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THE COMMUTATORS OF ANALYSIS AND INTERPOLATION

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Abstract. The boundedness properties of commutators for operators are of central importance in Mathematical Analysis, and some of these commutators arise in a natural way from interpolation theory. Our aim is to present a general abstract method to prove the boundedness of the commutator $[T, \Omega]$ for linear operators T and certain unbounded operators Ω that appear in interpolation theory, previously known and a priori unrelated for both real and complex interpolation methods, and also to show how the abstract result applies to some very concrete examples.

In Section 1 some examples are given to present some instances where these commutators are used in Analysis. Section 2 is the basic one and contains a general "commutator theorem" for operators of interpolation methods, and the basic idea is that Ω appears as a combination of two admissible interpolation methods, Φ and Ψ, that correspond to $\Phi(F) = F(\vartheta)$ and $\Psi(f) = F'(\vartheta)$ in the case of the complex method, with $\Omega(f) = \Psi(F)$ if $\Phi(F) = f$ (with a natural boundedness condition over the norms). Section 3 deals with the complex interpolation method and contains typical applications to commutators with pointwise multipliers. Section 4 refers to the real method, and an application to commutators with Fourier multipliers is included.

Contents

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1. Introduction

By an operator T between two (complex Banach) spaces, A and B, we understand a mapping (usually linear) from a dense subspace $D(T)$ of A to B. We write $T \in \mathcal{L}(A, B)$ or $T : A \to B$ to mean that T is bounded and linear, if no further indication is given.

We use the notation "$X \lesssim Y$" instead of "$X \leq cY$ for some constant $c > 0$", and "$X \simeq Y$" for "$X \lesssim Y$ and $Y \lesssim X$". Thus, $\|T(x)\|_B \lesssim \|x\|_A$ means that the operator T is bounded.

The mapping properties of commutators $[T, M] = TM - MT$ for operators such as the following ones (and their natural extensions to several variables and to different function spaces), are of central importance in Analysis.

- Pointwise multipliers, $M_v f = vf$, multiplication by a function v. Recall that $M_v : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ if and only if $v \in L^\infty(\mathbb{R})$, and then $\|M_v\| = \|v\|_\infty$.
- Fourier multipliers, T_μ, where its "symbol" μ is also a given function and $\widehat{T_\mu f} = \mu \widehat{f}$, where

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i x\xi}\, dx,$$

i.e., μ multiplies at "the other side" of the Fourier transform. Again $T_\mu : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ if and only if $\mu \in L^\infty$, and $\|T_\mu\| = \|\mu\|_\infty$.

- Singular integrals

$$(Tf)(x) = \int_{\mathbb{R}} K(x,y)f(y)\,dy,$$

defined by a kernel $K(x, y)$ that may have a singularity concentrated for y near x, such as Calderón-Zygmund operators.

- Pseudo-differential operators, formally

$$(\Psi_a f)(x) = \int_{\mathbb{R}} a(x,\xi)\widehat{f}(\xi)\,\mathrm{e}^{2\pi i x\xi}\,d\xi.$$

Depending on the symbol $a(x, \xi)$, the operator Ψ_a may be a pointwise multiplier, a Fourier multiplier, a singular integral or a differential operator.

1.1. An easy example from quantum theory

If T and M are continuous operators on a space E, then obviously $[T, M]$ is also continuous. But assume that they are not both bounded. Then the domain of $[T, M]$,

$$D[T,M] := \{f \in D(M) : T(f) \in D(M)\},$$

may not be dense in E; it can even be equal to $\{0\}$.

But in some important situations $D[T, M]$ is dense and the commutator is bounded, i.e., it has a well-defined bounded extension (still denoted $[T, M]$) to the whole space E.

This is easily understood with the elementary example of the commutator $[p, q]$ on the "states space" $L^2 = L^2(\mathbb{R})$ for the moment and position operators p and q for a single particle constrained to one dimension in quantum mechanics. They are the self-adjoint operators $p(f) = -if'$ (distributional derivative) and $q(f)(x) = xf(x)$ ($q = M_x$), with domains $\{f \in L^2 : f' \in L^2\}$ and $\{f \in L^2 : q(f) \in L^2\}$. They are both unbounded but $D[p, q]$ contains all test functions g in the Schwartz class $\mathcal{S}$, a dense subspace of L^2.

An obvious computation,

$$[p,q]f(x) = -i(xf(x))' + xif'(x) = -if(x),$$

shows that the cancellation given by the derivative provides a unique continuous extension $-i\,\mathrm{Id}$ of $[p, q]$, and we may say that this commutator is bounded on L^2 and write $[p, q] = -i\,\mathrm{Id}$.

1.2. Commutators of pseudo-differential operators

The same happens with pseudo-differential operators that arise in a natural way when using the Fourier integral in the theory of partial differential equations. We refer to [St2] for the details about the following facts.

As we have said the pseudo-differential operators admit the description (in the one variable case)

$$(\Psi_a f)(x) = \int_{\mathbb{R}} a(x,\xi)\widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi, \tag{1}$$

with some restrictions on the symbol, $a(x,\xi)$, that allow to define the above integral for functions that belong to $\mathcal{S}$, which is dense in many function spaces. If $a(x,\xi)$ is C^∞ and satisfies the estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-\beta}$$

for all indices α and β, we say that it is a standard symbol of order m and write $a \in S^m$; it is easily checked that the integral (1) is then absolutely convergent and infinitely differentiable, and integration by parts shows that $\Psi_a(\mathcal{S}) \subset \mathcal{S}$. As fundamental examples, let us mention polynomials $P = \sum_{\alpha=0}^m a_\alpha(x)(2\pi i\xi)^\alpha$ of degree m in ξ where the coefficients $a_\alpha(x)$ are bounded C^∞ functions with bounded derivatives of all orders. In this case, it follows from the properties of the Fourier integral that

$$\begin{aligned}(\Psi_P f)(x) &= \int_{\mathbb{R}} P\widehat{f}(\xi)\, e^{-2\pi i x\xi}\, d\xi \\ &= \sum_{\alpha=0}^m a_\alpha(x) \int_{\mathbb{R}} (2\pi i\xi)^\alpha \widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi \quad (f\in\mathcal{S}),\end{aligned}$$

hence, $(\Psi_P f)(x) = \sum_{\alpha=0}^m a_\alpha(x) f^{(\alpha)}(x)$, and $\Psi_P = P(x,D)$ is a differential operator of order m with variable coefficients.

If the symbol does not depend on ξ, $a(x,\xi) = v(x)$, then the Fourier inversion theorem gives

$$(\Psi_v f)(x) = v(x)\int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi = (M_v f)(x),$$

a pointwise multiplier. If it does not depend on x, $a(x,\xi) = \mu(\xi)$, we get

$$(\Psi_\mu f)(x) = \int_{\mathbb{R}} \mu(\xi)\widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi = (T_\mu f)(x),$$

a Fourier multiplier.

As the first fact concerning these operators let us mention that, if $a \in S^0$, then Ψ_a is bounded on L^r $(1 < r < \infty)$, and if $a \in S^m$, then $\Psi_a : W^{k,r} \to W^{k-m,r}$ $(1 < r < \infty,\ k \geq m)$. Here $W^{k,r} = W^{k,r}(\mathbb{R})$ denotes the usual Sobolev space of all $f \in L^r$ such that the derivatives $f^{(\alpha)}$ $(\alpha \leq k)$ satisfy $\|f\|_{k,r} := \big(\sum_{\alpha=0}^{k} \|f^{(\alpha)}\|_r^r\big)^{1/r} < \infty$.

The moment operator p is the pseudo-differential operator with the symbol $2\pi\xi$ in S^1. It follows that $p : W^{k,r} \to W^{k-1,r}$ and it is unbounded on L^s for all s, but for any $a = a(x,\xi) \in S^m$, a cancellation originated by the derivative appears again in the commutator

$$\begin{aligned}[\Psi_a, p]f(x) &= -i \int_{\mathbb{R}} \Big[a\widehat{f'}(\xi)\,\mathrm{e}^{2\pi i x\xi} + \widehat{f}(\xi)\frac{\partial}{\partial x}(a\,\mathrm{e}^{2\pi i x\xi})\Big]\,d\xi \\ &= -i \int_{\mathbb{R}} \frac{\partial a(x,\xi)}{\partial x}\widehat{f}(\xi)\,\mathrm{e}^{2\pi i x\xi}\,d\xi,\end{aligned}$$

i.e., $[p, \Psi_a] = \Psi_{i\partial a/\partial x}$, another pseudo-differential operator with standard symbol also of order m. Hence, if a is of order 0, then the same happens for $[p, \Psi_a]$ and it is bounded on L^r $(1 < r < \infty)$; if $a(x,\xi) = v(x)$, then $[p, M_v] = [p, \Psi_a] = iM_{v'}$ is also bounded on L^r $(1 < r < \infty)$.

Similarly, for the position operator $q = M_x$ we have $[\Psi_a, q] = \Psi_b$ with the symbol $b = -(2\pi i)^{-1}\,\partial a/\partial\xi$ of order $m-1$ if $a \in S^m$.

1.3. Calderón commutators

Some other well-known examples arise with the Cauchy singular integral.

Let γ be a simple closed C^1 curve in the complex plane $\mathbb{C}$. The Cauchy integral of a function f integrable on γ is

$$(C_\gamma f)(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad (z \notin \gamma).$$

Hence, if γ is the oriented boundary of a domain D, if f is continuous on $\overline{D}$ and analytic on D, and if $z \in D$, then the Cauchy integral formula reads $(C_\gamma f)(z) = f(z)$.

If $z = \gamma(t)$ is on the curve, a singular integral in the sense of Cauchy principal value appears,

$$(S_\gamma f)(x) = \frac{1}{\pi i}\int \frac{f(t)\gamma'(t)}{\gamma(x) - \gamma(t)}\,dt,$$

where we write $f(t)$ instead of $f(\gamma(t))$. This singular integral is associated with the problem of finding the inner and outer non-tangential limits, $C_\gamma^+ f$

and $C_\gamma^- f$, as $z \to \gamma$. As a matter of fact, $C_\gamma^+ f = f$ if and only if $C_\gamma^+ f \in L^1(\gamma)$ and $S_\gamma(C_\gamma^+ f) = C_\gamma^+ f$. PRIVALOV proved (see [St2]) that

$$C_\gamma^\pm f(z) = \pm\frac{f(z)}{2} + \frac{1}{2} S_\gamma f(z) \quad (z \in \gamma).$$

In this setting, the basic problem is to obtain the L^2-boundedness of $C_\gamma^\pm$, i.e., of S_γ. If γ is C^2, then this is equivalent to the L^2-boundedness of the Hilbert transform

$$Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy,$$

which is the Fourier multiplier with the symbol $\mu(x) = -i\,\mathrm{sgn}(x)$, and it is a bounded operator on L^p if $1 < p < \infty$. However, if γ is a C^1 curve, then the problem is much more involved and leads to curves $y = A(x)$ with a bounded derivative, i.e.

$$\gamma(x) = x + iA(x),\ \gamma'(x) = 1 + iA'(x) \quad (A' \in L^\infty(\mathbb{R})).$$

Hence, we need to deal with the singular integral

$$(S_\gamma f)(x) := \int_{-\infty}^{+\infty} \frac{f(t)(1 + ia(t))}{x - t + i(A(x) - A(t))}\, dt,$$

where we may incorporate the bounded factor $1 + iA'(t)$ to $f(t)$ and then the kernel is

$$K(x,y) = \frac{1}{x-y}\frac{1}{1 + i\frac{A(x)-A(y)}{x-y}}.$$

When $|A'(x)| \le M < 1$, we have the decomposition

$$K(x,y) = \sum_{k=0}^\infty (-i)^k K_k(x,y), \quad K_k(x,y) = \frac{1}{x-y}\left(\frac{A(x)-A(y)}{x-y}\right)^k,$$

and to study the L^2-boundedness of S_γ we may consider the singular integrals

$$(S_k f)(x) = \int_{\mathbb{R}} K_k(x,y) f(y)\, dy.$$

Note that $\pi S_0 = H$, the Hilbert transform, and that

$$S_1 f(x) = A(x)\int \frac{f(y)}{(x-y)^2}\, dy - \int \frac{A(y)f(y)}{(x-y)^2}\, dy = [H_2, M_A]f(x),$$

is the commutator of the pointwise multiplier M_A with the singular integral $H_2 f(x) = \int (x-y)^{-2} f(y)\, dy$. If $H_k f(x) = \int (x-y)^{-k} f(y)\, dy$, then S_k is a higher order commutator; e.g., $S_2 f = A^2 H_3 f - 2AH_3(Af) + H_3(A^2 f) = [[H_3, M_A], M_A]f$.

The operators S_k are the Calderón commutators and the proof of the L^2-boundedness of S_γ follows from estimates

$$\|S_k\| \leq C L^k M^k.$$

In 1977, A. P. Calderón obtained the boundedness of S_γ if M is small, and previously (in 1965) he had proved that the first commutator S_1 is bounded. The complete result, for any M, was achieved by R. Coifman, A. McIntosh and Y. Meyer in 1982. We refer to [St2] for the full description of these facts.

2. The commutator theorem of interpolation theory

2.1. Interpolation

Let us quickly recall some facts of interpolation theory (we refer to [BK], [BL], [BS], [KPS] and [Tr] for more details).

With the notation $T : \bar{A} \to \bar{B}$ or $T \in \mathcal{L}(\bar{A}, \bar{B})$ we represent a bounded linear operator between two couples of spaces in the sense of interpolation theory, where

(a) $\bar{A} = (A_0, A_1)$ (and the same for $\bar{B}$) is a Banach couple, in the sense that A_0 and A_1 are two (complex) Banach spaces continuously embedded in a common Hausdorff topological linear space, that allows to endow the sum space $\Sigma(\bar{A}) = A_0 + A_1$ with the norm

$$\|a\|_{\Sigma(\bar{A})} := \inf_{a=a_0+a_1} (\|a_0\|_0 + \|a_1\|_1) \quad (a \in \Sigma(\bar{A}))$$

(we set $\|\cdot\|_j = \|\cdot\|_{A_j}$, and $\|x\|_j = \infty$ if $x \notin A_j$);

(b) $T : \Sigma(\bar{A}) \to \Sigma(\bar{B})$ and $\|T(a)\|_j \leq M_j \|a\|_j$ ($0 \leq M_j < \infty$ and $a \in A_j$ for $j = 0, 1$). The norm of T is $\|T\| := \max(\|T\|_0, \|T\|_1)$ where $\|T\|_j$ denotes the norm of the restriction $T : A_j \to B_j$.

