2.003E-03	1.475
3	8.839E-02	9.856E-03	1.447	2.149E-03	1.491	9.985E-04	1.481
4	5.657E-02	5.116E-03	1.469	1.103E-03	1.493	5.145E-04	1.486
5	4.419E-02	3.552E-03	1.478	7.629E-04	1.495	3.562E-04	1.489

Table 2: Computational errors and the corresponding experimental orders of convergence of the SIPG method for $\alpha = -3/2$

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