An interpolation method associates with $\bar{A}$ and $\bar{B}$ two Banach spaces, A and B, continuously included in $\Sigma(\bar{A})$ and $\Sigma(\bar{B})$, respectively, such that $T : A \to B$ whenever $T : \bar{A} \to \bar{B}$ (this is referred to as an interpolation theorem of the interpolation method).

Complex interpolation methods are the abstract counterpart of the Riesz-Thorin convexity theorem (see Example 1 below).

For the Calderón complex method ([Ca]), that will be our model example, for a given ϑ, $0 < \vartheta < 1$, and for every couple $\bar{A}$, a certain Banach space $\mathcal{F}(\bar{A})$ of vector-valued functions is considered. It contains all bounded $\Sigma(\bar{A})$-valued continuous functions on the unit strip $\bar{\boldsymbol{S}} = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$,

$$F : \bar{\boldsymbol{S}} \to \Sigma(\bar{A}),$$

which are analytic on $\boldsymbol{S} = \{z \in \mathbb{C} : 0 < \Re z < 1\}$ and such that $F_j(t) := F(j + it)$ define two bounded continuous functions $F_j : \mathbb{R} \to A_j$ with the property $\lim_{t\to\infty} \|F_j(t)\|_j = 0$, where again we set $\|\cdot\|_j = \|\cdot\|_{A_j}$ $(j = 0, 1)$. The norm on $\mathcal{F}(\bar{A})$ is

$$\|F\|_{\mathcal{F}} := \max_{j=0,1}(\sup_{t\in\mathbb{R}} \|F_j(t)\|_j).$$

Then we have the interpolated space

$$[\bar{A}]_\vartheta := \{F(\vartheta) : F \in \mathcal{F}(\bar{A})\} \tag{2}$$

with the norm $\|a\|_{[\vartheta]} := \inf\{\|F\|_{\mathcal{F}} : F(\vartheta) = a\}$.

It is very easy to see that $T(F) := T \circ F \in \mathcal{F}(\bar{B})$ and $\|T(F)\|_{\mathcal{F}} \leq \|T\|\,\|F\|_{\mathcal{F}}$ if $T : \bar{A} \to \bar{B}$ and $F \in \mathcal{F}(\bar{A})$. Obviously, the interpolation theorem follows from this fact.

Given Banach spaces A and B and an operator T, the main goal is to prove that $T : A \to B$, by showing that A and B are interpolated spaces ($A = [\bar{A}]_\vartheta$ and $B = [\bar{B}]_\vartheta$ for some ϑ in the case of the complex method) of two convenient couples $\bar{A}$ and $\bar{B}$, for which it is known that $T : \bar{A} \to \bar{B}$.

Thus, a basic problem is to identify $A = [\bar{A}]_\vartheta$ and $B = [\bar{B}]_\vartheta$, at least by equivalence of norms showing that $\|a\|_{[\vartheta]} \simeq \|a\|_A$ for $a \in \Sigma(\bar{A})$. This means that for every a we must find some $F_a \in \mathcal{F}(\bar{A})$ such that $F_a(\vartheta) = a$ and $\|a\|_A \simeq \|F_a\|_{\mathcal{F}}$ (i.e., $\|a\|_A \leq \|F_a\|_{\mathcal{F}} \leq c\|a\|_A$ for some $c = c_{\bar{A}} > 1$), and then we say that F_a is "almost optimal" (for $\|a\|_A$).

Let us describe the case of interpolation of couples of L^p-spaces of vector-valued function by the Calderón method that will be useful in the sequel. We always assume that $0 < \vartheta < 1$, $p_0, p_1, p \geq 1$, and $p(\vartheta)$ is such that $1/p(\vartheta) = (1 - \vartheta)/p_0 + \vartheta/p_1$. A weight ω is a locally integrable positive function (on a given σ-finite measure space) and, if E is a Banach space and $|f|(\cdot) = \|f(\cdot)\|_E$, then $L^p(E, \omega)$ is defined by the condition $\int |f|^p\omega < \infty$.

Theorem 1. *Let ω_0, ω_1 be two weights, $\omega_0, \omega_1 > 0$, and let E be a complex Banach space. Then*

$$[L^{p_0}(\omega_0, E), L^{p_1}(\omega_1, E)]_\vartheta = L^p(\omega, E),$$

where $p = p(\vartheta)$ *and* $\omega = \omega_0^{(1-\vartheta)p/p_0}\omega_0^{\vartheta p/p_1}$. *An almost optimal selection in* $\mathcal{F}(L^{p_0}(\omega_0, E), L^{p_1}(\omega_1, E))$ *for the norm of* $f \in [L^{p_0}(\omega_0), L^{p_1}(\omega_1)]_\vartheta$ *is*

$$F_f(z) = \frac{f}{\|f(\cdot)\|_E}\Big(\frac{\|f(\cdot)\|_E}{\|f\|_{L^p(\omega,E)}}\Big)^{((1-z)/p_0+z/p_1)p}\|f\|_{L^p(\omega,E)}\Big(\frac{\omega_1}{\omega_0}\Big)^{p(\vartheta-z)/(p_0p_1)}.$$

P r o o f. Obviously, $F_f(\vartheta) = f$. If $|f| := \|f(\cdot)\|_E$, $f_0 := |F_f(it)|$ and $f_1 := |F_f(1+it)|$, then straightforward computations show that

$$f_0 = \|f\|_{L^p(\omega,E)}^{1-p/p_0}|f|^{p/p_0}\Big(\frac{\omega_1}{\omega_0}\Big)^{p\vartheta/(p_0p_1)}$$

and

$$f_1 = \|f\|_{L^p(\omega,E)}^{1-p/p_1}|f|^{p/p_1}\Big(\frac{\omega_1}{\omega_0}\Big)^{p(1-\vartheta)/(p_0p_1)}$$

do not depend on t. It is an easy exercise to show that

$$\|F_f\|_{\mathcal{F}} = \max(\|f_0\|_{L^{p_0}(\omega_0)}, \|f_1\|_{L^{p_1}(\omega_1)}) = \|f\|_{L^p(\omega,E)}.$$

□

We have found that F_f is not only almost optimal for $\|f\|_{[\vartheta]}$, but also $\|F_f\|_{\mathcal{F}} = \|f\|_{L^p(\omega,E)}$.

As special cases we have the following examples:

E x a m p l e 1 (Riesz-Thorin theorem). Let $p = p(\vartheta)$. Then $[L^{p_0}, L^{p_1}]_\vartheta = L^p$ and

$$F_f(z) := \frac{f}{|f|}\left(\frac{|f|}{\|f\|_p}\right)^{((1-z)/p_0+z/p_1)p}\|f\|_p$$

is an almost optimal selection for $\|f\|_{[\vartheta]} \simeq \|f\|_{L^p}$.

E x a m p l e 2. If $\omega = \omega_0^{1-\vartheta}\omega_1^\vartheta$, then $[L^p(\omega_0, E), L^p(\omega_1, E)]_\vartheta = L^p(\omega, E)$ and

$$F_f(z) := \omega_0^{(z-\vartheta)/p}\omega_1^{(\vartheta-z)/p}f$$

is an almost optimal selection for $\|f\|_{[\vartheta]} \simeq \|f\|_{L^p(\omega,E)}$.

R e m a r k 1. If $1 < p < \infty$, then there is a class of weights ω (the Muckenhoupt A_p-weights) such that the singular integral operators of the Calderón-Zygmund class (e.g., the Hilbert transform) are bounded on $L^p(\omega)$. Example 2 shows that, if $\omega_0, \omega_1 \in A_p$, then also $\omega := \omega_0^{1-\vartheta}\omega_1^\vartheta \in A_p$ for all $0 < \vartheta < 1$.

2.2. The Rochberg and Weiss commutator theorem

In [RW], R. Rochberg and G. Weiss considered operators $\Omega(f) := F'_f(\vartheta)$ to analyse the rate of change of the interpolated norms and obtained estimates for $[T, \Omega]$.

In order to explain the basic ideas, we start with comparing the derivatives of the functions that appear along Thorin's proof of the Riesz-Thorin theorem with those of certain modifications of these functions. This will be useful to show how cancellation, optimal selection and a second interpolation method are involved. Note that the Riesz-Thorin theorem is the Calderón complex method applied to the couple $(L^{p_0}(\lambda), L^{p_1}(\lambda))$.

We consider the "diagonal case" and an operator

$$T : (L^{p_0}(\lambda), L^{p_1}(\lambda)) \to (L^{p_0}(\mu), L^{p_1}(\mu)),$$

linear and bounded, i.e. $\|Tf\|_{p_j} \leq M_j \|f\|_{p_j}$. Then $T : L^p(\lambda) \to L^p(\mu)$ with a boundedness constant M satisfying $M \leq M_0^{1-\vartheta} M_1^{\vartheta}$ if $1 \leq p_0 < p < p_1 \leq \infty$ and $1/p = (1-\vartheta)/p_0 + \vartheta/p_1$. This means that, for any simple function f such that $\|f\|_p = 1$,

$$\Big| \int gTf \, d\mu \Big| \leq M \quad (g \text{ simple and } \|g\|_{p'} = 1).$$

In Thorin's proof this estimate is obtained as an application of the three-lines theorem to the function

$$F(z) := \int g_z T f_z \, d\mu$$

with

$$f_z = |f|^{\alpha(z)} \operatorname{sgn} f, \quad g_z = |g|^{(1-\alpha(z))p'} \operatorname{sgn} g, \quad \alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1}$$

(hence $p = 1/\alpha(\vartheta)$ and $F(\vartheta) = \int g(Tf)\, d\mu$). Let also

$$G(z) := \int g_z (Tf)_z \, d\mu$$

with $(Tf)_z = |Tf|^{\alpha(z)p} \operatorname{sgn}(Tf)$ and compare the derivatives

$$F'(\vartheta) = \int \Big[\Big(\frac{p'}{p_0} - \frac{p'}{p_1} \Big) (g \log |g|) Tf - \Big(\frac{p}{p_0} - \frac{p}{p_1} \Big) g T(f \log |f|) \Big] \, d\mu,$$

$$G'(\vartheta) = \int \Big[\Big(\frac{p'}{p_0} - \frac{p'}{p_1} \Big) (g \log |g|) Tf - \Big(\frac{p}{p_0} - \frac{p}{p_1} \Big) g Tf \log |Tf| \Big] \, d\mu.$$

If we denote $Lh = h \log |h|$, we obtain

$$G'(\vartheta) - F'(\vartheta) = \Big(\frac{p}{p_0} - \frac{p}{p_1}\Big) \int g[T(Lf) - L(Tf)]\, d\mu, \tag{3}$$

and for the circle $\gamma = \{z \in \mathbb{C} : |z - \vartheta| = r\}$ with $r = d(\vartheta, \partial \boldsymbol{S})$,

$$|F'(\vartheta)| = \Big|\frac{1}{2\pi i} \int_\gamma \frac{F(z)}{(z-\vartheta)^2}\, dz\Big| \leq \frac{M}{r}, \quad |G'(\vartheta)| \leq \frac{M}{r}. \tag{4}$$

From (3) and (4) it follows that

$$\|[T, L]f\|_p \leq C \quad (\|f\|_p = 1). \tag{5}$$

Although L is not homogeneous, the commutator $[T, L] = TL - LT$ satisfies $[T, L](\lambda f) = \lambda [T, L]f$ and (5) is equivalent to

$$\|[T, L]f\|_p \leq C\|f\|_p. \tag{6}$$

For the homogeneous operator

$$\Omega(h) = h \log \frac{|h|}{\|h\|_p}, \tag{7}$$

$L - \Omega : L^p(\lambda) \to L^p(\mu)$ is bounded, since $\|(L-\Omega)h\|_p = \|h\|_p \log \|h\|_p$; thus, from the Riesz-Thorin theorem and from (6) we obtain

$$\|[T, \Omega]f\|_p \leq C\|f\|_p, \tag{8}$$

which is the *commutator theorem.*

This is how R. Rochberg and G. Weiss explain in [RW] that the derivatives of some analytic families of operators in complex interpolation theory lead to estimates for $[T, \Omega]$, where Ω can be unbounded and non-linear.

The following facts were basic in their method:

1. With the evaluation $\delta_\vartheta : F \mapsto F(\vartheta)$, the evaluation of the derivatives, $\delta'_\vartheta : F \mapsto F'(\vartheta)$, is used.
2. The functionals δ_ϑ and δ'_ϑ are combined through a cancellation property.
3. Almost optimal selections F_f are needed to identify the interpolated spaces, such as $[L^{p_0}, L^{p_1}]_\vartheta = L^p$, and an Ω-operator is defined by applying δ'_ϑ to these selections.

The functionals δ_ϑ and δ'_ϑ can be used in the abstract frame of Calderón's method (2) for Banach couples $\bar{A}$ by applying δ'_ϑ to an almost optimal function F_a for every $a \in [\bar{A}]_\vartheta$.

Real interpolation methods are the abstract counterpart of the Marcinkiewicz interpolation theorem. We refer to Sections 4.1 and 4.2, where we show that these methods can be described following the pattern of the complex method.

A corresponding study for the real method was carried out by B. Jawerth, R. Rochberg and G. Weiss ([JRW]), with strong formal analogies to the complex method, but with very different details.

2.3. An abstract commutator theorem

In order to obtain a unified and extended method, using the interpolation theory previously defined in [W], we say as in [CCS1], [CCS2] and [CCS3], that (H, Φ) (or Φ) is an *interpolator* over the *functional spaces* spaces $H(\bar{A})$ if H is a functor from Banach couples to normed spaces,

$$H : \bar{A} \mapsto H(\bar{A}), \quad H : \mathcal{L}(\bar{A}; \bar{B}) \mapsto \mathcal{L}(H(\bar{A}); H(\bar{B})),$$

and Φ is a family of bounded linear operators

$$\Phi_{\bar{A}} \in \mathcal{L}(H(\bar{A}); \Sigma(\bar{A}))$$

such that

$$T\Phi_{\bar{A}} = \Phi_{\bar{B}} H(T) \quad (T \in \mathcal{L}(\bar{A}; \bar{B})). \tag{9}$$

Then, as in the case of the complex method (2), we obtain an interpolation method,

$$\bar{A}_\Phi := \Phi_{\bar{A}}(H(\bar{A})), \quad \|a\|_\Phi := \inf\{\|f\| : a = \Phi_{\bar{A}}(f)\}$$

and, for a fixed $c = c_{\bar{A}} > 1$, we can associate with every $a \in \bar{A}_\Phi$ an element $h_a \in H(\bar{A})$ such that $\Phi_{\bar{A}}(h_a) = a$ and $\|a\|_\Phi \le \|h_a\| \le c\|a\|_\Phi$ (i.e. $\|a\|_\Phi \simeq \|h_a\|$). We say that

$$a \in \bar{A}_\Phi \mapsto h_a \in H(\bar{A})$$

is an *almost optimal selection* for the interpolation method.

A *couple of interpolators* will be a pair (Φ, Ψ) of interpolators on the same functional spaces $H(\bar{A})$. This corresponds to $\Phi(F) = F(\vartheta) = \delta_\vartheta(F)$ and $\Psi(f) = F'(\vartheta) = \delta'_\vartheta(F)$ of the complex method. We define an associated Ω-operator,

$$\Omega_{\bar{A}} a := \Psi_{\bar{A}}(h_a) \in \bar{A}_\Psi \quad (a \in \bar{A}_\Phi),$$

and $[T, \Omega] := T\Omega_{\bar{A}} - \Omega_{\bar{B}} T = T\Omega - \Omega T$ (we suppress the subscripts $\bar{A}$, $\bar{B}$). Then

$$\Omega : \bar{A}_\Phi \to \bar{A}_\Psi \hookrightarrow \Sigma(\bar{A}). \tag{10}$$

Theorem 2. *If* (Φ, Ψ) *satisfies the cancellation condition*

$$\Psi_{\bar{A}}(\operatorname{Ker}\Phi_{\bar{A}}) \hookrightarrow \operatorname{Im}\Phi_{\bar{A}}, \tag{11}$$

a bounded inclusion, and $T \in \mathcal{L}(\bar{A}; \bar{B})$, *then* $[T, \Omega] : \bar{A}_\Phi \to \bar{B}_\Phi$ *is a bounded (possibly non-linear) operator.*

P r o o f. Let us denote

$$\bar{B}_{\Psi,(\Phi)} := \{b = \Psi_{\bar{B}}(f) : f \in H(\bar{B}),\ \Phi_{\bar{B}}(f) = 0\} = \Psi_{\bar{B}}(\operatorname{Ker}\Phi_{\bar{B}})$$

with $\|b\|_{\Psi,(\Phi)} = \inf\{\|f\| : \Phi_{\bar{B}}(f) = 0,\ b = \Psi_{\bar{B}}(f)\}$. By condition (11),

$$\|b\|_\Phi \le C\|b\|_{\Psi,(\Phi)}.$$

Since $\Phi_{\bar{B}}(H(T)h_a - h_{Ta}) = T\Phi_{\bar{A}} f_a - \Phi_{\bar{B}} h_{Ta} = 0$ and $[T, \Omega]a = T\Psi h_a - \Psi h_{Ta}$ it follows that $[T, \Omega]a \in \bar{B}_{\Psi,(\Phi)}$ and

$$\|[T, \Omega]a\|_{\Psi,(\Phi)} \le \|H(T)h_a - h_{Ta}\| \lesssim \|H(T)\|\,\|a\|_\Phi + \|Ta\|_\Phi \lesssim \|a\|_\Phi.$$

Hence, $\|[T, \Omega]a\|_\Phi \lesssim \|a\|_\Phi$. □

We also say that $\tilde{\Omega} : \bar{A}_\Phi \to \Sigma(\bar{A})$ is an Ω-operator for the couple of interpolators if $\tilde{\Omega} - \lambda\Omega$ is bounded on the interpolated spaces $\bar{A}_\Phi$ for some λ. In this case we still have the commutator theorem

$$[T, \tilde{\Omega}] : \bar{A}_\Phi \to \bar{B}_\Phi \quad (T \in \mathcal{L}(\bar{A}; \bar{B})),$$

since $[T, \tilde{\Omega}] = [T, \lambda\Omega] + T(\tilde{\Omega} - \lambda\Omega) + (\lambda\Omega - \tilde{\Omega})T$.

For another almost optimal selection $a \mapsto \tilde{h}_a$ we have another operator $\tilde{\Omega}$, but Ω and $\tilde{\Omega}$ are equivalent, since, for any $a \in \bar{A}_\Phi$, $\Phi(h_a - \tilde{h}_a) = 0$ and $(\tilde{\Omega} - \Omega)a = \Psi(h_a - \tilde{h}_a) \in \bar{A}_{\Psi,(\Phi)}$ with

$$\|(\tilde{\Omega} - \Omega)a\|_{\Psi,(\Phi)} \le \|h_a - \tilde{h}_a\| \le 2c\|a\|_\Phi.$$

R e m a r k 2. Without the cancellation condition (11), we still have $[T, \Omega] : \bar{A}_\Phi \to \bar{B}_{\Psi,(\Phi)}$.

We say that (Φ, Ψ) is *almost compatible* if condition (11) holds. In some examples we have the more complete cancellation property $\Psi_{\bar{A}}(\operatorname{Ker}\Phi_{\bar{A}}) = \operatorname{Im}\Phi_{\bar{A}}$ (with $\|a\|_\Phi \simeq \|a\|_{\Psi,(\Phi)}$) and then we say that the couple of interpolators is *compatible*.

The operator Ω may be not only unbounded, but even non-linear. It is always equivalent to a homogeneous one (satisfying $\tilde{\Omega}(\lambda x) = \lambda\tilde{\Omega}(x)$), since we may take a homogeneous almost optimal selection (satisfying $h_{\lambda x} = \lambda h_x$), as we shall always assume.

We shall be mainly concerned with applications of Theorem 2 and with the domain and range spaces of Ω.

R e m a r k 3. A very interesting recent paper concerning a unified commutator theory is [CKMR]. It refers to the special case of operators Ω that can be described through derivatives.

2.4. Domain space

Assume that (Φ, Ψ) is an almost compatible couple of interpolators and that we have chosen a homogeneous almost optimal selection $h_{\lambda x} = \lambda h_x$.

Definition. On the set $\mathrm{Dom}(\Omega_{\bar{A}}) := \{a \in \bar{A}_\Phi : \Omega_{\bar{A}} a \in \bar{A}_\Phi\}$ we define

$$\|a\|_D := \|a\|_\Phi + \|\Omega_{\bar{A}} a\|_\Phi.$$

Observe that $\|a\|_D > 0$ when $a \neq 0$, and $\|\lambda a\|_D = |\lambda| \, \|a\|_D$, since Ω is assumed to be homogeneous. Let us see that it is also "quasi-additive".

Lemma 1. *If (Φ, Ψ) is almost compatible, then for $a, b \in \bar{A}_\Phi$,*

$$\Omega_{\bar{A}}(a+b) - \Omega_{\bar{A}} a - \Omega_{\bar{A}} b \in \bar{A}_\Phi,$$

and there is a constant $C = C_A$ such that

$$\|\Omega_{\bar{A}}(a+b) - \Omega_{\bar{A}} a - \Omega_{\bar{A}} b\|_\Phi \leq C(\|a\|_\Phi + \|b\|_\Phi).$$

P r o o f. We have $\Omega_{\bar{A}}(a+b) - \Omega_{\bar{A}} a - \Omega_{\bar{A}} b = \Psi(h_{a+b} - h_a - h_b)$, and $\Phi(h_{a+b} - h_a - h_b) = 0$. Hence, $\Psi(h_{a+b} - h_a - h_b) = \Phi(f) \in \bar{A}_\Phi$ and

$$\|\Omega_{\bar{A}}(a+b) - \Omega_{\bar{A}} a - \Omega_{\bar{A}} b\|_\Phi \leq \|f\| \lesssim \|h_{a+b} - h_a - h_b\| \lesssim \|a\|_\Phi + \|b\|_\Phi.$$

□

Theorem 3. (a) *If (Φ, Ψ) is almost compatible, then* $\mathrm{Dom}(\Omega_{\bar{A}})$ *is a quasi-normed space and* $\mathrm{Dom}(\Omega_{\bar{A}}) = \Phi_{\bar{A}}(\Psi^{-1}(\bar{A}_\Phi))$ *(equivalent "norms", and for another almost optimal selection,* $\mathrm{Dom}(\Omega_{\bar{A}}) = \mathrm{Dom}(\tilde{\Omega}_{\bar{A}})$*).*

(b) *If (Φ, Ψ) is compatible, then*

$$\mathrm{Dom}(\Omega_{\bar{A}}) = \{\Phi_{\bar{A}}(f) : f \in H(\bar{A}),\ \Psi_{\bar{A}}(f) = 0\} = \bar{A}_{\Phi,(\Psi)},$$

with $\|x\|_D \simeq \inf\{\|f\|_{H(\bar{A})} : x = \Phi_{\bar{A}}(f),\ \Psi_{\bar{A}}(f) = 0\}$.

P r o o f. (a) If $a, b \in \mathrm{Dom}(\Omega_{\bar{A}})$, then from Lemma 1 we obtain

$$\begin{aligned} \|a+b\|_D &= \|a+b\|_\Phi + \|\Omega_{\bar{A}}(a+b)\|_\Phi \\ &\leq \|a\|_\Phi + \|b\|_\Phi + \|\Omega_{\bar{A}}(a+b) - \Omega_{\bar{A}} a - \Omega_{\bar{A}} b\|_\Phi + \|\Omega_{\bar{A}} a\|_\Phi + \|\Omega_{\bar{A}} b\|_\Phi \\ &\lesssim \|a\|_\Phi + \|b\|_\Phi + \|a\|_D + \|b\|_D \lesssim \|a\|_D + \|b\|_D. \end{aligned}$$

To show that $\mathrm{Dom}(\Omega) = \Phi(\Psi^{-1}(\bar{A}_\Phi))$, suppose that $a \in \mathrm{Dom}(\Omega)$; then there exists $h_a \in H(\bar{A})$ such that $\Phi(h_a) = a$, $\|h_a\| \leq C\|a\|_\Phi$ and $\Psi(h_a) = \Omega(a) \in \bar{A}_\Phi$. Hence $h_a \in \Psi^{-1}(\bar{A}_\Phi)$, and $a \in \Phi(\Psi^{-1}(\bar{A}_\Phi))$.

Conversely, if $a = \Phi(h)$, $\Psi(h) = \Phi(h')$, and $\Omega(a) = \Psi(h_a)$, then we have $\Phi_{\bar{A}}(h_a - h) = 0$ and thus $\Psi(h_a - h) = \Phi(h'') \in \bar{A}_\Phi$. Hence, $\Omega(a) = \Psi(h) + \Phi(h'') = \Phi(h') + \Phi(h'') \in \bar{A}_\Phi$.

(b) Let now $X := \Phi_{\bar{A}}(\mathrm{Ker}\,\Psi)$ with

$$\|x\|_X = \inf\{\|f\| : x = \Phi(f),\ \Psi(f) = 0\}.$$

For any $x \in \mathrm{Dom}(\Omega)$ we have $x = \Phi(h_x) \in \bar{A}_\Phi$, $\Omega(x) = \Psi(h_x) = \Phi(h) = \Psi(g)$, with $\Phi(g) = 0$, $\|h\| \lesssim \|\Omega(x)\|_\Phi$, $\|g\| \lesssim \|h\|$. Then $x = \Phi(h_x - g)$, $\Psi(h_x - g) = 0$ and we have $x \in X$, with

$$\|x\|_X \leq \|h_x - g\| \lesssim \|x\|_\Phi + \|\Omega(x)\|_\Phi.$$

Hence, $\|x\|_X \lesssim \|x\|_D$.

Conversely, if $x \in X$, $x = \Phi(f)$, $\Psi(f) = 0$ and $\|f\| \lesssim \|x\|_X$, then $\Omega(x) = \Psi(h_x) = \Psi(h_x - f) = \Phi(h)$, with $\|h\| \lesssim \|h_x - g\|$ (observe that $\Phi(h_x - f) = 0$). Hence $\Omega(x) \in \bar{A}_\Phi$ and

$$\|\Omega(x)\|_\Phi \lesssim \|h_x - f\| \lesssim \|x\|_\Phi + \|x\|_X \lesssim \|g\| + \|x\|_X \lesssim \|x\|_X.$$

Finally,

$$\|x\|_D = \|x\|_\Phi + \|\Omega(x)\|_\Phi \lesssim \|f\|_{H(\bar{A})} + \|\Omega_{\bar{A}} x\|_\Phi \lesssim \|x\|_X.$$

□

Observe that, as a consequence of Theorem 3, the necessary and sufficient condition for $\mathrm{Dom}(\Omega_{\bar{A}}) = \bar{A}_\Phi$ is that $H(\bar{A}) = \Psi_{\bar{A}}^{-1}(\bar{A}_\Phi) + \mathrm{Ker}\,\Phi_{\bar{A}}$. We can also give a converse result for (b):

Proposition 1. (Φ, Ψ) *is compatible if and only if* (Φ, Ψ) *is almost compatible,* $\mathrm{Dom}(\Omega_{\bar{A}}) = \Phi_{\bar{A}}(\mathrm{Ker}\,\Psi_{\bar{A}})$ *and* $\bar{A}_\Phi \hookrightarrow \bar{A}_\Psi$.

Proof. If $\mathrm{Dom}(\Omega) = \Phi(\mathrm{Ker}\,\Psi) = \Phi(\Psi^{-1}(\bar{A}_\Phi))$, then given $h \in \Psi^{-1}(\bar{A}_\Phi)$ there exists $h' \in \mathrm{Ker}\,\Psi$ such that $h - h' \in \mathrm{Ker}\,\Phi$. Thus, $\Psi^{-1}(\bar{A}_\Phi) \subset \mathrm{Ker}\,\Phi + \mathrm{Ker}\,\Psi$. Hence, if $a \in \bar{A}_\Phi$ and $h \in H(\bar{A})$ such that $\Psi(h) = a$, we have $h = h^1 + h^2$, $\Phi(h^1) = \Psi(h^2) = 0$. Therefore, $a = \Psi(h^1) \in \Psi(\mathrm{Ker}\,\Phi)$.

Conversely, if (Φ, Ψ) is compatible, then, by Theorem 3, we only need to show that $\bar{A}_\Phi \hookrightarrow \bar{A}_\Psi$. But if $a \in \bar{A}_\Phi$, then $a = \Phi(h_a) = \Psi(g) \in \bar{A}_\Psi$ and $\|g\| \lesssim \|h_a\| \lesssim \|a\|_\Phi$. □

Remark 4. If the couple of interpolators (Φ,Ψ) is not almost compatible, we may define

$$\mathrm{Dom}(\Omega_{\bar A}) := \{a \in \bar A_\Phi : \Omega_{\bar A} a \in \bar A_{\Psi,(\Phi)}\}$$

and

$$\|a\|_D := \|a\|_\Phi + \|\Omega(a)\|_{\Psi,(\Phi)}.$$

Then we still have $\mathrm{Dom}(\Omega_{\bar A}) = \bar A_{\Phi,(\Psi)}$ and $\|a\|_D \simeq \|a\|_{\Phi,(\Psi)}$.

Indeed, if $a \in \bar A_{\Phi,(\Psi)}$, then $a = \Phi(f)$, $\Psi(f) = 0$ and $\|f\| \simeq \|a\|_{\Phi,(\Psi)}$. Since $\Omega(a) = \Psi(h_a)$, we get $\Omega(a) = \Psi(h_a - f)$ and $\Phi(h_a - f) = 0$. Thus, $\Omega(a) \in \bar A_{\Psi,(\Phi)}$ and $\|\Omega(a)\|_{\Psi,(\Phi)} \le \|h_a - f\| \lesssim \|a\|_\Phi + \|a\|_{\Phi,(\Psi)}$. Hence, $\|a\|_D = \|a\|_\Phi + \|\Omega(a)\|_{\Psi,(\Phi)} \lesssim \|a\|_{\Phi,(\Psi)}$.

Conversely, let $a \in \mathrm{Dom}(\Omega_{\bar A})$. Since $\Omega(a) \in \bar A_{\Psi,(\Phi)}$, we have $\Omega(a) = \Psi(h)$ with $\Phi(h) = 0$. Then $\Phi(h_a - h) = a$, $\Psi(h_a - h) = 0$, and it follows that $a \in \bar A_{\Phi,(\Psi)}$ and

$$\|a\|_{\Phi,(\Psi)} \le \|h_a - h\| \lesssim \|a\|_\Phi + \|\Omega(a)\|_{\Psi,(\Phi)} \lesssim \|a\|_D.$$

2.5. Range

Other important sets related with the Ω-operator are the range spaces:

Definition. $\mathrm{Rang}(\Omega_{\bar A}) := \{\Omega_{\bar A} a : a \in \bar A_\Phi\}$, endowed with the norm

$$\|x\|_R := \inf\{\|a\|_\Phi : \Omega_{\bar A} a = x\}.$$

In general, $\mathrm{Rang}(\Omega_{\bar A})$ is not a linear space and it depends on the almost optimal selection used to define Ω. It is easy to check that $\lambda x \in \mathrm{Rang}(\Omega_{\bar A})$ and $\|\lambda x\|_R = |\lambda|\,\|x\|_R$, if $x \in \mathrm{Rang}(\Omega_{\bar A})$.

We shall also consider $\bar A_\Phi + \mathrm{Rang}(\Omega_{\bar A})$ with

$$\|x\|_+ := \inf\{\|a\|_\Phi + \|\Omega_{\bar A} b\|_R : x = a + \Omega_{\bar A} b,\ a, b \in \bar A_\Phi\}.$$

It follows from the definitions that $\mathrm{Rang}(\Omega_{\bar A}) \hookrightarrow \bar A_\Psi$ and $\bar A_\Phi + \mathrm{Rang}(\Omega_{\bar A}) \hookrightarrow \bar A_\Phi + \bar A_\Psi$. In fact, for $x = \Omega_{\bar A} a$ with $\|a\|_\Phi \lesssim \|x\|_R$ we have

$$\|x\|_\Psi = \|\Psi(h_a)\|_\Psi \lesssim \|h_a\| \lesssim \|x\|_R.$$

We may also define $\bar A_{\Psi,(\Phi)} + \mathrm{Rang}(\Omega_{\bar A})$ in a similar way.

Theorem 4. (a) $\bar{A}_{\Psi,(\Phi)} + \mathrm{Rang}(\Omega_{\bar{A}}) = \bar{A}_\Psi$ *with equivalent norms. Hence, if* (Ψ, Φ) *is compatible, then* $\bar{A}_\Psi = \bar{A}_\Phi + \mathrm{Rang}(\Omega_{\bar{A}})$ *with equivalent "norms".*

(b) *If* (Φ, Ψ) *is almost compatible, then* $\bar{A}_\Psi \hookrightarrow \bar{A}_\Phi + \mathrm{Rang}(\Omega_{\bar{A}})$ *and, for any bounded linear operator* $T : \bar{A} \to \bar{B}$, $T : \bar{A}_\Phi + \mathrm{Rang}(\Omega_{\bar{A}}) \to \bar{B}_\Phi + \mathrm{Rang}(\Omega_{\bar{B}})$ *is bounded.*

Proof. (a) If $x = x_1 + x_2 \in \bar{A}_{\Psi,(\Phi)} + \mathrm{Rang}(\Omega)$ with $\|x\|_{\bar{A}_{\Psi,(\Phi)}+\mathrm{Rang}(\Omega)} \simeq \|x_1\|_{\Psi,(\Phi)} + \|x_2\|_R$ and $\|x_2\|_R \simeq \|x_3\|_\Phi$ with $\Omega(x_3) = x_2$, we can consider

$$x_1 = \Psi(f), \quad \Phi(f) = 0, \quad \|f\| \lesssim \|x_1\|_{\Psi,(\Phi)},$$
$$x_2 = \Omega(x_3), \quad x_3 \in \bar{A}_\Phi, \quad \|x_3\|_\Phi \lesssim \|x_2\|_R,$$

and

$$\Omega(x_3) = \Psi(h_{x_3}), \quad \Phi(h_{x_3}) = x_3, \quad \|h_{x_3}\| \lesssim \|x_3\|_\Phi.$$

It follows that $x = \Psi(f + h_{x_3}) \in \bar{A}_\Psi$ and

$$\|x\|_\Psi \le \|f\| + \|h_{x_3}\| \lesssim \|x_1\|_{\Psi,(\Phi)} + \|x_2\|_R.$$

Conversely, if $x \in \bar{A}_\Psi$, then $x = \Psi(f)$ with $\|f\| \lesssim \|x\|_\Psi$. Since $\Omega(\Phi(f)) = \Psi(h)$ with $\Phi(h) = \Phi(f)$ and $\|h\| \lesssim \|f\|$, we have $x - \Omega(\Phi(f)) = \Psi(f-h)$ with $\Phi(f-h) = 0$. Then $x - \Omega(\Phi(f)) \in \bar{A}_{\Psi,(\Phi)}$ and $\|x - \Omega(\Phi(f))\|_{\Psi,(\Phi)} \lesssim \|f\|_{H(\bar{A})}$. Therefore,

$$x = x - \Omega(\Phi(f)) + \Omega(\Phi(f)) \in \bar{A}_{\Psi,(\Phi)} + \mathrm{Rang}(\Omega)$$

with

$$\|x\|_{\bar{A}_{\Psi,(\Phi)}+\mathrm{Rang}(\Omega)} \lesssim \|f\| \lesssim \|x\|_\Psi.$$

(b) Since (Φ, Ψ) is almost compatible, $\bar{A}_{\Psi,(\Phi)} \hookrightarrow \bar{A}_\Phi$ and it follows from (a) that

$$\bar{A}_\Psi = \bar{A}_{\Psi,(\Phi)} + \mathrm{Rang}(\Omega_{\bar{A}}) \hookrightarrow \bar{A}_\Phi + \mathrm{Rang}(\Omega_{\bar{A}}).$$

Let $T : \bar{A} \to \bar{B}$. For any $x = a + \Omega_{\bar{A}} b \in \bar{A}_\Phi + \mathrm{Rang}(\Omega_{\bar{A}})$ with $\|a\|_\Phi + \|\Omega_{\bar{A}} b\|_R \lesssim \|x\|_+$ and $\|b\|_\Phi \lesssim \|\Omega_{\bar{A}} b\|_R$ we have $\|a\|_\Phi + \|b\|_\Phi \lesssim \|x\|_+$. It follows that

$$Tx = (Ta + [T, \Omega]b) + \Omega_{\bar{B}} Tb \in \bar{B}_\Phi + \mathrm{Rang}(\Omega_{\bar{B}})$$

with

$$\begin{aligned}\|Tx\|_+ &\le (\|T\|_{\Phi,\Phi} + \|[T,\Omega]\|_{\Phi,\Phi})(\|a\|_\Phi + \|b\|_\Phi)\\ &\le (1+\varepsilon)^2 (\|T\|_{\Phi,\Phi} + \|[T,\Omega]\|_{\Phi,\Phi}) \|x\|_+.\end{aligned}$$

Thus $\|T\|_{\bar{A}_\Phi + \mathrm{Rang}(\Omega_{\bar{A}}), \bar{B}_\Phi + \mathrm{Rang}(\Omega_{\bar{B}})} \le \|T\|_{\Phi,\Phi} + \|[T,\Omega]\|_{\Phi,\Phi}$. □

2.6. Twisted sums

Let us look at the relations of our commutators with KALTON's work in [Ka1] and [Ka2].

A *derivation* on a Banach space X is an operator

$$\Omega : X \to L$$

from X to a Hausdorff topological linear space L such that $X \hookrightarrow L$ satisfying the following conditions:

1. Ω is continuous at $0 \in X$.
2. Ω is homogeneous ($\Omega(\lambda x) = \lambda\Omega(x)$, hence $\Omega(0) = 0$)).
3. Ω is quasi-additive ($\|\Omega(x+y) - \Omega(x) - \Omega(y)\|_X \lesssim \|x\|_X + \|y\|_X$).

In Kalton's work, X is a Köthe space and $L = L_0$, the space of measurable functions.

The corresponding *derived space* is

$$X \oplus_\Omega X := \{(x, y) \in L \times L : \|(x, y)\|_\Omega := \|x\|_X + \|\Omega(x) - y\|_X < \infty\}.$$

Hence, $(x, y) \in X \oplus_\Omega X$ if and only if $x \in X$, $\Omega(x) - y \in X$.

Proposition 2. *$X \oplus_\Omega X$ is a quasi-Banach space, $X \oplus_\Omega X \hookrightarrow L \times L$, and*

$$0 \to X \xrightarrow{j} X \oplus_\Omega X \xrightarrow{q} X \to 0,$$

where $j(x) := (0, x)$ is an isometry and $q(x, y) := x$.

Proof. It follows from the subadditivity of Ω that

$$\begin{aligned} \|(x_1 + x_2, y_1 + y_2)\|_\Omega &\lesssim \|(x_1, y_1)\|_\Omega + \|(x_2, y_2)\|_\Omega \\ &\quad + \|\Omega(x_1 + x_2) - \Omega(x_1) - \Omega(x_2)\|_X \\ &\lesssim \|(x_1, y_1)\|_\Omega + \|(x_2, y_2)\|_\Omega \end{aligned}$$

and $\|\cdot\|_\Omega$ is a quasi-norm.

Let $\|(x_n, y_n)\|_\Omega \to 0$. Since $\Omega(x_n) \to 0$ and $\Omega(x_n) - y_n \to 0$ in L, it follows that $y_n \to 0$ and $x_n \to 0$ in L. Thus, $X \oplus_\Omega X \hookrightarrow L \times L$.

The linear subspace $F := j(X) = \{(0, x) : x \in X\}$ of $X \oplus_\Omega X$ is closed (if $(0, y_n) \to (x, y)$ in $X \oplus_\Omega X$, then $x = 0$ and $\|(0, y)\|_\Omega = \|y\|_X < \infty$) and it is complete ($j$ is an isometry), and so is $(X \oplus_\Omega X)/F$. But completeness is a three-space property, and $X \oplus_\Omega X$ will be also complete. □

Since $0 \to X \xrightarrow{j} X \oplus_\Omega X \xrightarrow{q} X \to 0$, $X \oplus_\Omega X$ is a *twisted sum* of X and X.

Every operator $\Omega_{\bar{A}}$ associated with a couple (Φ, Ψ) of interpolators is a derivation on $\bar{A}_\Phi$, since $\Omega_{\bar{A}} : \bar{A}_\Phi \to \Sigma(\bar{A})$ is continuous at 0, by (10), homogeneous and quasi-additive (Lemma 1).

As in [CJMR], we associate with every $T \in \mathcal{L}(\bar{A}; \bar{B})$ the operator $\tilde{T}(a, b) = (Ta, Tb)$.

Theorem 5. *The following properties are equivalent:*

(a) $[T, \Omega] : \bar{A}_\Phi \to \bar{B}_\Phi$, *bounded.*

(b) $\tilde{T} : \bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi \to \bar{B}_\Phi \oplus_\Omega \bar{B}_\Phi$, *bounded.*

Moreover, if (Φ, Ψ) *is compatible, then*

$$\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi = \mathrm{Im}(\Phi_{\bar{A}}, \Phi_{\bar{B}}),$$

where $(\Phi_{\bar{A}}, \Phi_{\bar{B}})f := (\Phi_{\bar{A}} f, \Phi_{\bar{B}} f)$.

P r o o f. Let $(a, b) \in \bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi$. Then

$$\begin{aligned} \|\tilde{T}(a, b)\|_{\bar{B}_\Phi \oplus_\Omega \bar{B}_\Phi} &= \|Ta\|_{\bar{B}_\Phi} + \|\Omega Ta - Tb\|_{\bar{B}_\Phi} \\ &\le C\|a\|_{\bar{A}_\Phi} + \|\Omega Ta - T\Omega a\|_{\bar{B}_\Phi} + \|T(\Omega a - b)\|_{\bar{B}_\Phi} \\ &\le C\big(\|a\|_{\bar{A}_\Phi} + \|\Omega a - b\|_{\bar{A}_\Phi}\big) = C\|(a, b)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi}. \end{aligned}$$

Conversely, let $a \in \bar{A}_\Phi$. Then

$$\begin{aligned} \|[T, \Omega]a\|_{\bar{B}_\Phi} &= \|\Omega Ta - T\Omega a\|_{\bar{B}_\Phi} \le \|(Ta, T\Omega a)\|_{\bar{B}_\Phi \oplus_\Omega \bar{B}_\Phi} \\ &= \|\tilde{T}(a, \Omega a)\|_{\bar{B}_\Phi \oplus_\Omega \bar{B}_\Phi} \le C\|(a, \Omega a)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi} = C\|a\|_{\bar{A}_\Phi}. \end{aligned}$$

Let now $\mathcal{E} := \{(a, b) : a = \Phi(f),\ b = \Psi(f),\ f \in H(\bar{A})\}$ endowed with the natural norm, and let $(a, b) \in \bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi$. Then $\Omega a = \Psi(h_a)$ with $\Phi(h_a) = a$, $\|h_a\|_{H(\bar{A})} \le C\|a\|_{\bar{A}_\Phi}$, $b - \Omega a = \Phi(g) = \Psi(h)$ with $\Phi(h) = 0$ and $\|h\|_{H(\bar{A})} \le C\|b - \Omega a\|_{\bar{A}_\Phi}$.

Therefore, $a = \Phi(h_a + h)$, $b = b - \Omega a + \Omega a = \Psi(h_a + h)$; thus, $(a, b) \in \mathcal{E}$ and

$$\|(a, b)\|_{\mathcal{E}} \le \|h_a + h\|_{H(\bar{A})} \le C\big(\|a\|_{\bar{A}_\Phi} + \|b - \Omega a\|_{\bar{A}_\Phi}\big) = C\|(a, b)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi}.$$

Let now $(a, b) \in \mathcal{E}$ and set $a = \Phi(h)$, $b = \Psi(h)$ and $\|h\|_{H(\bar{A})} \le C\|(a, b)\|_{\mathcal{E}}$. Then $a \in \bar{A}_\Phi$, $\Omega a = \Psi(h_a)$ and $\Omega a - b = \Psi(h_a - h)$, with $\Phi(h_a - h) = 0$. Therefore, $\Omega a - b \in \bar{A}_\Phi$ and

$$\begin{aligned} \|(a, b)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi} &= \|a\|_{\bar{A}_\Phi} + \|\Omega a - b\|_{\bar{A}_\Phi} \lesssim \|h\|_{H(\bar{A})} + \|h_a - h\|_{H(\bar{A})} \\ &\lesssim \|h\|_{H(\bar{A})} \lesssim \|(a, b)\|_{\mathcal{E}}. \end{aligned}$$

□

R e m a r k 5. It was observed in [Ka2] that for any derivation Ω on X which has an almost optimal selection $x \in X \mapsto y_x \in L$ (in the sense that $\|\Omega(x) - y_x\|_X \le c\|x\|_X$ for some constant $c > 0$) and for any operator T of L such that $T : X \to X$, the conditions

(a) $[T, \Omega] : X \to X$, bounded,
(b) $\tilde{T} : X \oplus_\Omega X \to X \oplus_\Omega X$, bounded

are equivalent.

It is also shown in [Ka2] that, for super-reflexive Köthe spaces X, many derivations (all "real centralizers") are Ω-operators associated with the complex interpolation method, $X = [X_0, X_1]_{1/2}$. In this case, $X \oplus_\Omega X$ is normable.

3. The complex method

3.1. The complex commutator theorem

Let S and R be two analytic functionals on the strip $\boldsymbol{S}$, such as δ_ϑ and δ'_ϑ. They are linear and bounded on the spaces $\mathcal{F}(\bar{A})$ and, by defining $\mathcal{F}(T)f = T \circ f$, (S, R) is a couple of interpolators on these functional spaces. We can consider the Lions-Schechter interpolation methods (cf. [Li] and [Sc]) such as

$$[\bar{A}]_S = S(\mathcal{F}(\bar{A})).$$

For a fixed almost optimal selection $a \in [\bar{A}]_S \mapsto h_a \in \mathcal{F}(\bar{A})$, the corresponding Ω-operator will be

$$\Omega^C_{\bar{A}}(a) = R(h_a),$$

and the commutator theorem (Theorem 2) reads

$$[T, \Omega^C] : [\bar{A}]_S \to [\bar{B}]_{R,(S)},$$

which turns into

$$[T, \Omega^C] : [\bar{A}]_S \to [\bar{B}]_S$$

if (S, R) is almost compatible.

In any case, by Remark 4,

$$\mathrm{Dom}(\Omega^C) = [\bar{A}]_{S,(R)}$$

if $\mathrm{Dom}(\Omega^C) := \{a \in [\bar{A}]_S : \Omega(a) \in [\bar{A}]_{R,(S)}\}$.

3.2. The basic example

The couple $(\delta_\vartheta, \delta'_\vartheta)$ of interpolators corresponds to the R. Rochberg and G. Weiss construction [RW] associated with the complex Calderón interpolation method (see Section 2.2).

Theorem 6. *The couple* $(\delta_\vartheta, \delta'_\vartheta)$ *is compatible and* $\Omega^C(a) := h'_a(\vartheta)$ *satisfies*

(a) $[T, \Omega^C] : \bar{A}_{[\vartheta]} \to [\bar{B}]_{[\vartheta]}$,

(b) $\mathrm{Dom}(\Omega^C_{\bar{A}}) = \{x = f(\vartheta) : f \in \mathcal{F}(\bar{A}),\ f'(\vartheta) = 0\}$,
$\|x\|_D = \inf\{\|f\| : f \in \mathcal{F}(\bar{A}),\ f(\vartheta) = x,\ f'(\vartheta) = 0\}$,

(c) $\bar{A}_{\delta'_\vartheta} = \bar{A}_{[\vartheta]} + \mathrm{Rang}(\Omega^C)$.

P r o o f. If $g \in \mathcal{F}(\bar{A})$ and $\delta_\vartheta(g) = g(\vartheta) = 0$, then $\delta'_\vartheta(g) = g'(\vartheta) = f(\vartheta)$ with $f(z) = \varphi'(\vartheta)g(z)/\varphi(z)$, where φ is a conformal mapping from the strip $\boldsymbol{S} = \{z \in \mathbb{C} : 0 < \Re z < 1\}$ onto the unit disk $\boldsymbol{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $\varphi(\vartheta) = 0$. We have $f \in \mathcal{F}(\bar{A})$ with $\|f\| = |\varphi'(\vartheta)|\|g\|$. Conversely, for every $f \in \mathcal{F}(\bar{A})$,

$$g := \frac{\varphi f}{\varphi'(\vartheta)} \in \mathcal{F}(\bar{A})$$

with $\|g\| = \|f\|/|\varphi'(\vartheta)|$, $\delta_\vartheta(g) = g(\vartheta) = 0$, and $\delta'_\vartheta(g) = g'(\vartheta) = f(\vartheta) = \delta_\vartheta(f)$. □

If F_f is as in Example 1, then

$$F'_f(\vartheta) = p\Big(\frac{1}{p_1} - \frac{1}{p_0}\Big) f \log \frac{|f|}{\|f\|_p}.$$

Hence, we can obtain the following:

E x a m p l e 3. An Ω-operator for $[L^{p_0}, L^{p_1}]_\vartheta = L^p$ is (equivalent to)

$$\Omega f = f \log \frac{|f|}{\|f\|_p},$$

which is the non-linear operator (7).

Similarly, if F_f is as in Example 2, then

$$F'_f(\vartheta) = \Big(\log \frac{\omega_0}{\omega_1}\Big) f$$

and we obtain:

E x a m p l e 4. An Ω-operator for $[L^p(\omega_0, E), L^p(\omega_1, E)]_\vartheta = L^p(\omega, E)$ is the linear operator

$$\Omega f = \Big(\log \frac{\omega_0}{\omega_1}\Big) f.$$

3.3. Application to pointwise multipliers

In order to guess a condition on $b \in L^1_{\rm loc}(\mathbb{R})$ to obtain a commutator theorem for M_b on $L^p(\mathbb{R})$, let us denote

$$\|b\|_* := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b_Q|\, dx \quad \Big(b_Q := \frac{1}{|Q|} \int_Q b\Big)$$

and assume that $[M_b, H]$ defines a bounded operator on some $L^p(\mathbb{R})$ ($1 < p < \infty$). We shall see that this implies $\|b\|_* \leq A\|[M_b, H]\|$ ($A > 0$ is a constant).[1]

We want to estimate $|Q|^{-1} \int_Q |b - b_Q|$ when Q is an interval and, by translation invariance, we may assume $Q = (-r, r)$. If

$$\Gamma(x) := \chi_Q(x) \operatorname{sgn}(b(x) - b_Q),$$

then

$$\begin{aligned} |Q|\, |b(x)\chi_Q(x) - \chi_Q(x) b_Q| &= |Q|\Gamma(x)\chi_Q(x)\Big(b(x) - \int_Q b(y) \frac{dy}{|Q|}\Big) \\ &= \int_Q (b(x) - b(y))\Gamma(x)\, dy \\ &= \int_Q \frac{b(x) - b(y)}{x - y} \big(x\Gamma(x)\chi_Q(y) - y\Gamma(x)\chi_Q(y)\big)\, dy \\ &= [M_b, H]\big(x\Gamma(x)\chi_Q - \mathrm{Id}\,\Gamma(x)\chi_Q\big) \\ &= x\Gamma(x)[M_b, H](\chi_Q) - \Gamma(x)[M_b, H](\mathrm{Id}\,\chi_Q). \end{aligned}$$

Hence,

$$|Q| \int_Q |b - b_Q| \lesssim \|x\Gamma(x)\|_{p'} \|\chi_Q\|_p + \|\Gamma\|_{p'} \|y\chi_Q(y)\|_p \simeq |Q|^2$$

and $\|b\|_* < \infty$.

This leads to consider the space

$$BMO := \{f \in L^1_{\rm loc}(\mathbb{R}^n) : \|f\|_* < \infty\}.$$

Let us recall that, if $1 < p < \infty$, then there exists a constant $C_0 > 0$ such that, whenever $\|b\|_* < C_0$, $\omega := \mathrm{e}^b$ is an A_p-weight (i.e., the Hardy-Littlewood maximal function is bounded on $L^p(\omega)$), and then

$$K : L^p(\omega) \to L^p(\omega)$$

where K is a Calderón-Zygmund operator (such as the Hilbert transform H if $n = 1$, and the Riesz transforms $R_j = T_{m_j}$ with $m_j(y) = -iy_j/\|y\|$ if $n \geq 1$). See [GR].

[1] The same result holds for $b \in L^1_{\rm loc}(\mathbb{R}^n)$ with $n > 1$, if $[M_b, R_i]$ are bounded, where R_i, $1 \leq i \leq n$, denote the Riesz transforms. See [CRW].

Theorem 7 (Coifman, Rochberg and Weiss). *If K is a Calderón-Zygmund operator on $\mathbb{R}^n$ and $b \in BMO$, then $[K, M_b] : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.*

P r o o f. By homogeneity, we can assume that $\|b/2\|_{BMO} < C_0$, so that $\mathrm{e}^{b/2}$ and $\mathrm{e}^{-b/2}$ are A_p-weights. Therefore,

$$K : (L^p(\mathrm{e}^{b/2}), L^p(\mathrm{e}^{-b/2})) \to (L^p(\mathrm{e}^{b/2}), L^p(\mathrm{e}^{-b/2})). \tag{12}$$

By Example 4, $\Omega_{(L^p(\mathrm{e}^{b/2}), L^p(\mathrm{e}^{-b/2}))} f = M_b f$, and so (12) together with Theorem 2 yield the result, since $[L^p(\omega_0), L^p(\omega_1)]_{1/2} = L^p(\omega_0^{1/2}\omega_1^{1/2})$. □

The BMO norm is related with the atomic Hardy space[2] by the duality defined through $\langle f, g\rangle := \int fg$. More precisely,

$$\|f\|_{H^1} = \sup_{\substack{\varphi \in \mathcal{S},\\ \|\varphi\|_* \leq 1}} |\langle \varphi, f\rangle|.$$

As an application of Theorem 7 given by R. Coifman, R. Rochberg and G. Weiss we have:

Corollary 1. *If K is a Calderón-Zygmund operator, K^* its adjoint, $1 < p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, then*

$$(Kf)g - fK^*g \in H^1(\mathbb{R}^n).$$

P r o o f. If $b \in \mathcal{S}$, then

$$\left| \int b\big((Kf)g - fK^*g\big) \right| = \left| \int g([b, K]f) \right| \lesssim \|b\|_* \|f\|_p \|g\|_{p'}.$$

Now, by duality,

$$\|(Kf)g - fK^*g\|_{H^1} = \sup_{\|b\|_* \leq 1} \left| \int b\big((Kf)g - fK^*g\big) \right| \lesssim \|f\|_p \|g\|_{p'},$$

which is finite. □

[2] There are several descriptions for the Hardy space. Namely, $H^1(\mathbb{R}^n) = \{f \in L^1 : f = \sum_{j=1}^\infty \alpha_j a_j, \sum_{j=1}^\infty |\alpha_j| < \infty, |a_j| \leq \frac{1}{|I_j|}\chi_{I_j}, \int a_j = 0 \ (I_j \text{ interval})\}$ and $\|f\|_{H^1} := \inf\{\sum_{j=1}^\infty |\alpha_j| : f = \sum_{j=1}^\infty \alpha_j a_j\}$. See [GR].

If $n = 1$, then H^1 is the image of $L^2 \times L^2$ for the bilinear mapping $(f, g) \mapsto (Hg)f + f(Hg)$. There is no similar fact for Calderón-Zygmund operators if $n = 2$ (or any $n > 1$), but an open problem is whether H^1 is the image of the Sobolev space $W^{1,2}(\mathbb{R}^2)^2$ by the Jacobian,

$$J(u^1, u^2) := \det(\nabla \boldsymbol{u}) = \partial_x u^1 \partial_y u^2 - \partial_x u^2 \partial_y u^1.$$

Corollary 1 can be used to obtain the following "Jacobian Theorem" (see [CLMS], where it is also proved that $\overline{[J(W^{1,2}(\mathbb{R}^2)^2)]} = H^1(\mathbb{R}^2)$).

Theorem 8. *If* $\boldsymbol{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)^2$ *and* $\nabla \boldsymbol{u} \in L^2(\mathbb{R}^2)^{2\times 2}$*, then* $J(\boldsymbol{u}) \in H^1(\mathbb{R}^n)$.

Proof. We may write $J(\boldsymbol{u}) = \nabla u^1 \cdot \boldsymbol{B} = \boldsymbol{E} \cdot \boldsymbol{B}$ with

$$\boldsymbol{B} \in L^2(\mathbb{R}^2)^2,\ \operatorname{div} \boldsymbol{B} = 0; \quad \boldsymbol{E} \in L^2(\mathbb{R})^2,\ \operatorname{curl} \boldsymbol{E} = 0.$$

It follows from this last condition that

$$E^1 = R_1 f,\ E^2 = R_2 f \quad (f \in L^2(\mathbb{R}))$$

and $\operatorname{div} \boldsymbol{B} = 0$ implies $R_1 B^1 + R_2 B^2 = \operatorname{div}(-\Delta)^{-1/2} \boldsymbol{B} = (-\Delta)^{-1/2} \operatorname{div} \boldsymbol{B} = 0$. Finally, an application of Corollary 1 ensures that

$$J(\boldsymbol{u}) = \boldsymbol{E} \cdot \boldsymbol{B} = \sum_{j=1}^{2} (R_j f) B^j = \sum_{j=1}^{2} \big((R_j f) B^j - f(R_j B^j)\big) \in H^1(\mathbb{R}^n).$$

□

3.4. The use of vector function spaces

As observed in [CCS3], some results by C. Segovia and J. L. Torrea (see [ST1] and [ST2]) concerning commutators of maximal functions can be obtained from Theorem 2.

Recall that if $1 < p < \infty$ and $\omega \in A_p(\mathbb{R}^n)$, then the Hardy-Littlewood maximal function $\mathcal{M}$ is bounded on $L^p(\omega)$.

Theorem 9. *If* $b \in BMO(\mathbb{R}^n)$*, then the maximal operator*

$$Sf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |b(y) - b(x)|\, |f(y)|\, dy$$

is bounded in $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$*. Here,* B *represents a ball in* $\mathbb{R}^n$*.*

P r o o f. We may assume that the BMO norm of b is sufficiently small and so $\mathrm{e}^b \in A_p$. Hence, $\mathcal{M}$ is bounded on $L^p(\mathrm{e}^{\pm b})$.

If we define

$$Tf(x) := \left(\frac{1}{|B|} \int_B f(x-y)h(y)\,dy \right)_{B \ni 0, \|h\|_\infty \le 1},$$

then

$$\|Tf\|_{L^p(\mathrm{e}^{\pm b}, L^\infty)} \le \|\mathcal{M}f\|_{L^p(\mathrm{e}^{\pm b})} \lesssim \|f\|_{L^p(\mathrm{e}^{\pm b})}$$

and, by Theorem 6, $[T,\Omega] : [\bar{A}]_{1/2} \to [\bar{B}]_{1/2}$ with $\bar{A} = \big(L^p(\mathrm{e}^b), L^p(\mathrm{e}^{-b})\big)$ and $\bar{B} = \big(L^p(\mathrm{e}^b, L^\infty), L^p(\mathrm{e}^{-b}, L^\infty)\big)$.

However, since $[\bar{A}]_{1/2} = L^p$ and $[\bar{B}]_{1/2} = L^p(L^\infty)$ (see Example 2), from Theorem 2 we obtain

$$[T,\Omega] : L^p \to L^p(L^\infty),$$

and, as in Theorem 7, we have $\Omega_{\bar{A}} f = bf$ and $\Omega_{\bar{B}}(f_B)_{B\ni 0} = (bf_B)_{B\ni 0}$.

Finally, $S : L^p \to L^p$ is equivalent to the boundedness of $[T,\Omega]$ since

$$[T,\Omega]f(x) = \left(\frac{1}{|B|} \int_B \{b(x-y) - b(x)\} f(x-y)h(y)\,dy \right)_{B \ni 0, \|h\|_\infty \le 1}$$

and $\|[T,\Omega]f(x)\|_\infty = Sf(x)$. □

As an application we recover the following result obtained in [MS] by real interpolation.

Corollary 2. *If $b \in BMO(\mathbb{R}^n)$, $b \ge 0$ and $1 < p < \infty$, then*

$$[\mathcal{M}, M_b] : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

P r o o f. If $0 \le b \in BMO$, then

$$\frac{1}{|B|} \int_B b(x-y)|f(x-y)|\,dy$$
$$= b(x) \frac{1}{|B|} \int_B |f(x-y)|\,dy + \frac{1}{|B|} \int_B (b(x-y) - b(x))|f(x-y)|\,dy$$

and $[\mathcal{M}, M_b]f \le Sf$. □

We set $S_I := T_{\chi_I}$. If a collection of intervals $I_j \subset \mathbb{R}$ is given, we shall consider the corresponding Fourier multipliers $S_j := S_{I_j}$. A weighted extension of the Littlewood-Paley inequality proved by J. L. Rubio de Francia in [Ru] allows us to obtain another commutator estimate.

Theorem 10. *Let $\{I_j\}_{j\in J}$ be a collection of disjoint intervals. If $b \in BMO(\mathbb{R})$ and $2 < p < \infty$, then*

$$\Big\|\Big(\sum_{j\in J}\big|[S_j, M_b]f\big|^2\Big)^{1/2}\Big\|_p \leq C\big\|f\big\|_p.$$

P r o o f. As seen in [Ru], if $p > 2$ and $\omega \in A_{p/2}$, then

$$\Big\|\Big(\sum_j |S_{I_j}f|^2\Big)^{1/2}\Big\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}$$

or, equivalently,

$$Tf(x) := (S_j f)_j$$

satisfies $T : L^p(\omega) \to L^p(\omega, \ell^2)$.

Hence, assuming that $\|b\|_*$ is small, we have $T : L^p(\mathrm{e}^{\pm b}) \to L^p(\mathrm{e}^{\pm b}, \ell^2)$ and the proof continues as in Theorem 9. □

3.5. Lions-Schechter complex methods

On the same functional spaces $\mathcal{F}(\bar{A})$ we may consider higher order derivatives $\delta_\vartheta^{(m)}$.

A couple $(\delta_\vartheta^{(m)}, \delta_\vartheta^{(n)})$ is not necessarily almost compatible, but we have the following special cases.

Theorem 11. *Let $\bar{A} = (L^{p_0}(\mu), L^{p_1}(\mu))$ on a given measure space. Then $(\delta_\vartheta^{(n)}, \delta_\vartheta^{(n+1)})$ are almost compatible couples of interpolators over $\mathcal{F}(\bar{A})$ for all n and*

$$\Omega(u) := \frac{n + \big|\log|u|\big|^n}{1 + \big|\log|u|\big|^n}\, u \log|u|$$

defines an Ω-operator for this couple.

P r o o f. For $p = p(\vartheta)$, it is known that (cf. [CC1])

$$[L^{p_0}, L^{p_1}]_{\delta_\vartheta^{(m)}} = L^p(\log L)^{-p} = \Big\{u : \int \Big(\frac{|u(x)|}{1 + \log|u(x)|}\Big)^p d\mu(x) < \infty\Big\}.$$

That the interpolators are almost compatible, i.e.,

$$[L^{p_0}, L^{p_1}]_{\delta_\vartheta^{(n+1)}, (\delta_\vartheta^{(n)})} \hookrightarrow [L^{p_0}, L^{p_1}]_{\delta_\vartheta^{(n)}},$$

is proved in [CCMS]. It is done by associating the function

$$W(z) := \|F(\vartheta)\|_p \frac{F(\vartheta)}{|F(\vartheta)|}\left[\frac{|F(\vartheta)|}{\|F(\vartheta)\|_p}\right]^{((1-z)/p_0+z/(p_1))p}$$

with every almost optimal $f \in \mathcal{F}(\bar{A})$ such that $x = f^{(n+1)}(\vartheta)$ and $f^{(n)}(\vartheta) = 0$ ($\|f\| \simeq \|x\|_{\delta_\vartheta^{(n+1)},(\delta_\vartheta^{(n)})}$).

Let ψ be a conformal mapping from $\boldsymbol{S}$ onto the unit disc such that $\psi(\vartheta) = 0$, $G := (f - W)/\psi \in \mathcal{F}(\bar{A})$. Since $(\psi G)(\vartheta) = f(\vartheta) - W(\vartheta) = 0$,

$$W^{(n)}(\vartheta) = -(\psi G)^{(n)} \in [L^{p_0}, L^{p_1}]_{\delta_\vartheta^{(n-1)}}.$$

On the other hand,

$$x = W^{(n+1)}(\vartheta) + (\psi G)^{(n+1)}(\vartheta),$$

where $(\psi G)^{(n+1)}(\vartheta) \in [L^{p_0}, L^{p_1}]_{\delta_\vartheta^{(n)}}$ since $(\psi G)(\vartheta) = 0$, and $W^{(n+1)}(\vartheta) \in L^p(\log L)^{-p}$ since a direct computation shows that

$$\|h\|_{L^p(\log L)^{-p}} \leq \|\psi G\| \lesssim \|f(\vartheta)\|_p \lesssim \|f\|.$$

To obtain an almost optimal selection h_u for $u \in [L^{p_0}, L^{p_1}]_{\delta_\vartheta^{(n)}}$, let φ be again an analytic bounded function on $\boldsymbol{S}$ such that $\varphi^{(j)}(\vartheta) = 0$ if $j \in \{0, \dots, n-1, n+1\}$ but $\varphi^{(n)}(\vartheta) = 1$. Then define

$$h_u(z) := \operatorname{sgn} u \frac{\left[c_n(\operatorname{sgn} \log |u|)^n + \varphi(z)\right]|u|^{((1-z)/p_0+z/p_1)p}}{1 + |\log |u||^n}$$

with $c_n = (p/p_1 - p/p_0)^{-n}$.

Hence a possible choice for Ω is

$$\begin{aligned}\Omega u &= h_u^{(n+1)}(\vartheta) \\ &= \frac{|\log |u||^n \log |u|\left(\frac{p}{p_1} - \frac{p}{p_0}\right) + n \log |u|\left(\frac{p}{p_1} - \frac{p}{p_0}\right)}{1 + |\log |u||^n} u \\ &= \left(\frac{p}{p_1} - \frac{p}{p_0}\right)\frac{n + |\log |u||^n}{1 + |\log |u||^n} u \log |u|.\end{aligned}$$

□

Theorem 12. *Let $\bar{A} = (L^p(\omega_0), L^p(\omega_1))$ on a given measure space. Then $(\delta_\vartheta^{(n)}, \delta_\vartheta^{(n+1)})$ are almost compatible couples of interpolators over $\mathcal{F}(\bar{A})$ for all n and*

$$\Omega(u) := \frac{n+1+\big|\log(\omega_0/\omega_1)\big|^n}{1+\big|\log(\omega_0/\omega_1)\big|^n}\Big(\log\frac{\omega_0}{\omega_1}\Big)u$$

defines an Ω-operator for these Banach couples.

Proof. It is known (cf. [CC2]) that, if $1 \le p < \infty$,

$$[L^p(\omega_0), L^p(\omega_1)]_{\delta_\vartheta^{(n)}} = L^p(\omega_0^{1-\vartheta}\omega_1^\vartheta \widehat{\omega}^{-np})$$

with $\omega = \omega_0^{1-\vartheta}\omega_1^\vartheta\widehat{\omega}^{-np}$, where $\widehat{\omega} = 1 + \big|\log(\omega_0/\omega_1)\big|$.

To prove that

$$[L^p(\omega_0), L^p(\omega_1)]_{\delta_\vartheta^{(n+1)},(\delta_\vartheta^{(n)})} \hookrightarrow [L^p(\omega_0), L^p(\omega_1)]_{\delta_\vartheta^{(n)}},$$

again, as in [CCMS], we associate the function

$$W(z) := \Big(\frac{\omega_0}{\omega_1}\Big)^{z-\vartheta}$$

with every almost optimal $f \in \mathcal{F}(\bar{A})$ such that $x = f^{(n+1)}(\vartheta)$ and $f^{(n)}(\vartheta) = 0$ ($\|f\| \simeq \|x\|_{\delta_\vartheta^{(n+1)},(\delta_\vartheta^{(n)})}$), and then, if ψ is as in Theorem 11, we set $G := (f - W)/\psi \in \mathcal{F}(\bar{A})$. Since $(\psi G)(\vartheta) = f(\vartheta) - W(\vartheta) = 0$,

$$W^{(n)}(\vartheta) = f(\vartheta)\Big(\log\frac{\omega_0}{\omega_1}\Big)^n = -(\psi G)^{(n)} \in [L^p(\omega_0), L^p(\omega_1)]_{\delta_\vartheta^{(n-1)}}.$$

On the other hand,

$$x = W^{(n+1)}(\vartheta) + (\psi G)^{(n+1)}(\vartheta) = f(\vartheta)\Big(\log\frac{\omega_0}{\omega_1}\Big)^{(n+1)} + v,$$

and $v \in [L^p(\omega_0), L^p(\omega_1)]_{\delta_\vartheta^{(n)}}$ since $(\psi G)(\vartheta) = 0$. If h_0 is such that

$$h := f(\vartheta)\Big(\log\frac{\omega_0}{\omega_1}\Big)^{n+1} = h_0 \log\frac{\omega_0}{\omega_1},$$

then

$$h_0 \in L^p\big(\omega_0^{1-\vartheta}\omega_1^\vartheta\widehat{\omega}^{-(n-1)p}\big) = L^p(\omega)$$

and an easy computation shows that

$$\|h\|_{L^p(\omega)} \leq \|h_0\|_{\delta_\vartheta^{n-1}} \leq \|\psi G\| \lesssim \|f(\vartheta)\|_p \lesssim \|f\|.$$

To obtain an almost optimal selection for $u \in [L^p(\omega_0), L^p(\omega_1)]_{\delta_\vartheta^{(n)}}$, choose φ again as in Theorem 11, such that $\varphi^{(j)}(\vartheta) = 0$ if $j \in \{0, \ldots, n-1, n+1\}$ and $\varphi^{(n)}(\vartheta) = 1$. Then

$$h_u(\vartheta) := \frac{\big(\operatorname{sgn} \log(\omega_0/\omega_1)\big)^n + \varphi(z)}{1 + |\log(\omega_0/\omega_1)|^n} \Big(\frac{\omega_0}{\omega_1}\Big)^{z-\vartheta} u$$

satisfies $h_u^{(n)}(\vartheta) = u$ and $\|h_u\| \leq \|u\|_{L^p(\omega_0^{1-\vartheta}\omega_1^{\vartheta}\widehat{\omega}^{-np})}$, and

$$h_u^{(n+1)}(\vartheta) = \frac{n + 1 + |\log(\omega_0/\omega_1)|^n}{1 + |\log(\omega_0/\omega_1)|^n} \Big(\log \frac{\omega_0}{\omega_1}\Big) u = \Omega(u).$$

□

R e m a r k 6. The same result, with the same proof, holds for the Banach couples $(L^p(\omega_0, E), L^p(\omega_1, E))$ of vector-valued functions.

Let us apply the previous theorems to obtain some extensions of the commutator estimates of pointwise multipliers (Theorem 7) and of the Littlewood-Paley inequality (Theorem 10).

Proposition 3. *Let K be a Calderón-Zygmund operator on $\mathbb{R}^m$, $b \in BMO$ and let $\alpha \geq 0$ be a constant. Then*

$$[K, M_b] : L^p((1 + |b|)^{-\alpha}) \to L^p((1 + |b|)^{-\alpha}).$$

P r o o f. Let

$$M_n f = b \frac{n + 1 + |b|^n}{1 + |b|^n} f = M_b f + \frac{nb}{1 + |b|^n} f.$$

We have

$$[K, M_n]f = [K, M_b]f - n[K, M_b]\Big(\frac{f}{1 + |b|^n}\Big) - nbK\Big(\frac{f}{1 + |b|^n}\Big) + \frac{nb}{1 + |b|^n} Kf.$$

It follows from Theorem 12 that

$$[K, M_n] : L^p\Big(\frac{1}{1 + |b|^n}\Big) \to L^p\Big(\frac{1}{1 + |b|^n}\Big)$$

and from Theorem 7 that

$$[K, M_b]\Big(\frac{f}{1+|b|^n}\Big) \in L^p \subset L^p\Big(\frac{1}{1+|b|^n}\Big) \quad \text{if} \quad f \in L^p\Big(\frac{1}{1+|b|^n}\Big).$$

Moreover, since $b/(1+|b|^n)$ is bounded, we have

$$K\Big(\frac{f}{1+|b|^n}\Big) \in L^p, \quad b\frac{f}{1+|b|^n} \in L^p\Big(\frac{1}{1+|b|^n}\Big)$$

and

$$\frac{b}{1+|b|^n}K(f) \in L^p \subset L^p\Big(\frac{1}{1+|b|^n}\Big).$$

Thus

$$[K, M_b] : L^p\Big(\frac{1}{1+|b|^n}\Big) \to L^p\Big(\frac{1}{1+|b|^n}\Big),$$

and it follows by interpolation that

$$[K, M_b] : L^p\Big(\frac{1}{1+|b|^\alpha}\Big) \to L^p\Big(\frac{1}{1+|b|^\alpha}\Big)$$

for any $\alpha \geq 0$. □

In the same way we obtain:

Proposition 4. *Let $2 < p < \infty$ and $b \in BMO(\mathbb{R})$. Then*

$$\Big\| \Big(\sum_j \big| [S_{I_j}, b] f \big|^2 \Big)^{1/2} \Big\|_{L^p((1+|b|)^{-\alpha})} \leq C_\alpha \|f\|_{L^p((1+|b|)^{-\alpha})}$$

for any collection $(I_j)_j$ of disjoint intervals and for every $\alpha \geq 0$.

4. Real methods

We assume $0 < \vartheta < 1$ and $1 \leq p \leq \infty$.

The real interpolation methods are the abstract counterpart of the Marcinkiewicz interpolation theorem. As shown by J. Peetre, they admit equivalent definitions, using the K-functional or the J-functional.

4.1. The J-method

For a given Banach couple $\bar{A}$, we denote $\Delta(\bar{A}) = A_0 \cap A_1$ and

$$J(t, a) = J(t, a; \bar{A}) = \max(\|a\|_0, t\|a\|_1) \quad (a \in \Delta(\bar{A}),\ t > 0).$$

The *J-method* corresponds to the interpolator Φ^J on the functional spaces

$$H^J(\bar{A}) = \{u : \mathbb{R}^+ \to \Delta(\bar{A}) \text{ measurable} : \|u\| = \|t^{-\vartheta} J(t; u(t))\|_{L^p(dt/t)} < \infty\}$$

defined as

$$\Phi^J_{\bar{A}}(u) = \int_0^\infty u(t)\,\frac{dt}{t} \quad (\Sigma(\bar{A})\text{-valued}).$$

Again, $H^J(T) = T \circ u$ if $T \in \mathcal{L}(\bar{A}; \bar{B})$.

In this case,

$$\bar{A}_{\Phi^J} = \left\{a \in \Sigma(\bar{A}) : a = \int_0^\infty u(t)\,\frac{dt}{t},\ u \in H^J(\bar{A})\right\} = \bar{A}_{\vartheta,p}$$

and we consider an almost optimal selection

$$u_a \in H^J(\bar{A}), \quad \int_0^\infty u_a(t)\,\frac{dt}{t} = a, \quad \|u_a\| \le c\|a\|_{\vartheta,p}.$$

To define the Ω-operator we need to associate Φ^J with another interpolator Ψ^J on the same functional spaces $H^J(\bar{A})$. By relating the J-method with the complex method, we shall see that

$$\Psi^J_{\bar{A}}(u) := \int_0^\infty (\log t) u(t)\,\frac{dt}{t}$$

is a convenient definition.

The relationship is given by the mixed reiteration formula due to J.-L. Lions (cf. [BL, Theorem 4.2.7]),

$$[\bar{A}_{\vartheta_0,p_0}, \bar{A}_{\vartheta_1,p_1}]_\lambda = \bar{A}_{\vartheta,p} \tag{13}$$

with $\vartheta = (1-\lambda)\vartheta_0 + \lambda\vartheta_1$. One inclusion is obtained by means of

$$f_a(z) := \int_0^\infty t^{(\vartheta_1-\vartheta_0)(z-\lambda)} u_a(t)\,\frac{dt}{t}$$

for every $a \in \bar{A}_{\vartheta,p}$; then

$$\Omega^C(a) = f_a'(\lambda) = (\vartheta_1 - \vartheta_0)\int_0^\infty (\log t) u_a(t)\,\frac{dt}{t}.$$

Thus, we are led to define $\Psi^J_{\bar{A}}(u) = \int_0^\infty (\log t) u(t)\,\frac{dt}{t}$.

Theorem 13. *The couple of interpolators* (Φ^J, Ψ^J) *is compatible and*

$$\bar{A}_{\Psi^J} = \left\{ a = \int_0^\infty v(t)\,\frac{dt}{t} : v : \mathbb{R}^+ \to \Delta(\bar{A}) \text{ measurable}, \right.$$
$$\left. \left\| t^{-\vartheta} \frac{J(t, v(t))}{1 + |\log t|} \right\|_{L^p(dt/t)} < \infty \right\}$$

with

$$\|a\|_{\Psi^J} = \inf \left\| t^{-\vartheta} \frac{J(t, v(t))}{1 + |\log t|} \right\|_{L^p(dt/t)},$$

the infimum being taken over all representations $a = \int_0^\infty v(t)\,\frac{dt}{t}$.

Proof. Ψ^J is well defined and bounded from $H(\bar{A})$ to $\Sigma(\bar{A})$:

$$\begin{aligned}
\|\Psi^J_{\bar{A}}(u)\|_\Sigma &\le \int_0^1 |\log t|\,\|u\|_0 \frac{dt}{t} + \int_1^\infty (\log t)\|u\|_1 \frac{dt}{t} \\
&\le \int_0^1 |\log t|\, J(t, u(t))\,\frac{dt}{t} + \int_1^\infty \frac{\log t}{t} J(t, u(t))\,\frac{dt}{t} \\
&\le \left[\left(\int_0^1 (|\log t|\, t^\vartheta)^{p'} \frac{dt}{t} \right)^{1/p'} + \left(\int_1^\infty \left(\frac{\log t}{t^{1-\vartheta}} \right)^{p'} \frac{dt}{t} \right)^{1/p'} \right] \\
&\quad \times \|t^{-\vartheta} J(t, u)\|_{L^p(dt/t)} \\
&= C\|u\|_{H(\bar{A})}.
\end{aligned}$$

To see that Φ^J and Ψ^J are compatible, first assume that $\int_0^\infty u(t)\,\frac{dt}{t} = 0$, (with $u \in H(\bar{A})$) and define

$$F(z) = \int_0^\infty t^z u(t)\,\frac{dt}{t}$$

on the strip $\{z \in \mathbb{C} : -\varepsilon < \Re z < \varepsilon\}$ with ε such that $0 < \vartheta - \varepsilon < \vartheta + \varepsilon < 1$. It is easily seen that $F(\pm\varepsilon \pm ti) \in \bar{A}_{\vartheta\pm\varepsilon,p}$, $\|F(\pm\varepsilon \pm ti)\|_{\vartheta\pm\varepsilon,p} \le C\|u\|_{H(\bar{A})}$, and that, since $F(0) = 0$, we have

$$F'(0) = \int_0^\infty (\log t)\, u(t)\,\frac{dt}{t} = \Psi^J_{\bar{A}}(u) \in \left[\bar{A}_{\vartheta-\varepsilon,p}, \bar{A}_{\vartheta+\varepsilon,p}\right]_0 = \bar{A}_{\vartheta,p} = \bar{A}_{\Phi^J}$$

with $\|\Psi^J_{\bar{A}}(u)\|_{\Phi^J} \le C\|u\|_{H(\bar{A})}$.

For the converse inclusion $\operatorname{Im}\Phi^J_{\bar A} \hookrightarrow \Psi^J_{\bar A}(\operatorname{Ker}\Phi^J_{\bar A})$, let $u \in H(\bar A)$ be given and consider $v(t) = u(t) - u(\mathrm{e}\,t)$. Then we have $v \in H(\bar A)$, $F^J_{\bar A}(v) = \int_0^\infty v(t)\,\frac{dt}{t} = 0$,

$$\begin{aligned}\Psi^J_{\bar A}(v) &= \int_0^\infty (\log t)(u(t) - u(\mathrm{e}\,t))\,\frac{dt}{t} \\ &= \int_0^\infty (\log t)\,u(t)\,\frac{dt}{t} - \int_0^\infty \left(\log\frac{t}{\mathrm{e}}\right) u(t)\,\frac{dt}{t} \\ &= \int_0^\infty u(t)\,\frac{dt}{t} = \Phi^J_{\bar A}(u)\end{aligned}$$

and $\|v\|_{H(\bar A)} \le C\|u\|_{H(\bar A)}$. For the last part of the theorem, let

$$B = \left\{ a = \int_0^\infty v(t)\,\frac{dt}{t} : \left(\int_0^\infty \left(\frac{J(t,v(t))}{t^\vartheta(1+|\log t|)}\right)^p \frac{dt}{t}\right)^{1/p} < \infty \right\}$$

and let $a \in \bar A_{\Psi^J}$ be such that $a = \int_0^\infty (\log t)u(t)\,\frac{dt}{t}$ and

$$\left(\int_0^\infty \left(\frac{J(t,u(t))}{t^\vartheta}\right)^p \frac{dt}{t}\right)^{1/p} \le \|a\|_{\Psi^J} + \varepsilon.$$

Then $a = \int_0^\infty v(t)\,\frac{dt}{t}$, $v(t) = (\log t)\,u(t)$, and

$$\|a\|_B \le \left\| t^{-\vartheta}\frac{J(t,v(t))}{1+|\log t|}\right\|_{L^p(dt/t)} \le \|t^{-\vartheta}J(t,u(t))\|_{L^p(dt/t)} \le \|a\|_{\Psi^J} + \varepsilon.$$

To show that $B \hookrightarrow \bar A_{\Psi^J}$, we observe that for any $a \in B$,

$$a = \int_0^\infty v(t)\,\frac{dt}{t} = \int_0^\infty \frac{v(t)}{1+|\log t|}\,\frac{dt}{t} + \int_0^\infty \frac{|\log t|\,v(t)}{1+|\log t|}\,\frac{dt}{t} = b + c,$$

where

$$\left\| t^{-\vartheta}\frac{J(t,v(t))}{1+|\log t|}\right\|_{L^p(dt/t)} \le \|a\|_B + \varepsilon$$

and $b \in \bar A_{\Phi^J} \hookrightarrow \bar A_{\Psi^J}$ such that $\|b\|_\Psi \le C\|b\|_\Phi \le C(\|a\|_B + \varepsilon)$. On the other hand,

$$c = \int_0^\infty (\log t)\,w(t)\,\frac{dt}{t}$$

with $w(t) = \operatorname{sgn}(\log t)v(t)/(1+|\log t|)$, and $\Phi_{\vartheta,p}(t,w(t)) \le \|a\|_B + \varepsilon$. It follows that $c \in \bar A_{\Psi^J}$ and $\|c\|_\Psi \le \|a\|_B + \varepsilon$. Hence $a \in \bar A_{\Psi^J}$ and $\|a\|_\Psi \le C\|a\|_B$. □

Let now Ω^J be the Ω-operator associated with the pair (Φ^J, Ψ^J) and with our given almost optimal selection $a \mapsto u_a$. By Theorem 2,

$$[T, \Omega^J] : \bar{A}_{\vartheta,p} \to \bar{B}_{\vartheta,p}$$

if $T \in \mathcal{L}(\bar{A}; \bar{B})$, and, as an application of Theorem 4,

$$\bar{A}_{\vartheta,p;J} + \mathrm{Rang}(\Omega^J_{\bar{A}}) = \bar{A}_{\Psi^J}$$

and

$$\mathrm{Dom}(\Omega^J_{\bar{A}}) = \left\{ a = \int_0^\infty u(t)\,\frac{dt}{t} : \int_0^\infty (\log t)\, u(t)\,\frac{dt}{t} = 0,\ u \in H(\bar{A}) \right\} \tag{14}$$

which has the following description (see [CJM] for another proof):

Theorem 14.

$$\mathrm{Dom}(\Omega^J_{\bar{A}}) = \Big\{ a = \int_0^\infty u(t)\,\frac{dt}{t} : u \in H(\bar{A}),$$
$$\big\| t^{-\vartheta}(1 + |\log t|) J(t, u(t)) \big\|_{L^p(dt/t)} < \infty \Big\}.$$

P r o o f. Let $\mathcal{E}$ be the right-hand side space with the natural norm. Choose $a = \int_0^\infty u(t)\,\frac{dt}{t} \in \mathcal{E}$ such that $u \in H(\bar{A})$ and

$$\big\| t^{-\vartheta}(1 + |\log t|) J(t, u(t)) \big\|_{L^p(dt/t)} \le C\|a\|_{\mathcal{E}}.$$

Then $\Phi_{\vartheta,p}(J(t,u(t)) < \infty$ and $a \in \bar{A}_{\Phi^J}$. Also $a = \int_0^\infty u_a(t)\,\frac{dt}{t}$, $\Omega^J_{\bar{A}} a = \int_0^\infty (\log t) u_a(t)\,\frac{dt}{t}$ and then $\int_0^\infty (u(t) - u_a(t))\,\frac{dt}{t} = 0$. Thus, since $(\log t)u(t) \in H(\bar{A})$, we obtain

$$b = \int_0^\infty (\log t) u(t)\,\frac{dt}{t} \in \bar{A}_{\Phi^J}, \quad \Omega^J_{\bar{A}} a - b = \int_0^\infty (\log t)(u_a(t) - u(t))\,\frac{dt}{t} \in \bar{A}_{\Phi^J},$$

hence

$$\Omega^J_{\bar{A}} a = \int_0^\infty (\log t) u_a(t)\,\frac{dt}{t} \in \bar{A}_{\Phi^J},$$

and

$$\begin{aligned} \|a\|_D = \|a\|_\Phi + \|\Omega^J_{\bar{A}} a\|_\Phi &\le \|u\|_{H(\bar{A})} + \|\Omega^J_{\bar{A}} a - b\|_\Phi + \|b\|_\Phi \\ &\le \|u\|_{H(\bar{A})} + C\|u - u_a\|_{H(\bar{A})} + C\|u\|_{H(\bar{A})} \le C\|a\|_{\mathcal{E}}. \end{aligned}$$

To show that $\mathrm{Dom}(\Omega^J_{\bar{A}}) \hookrightarrow \mathcal{E}$, we shall use the following facts:

(i) $(\bar{A}_{\vartheta_0,q_0}, \bar{A}_{\vartheta_1,q_1})$ is a partial retract of the couple $(l^{q_0}(2^{-n\vartheta_0}), l^{q_1}(2^{-n\vartheta_1}))$ (cf. [Cw] and [CJM]). Recall that $\bar{A}$ is a partial retract of $\bar{B}$ if, for every $x \in \Sigma(\bar{A})$, there exists a pair of bounded linear operators, $F_x : \bar{A} \to \bar{B}$ and $P_x : \bar{B} \to \bar{A}$, such that $P_x \circ F_x x = x$ and $\sup_x \|F_x\| < \infty$, $\sup_x \|P_x\| < \infty$.

(ii) $[l^p(2^{-n\vartheta_0}), l^p(2^{-n\vartheta_1})]^{\delta'_\mu} = l^p((1+|n|)2^{-n\vartheta})$, with $\vartheta = (1-\mu)\vartheta_0 + \mu\vartheta_1$ (cf. [CC2]).

(iii) $(\bar{A}_{\vartheta_0,q_0}, \bar{A}_{\vartheta_1,q_1})_{\varphi_\mu,p} = \bar{A}_{\varphi_\vartheta,p}$, with $\varphi_\lambda(x) = (1+|\log x|)x^{-\lambda}$ (cf. [G]).

Let now $a \in \mathrm{Dom}(\Omega^J_{\bar{A}})$ and $u \in H(\bar{A})$ be such that

$$a = \int_0^\infty u(t)\,\frac{dt}{t}, \quad \int_0^\infty (\log t)u(t)\,\frac{dt}{t} = 0, \quad \left\|t^{-\vartheta} J(t, u(t))\right\|_{L^p(dt/t)} < \infty.$$

Then

$$F(z) = \int_0^\infty t^z u(t)\,\frac{dt}{t} \in \mathcal{F}(\bar{A}_{\vartheta-\varepsilon,p}, \bar{A}_{\vartheta+\varepsilon,p})$$

on the strip $\vartheta - \varepsilon < \Re z < \vartheta + \varepsilon$, $F(0) = a$ and $F'(0) = 0$. Hence,

$$a \in [\bar{A}_{\vartheta-\varepsilon,p}, \bar{A}_{\vartheta+\varepsilon,p}]^{\delta'_0} \quad \text{with} \quad \|a\|_{[\bar{A}_{\vartheta-\varepsilon,p}, \bar{A}_{\vartheta+\varepsilon,p}]^{\delta'_0}} \le \|F\|_{\mathcal{F}} \le \|u\|_{H(\bar{A})}.$$

Keeping the notation of (i), let

$$F : (\bar{A}_{\vartheta-\varepsilon,p}, \bar{A}_{\vartheta+\varepsilon,p}) \to (l^p(2^{-n(\vartheta-\varepsilon)}), l^p(2^{-n(\vartheta+\varepsilon)}))$$

and

$$P : (l^p(2^{-n(\vartheta-\varepsilon)}), l^p(2^{-n(\vartheta+\varepsilon)})) \to (\bar{A}_{\vartheta-\varepsilon,p}, \bar{A}_{\vartheta+\varepsilon,p})$$

be a pair of bounded linear mappings such that $PFa = a$. Then, by interpolation (we use (ii) and (iii)),

$$F : [\bar{A}_{\vartheta-\varepsilon,p}, \bar{A}_{\vartheta+\varepsilon,p}]^{\delta'_0} \to l^p((1+|n|)2^{-n\vartheta})$$

and

$$P : l^p((1+|n|)2^{-n\vartheta}) \to \bar{A}_{\varphi_\vartheta,p}.$$

Hence,

$$\|a\|_{\varphi_\vartheta,p} = \|PFa\|_{\varphi_\vartheta,p} \le C\|Fa\|_{l^p((1+|n|)2^{-n\vartheta})} \le C\|a\|_{[\bar{A}_{\vartheta-\varepsilon,p}, \bar{A}_{\vartheta+\varepsilon,p}]^{\delta'_0}},$$

we have $a \in \bar{A}_{\varphi_\vartheta,p}$, and there exists $v \in H(\bar{A})$ such that

$$a = \int_0^\infty v(t)\,\frac{dt}{t} \quad \text{and} \quad \left\|t^{-\vartheta}(1+|\log t|)J(t, v(t))\right\|_{L^p(dt/t)} < \infty.$$

Thus $a \in \mathcal{E}$. □

R e m a r k 7. From (i)–(iii) we obtain the reiteration property

$$[\bar{A}_{\vartheta_0,q_0}, \bar{A}_{\vartheta_1,q_1}]^{\delta'_\mu} = \bar{A}_{\varphi_\vartheta,q},$$

where $1/q = (1-\vartheta)/q_0 + \vartheta/q_1$.

4.2. The *K*-method

It is well known that the real interpolation method can be equivalently defined by the *K-functional*

$$K(t,x) = K(t,x;\bar{A}) := \inf_{x=a_0+a_1}(\|a_0\|_0 + t\|a_1\|_1),$$

and $\|x\|_{\vartheta,p} \simeq \|x\|_{\vartheta,p;K}$, where

$$\|x\|_{\vartheta,p;K} := \|t^{-\vartheta}K(t,x)\|_{L^p(dt/t)}.$$

Given an *almost optimal decomposition* $x = a_0(t)+a_1(t)$ for the K-functional such that $K(t,x) \simeq \alpha(a_0,a_1)$, where

$$\alpha(a_0,a_1)(t) := \|a_0(t)\|_0 + t\|a_1(t)\|_1,$$

we say that

$$P_t^{\bar{A}}x = P_t x := a_0(t)$$

is an *almost optimal projection.* We may assume that $a_0 : \mathbb{R} \to A_0$ is continuous by choosing $a_0(2^n)$ for each $n \in \boldsymbol{Z}$ and then $a_0(t)$ linear on $[2^n, 2^{2^{n+1}}]$, but it is not always linear in x; if it can be chosen linear, then $\bar{A}$ is said to be *quasi-linearizable.*

E x a m p l e 5. For the K-functional of the couple (L^1, L^∞),

$$P_t f := (|f| - f^*(t))\,\mathrm{sgn}\, f\chi_{\{|f|>f^*(t)\}}$$

defines an almost optimal projection. See [BL], Theorem 5.2.1.

R e m a r k 8. Let $\bar{A}$ be a couple of Banach function spaces on a measure space. Given $f \in \Sigma(\bar{A})$ let $f = f_0 + f_1$ be an almost optimal decomposition for the K-functional. If we take $E_f(t) := \{\omega : |f_0(\omega)| > |f_1(\omega)|\}$, we obtain another almost optimal projection of the type

$$P_t f = f\chi_{E_f(t)}.$$

Obviously, $|f\chi_{E_f(t)}| \le 2|f_0|$ and $|f\chi_{E_f(t)^c}| \le 2|f_1|$, and so

$$\|f\chi_{E_f(t)}\|_0 + t\|f\chi_{E_f(t)^c}\|_1 \le 2\alpha(f_0, f_1) \lesssim K(t, f).$$

R e m a r k 9. If we have an almost optimal projection $P_t^{\bar{A}}$ for a couple $\bar{A}$ and $\bar{X} = (\bar{A}_{\vartheta_0,q_0}, \bar{A}_{\vartheta_1,q_1})$ $(\vartheta_0 < \vartheta_1)$ is obtained by interpolation, then the Holmstedt reiteration formula

$$K(t^\varrho, x; \bar{X}) \simeq \Big(\int_0^t \big(s^{-\vartheta_0} K(s, x; \bar{A})\big)^{q_0} \frac{ds}{s} \Big)^{1/q_0} + t^\varrho \Big(\int_t^\infty \big(s^{-\vartheta_1} K(s, x; \bar{A})\big)^{q_1} \frac{ds}{s} \Big)^{1/q_1}$$

with $\varrho = \vartheta_1 - \vartheta_0$ (cf. [BL, Theorem 3.6.1]) allows to obtain the almost optimal projection for $\bar{X}$,

$$P_t^{\bar{X}} x := P_{t^{1/\varrho}}^{\bar{A}} x.$$

To see that this *K-method* is defined by an interpolator, it is natural to consider the functional spaces

$$H^K(\bar{A}) = \{(a_0, a_1) : \mathbb{R}^+ \to A_0 \times A_1 : a_0, a_1 \text{ measurable, } a_0(t) + a_1(t) = \text{const., } \|(a_0, a_1)\| < \infty\}$$

with

$$\|(a_0, a_1)\| := \|t^{-\vartheta} \alpha(a_0, a_1)(t))\|_{L^p(dt/t)}$$

and $H^K(T)(a_0, a_1) := (T \circ a_0, T \circ a_1)$.

Then the functional

$$\Phi^K(a_0, a_1) := a_0 + a_1$$

acting on $H^K(\bar{A})$ defines an interpolator on these functional spaces and obviously $\bar{A}_{\Phi^K} = \bar{A}_{\vartheta,p;K} = \bar{A}_{\vartheta,p}$.

An almost optimal decomposition for the K-functional is clearly also an almost optimal selection

$$a_x = (a_0, a_1) = (P_t x, (I - P_t)x) \in H^K(\bar{A}),$$
$$a_0(t) + a_1(t) = x, \ \|(a_0, a_1)\| \le c\|x\|_{\vartheta,p},$$

for this K-method.

Again we can look for an appropriate second interpolator Ψ^K by observing that in the reiteration result (13), if for any

$$a = f(\lambda) \in [\bar{A}_{\vartheta_0,p_0}, \bar{A}_{\vartheta_1,p_1}]_\lambda$$

we choose an almost optimal $f = f_a \in \mathcal{F}(\bar{A}_{\vartheta_0,p_0}, \bar{A}_{\vartheta_1,p_1})$ for the complex method and define

$$g_t(z) = t^{(z-\lambda)(\vartheta_1-\vartheta_0)} f(z),$$

then $g_t(\lambda) = a$ and

$$a = \int_{-\infty}^{+\infty} g_t(is) P_0(\lambda, s)\, ds + \int_{-\infty}^{+\infty} g_t(1+is) P_1(\lambda, s)\, ds = a_0(t) + a_1(t)$$

with $a_j(t) \in \bar{A}_{\vartheta_j,p_j}$ $(j = 0, 1)$. Now we can compute the derivative (as in [CCMS]) and we get

$$\Omega^C(a) = f'(\lambda) = \int_0^1 a_0(t)\, \frac{dt}{t} - \int_1^\infty a_0(t)\, \frac{dt}{t}.$$

This suggests the definition

$$\Psi_{\bar{A}}^K(a_0, a_1) := \int_0^1 a_0(t)\, \frac{dt}{t} - \int_1^\infty a_0(t)\, \frac{dt}{t}.$$

Theorem 15. *The couple* (Φ^K, Ψ^K) *is a compatible pair of interpolators such that*

$$\Omega^K(x) := \int_0^1 a_0(t)\, \frac{dt}{t} - \int_1^\infty a_0(t)\, \frac{dt}{t}$$

for our almost optimal selection, and $\Omega^K = -\Omega^J$ *for a convenient almost optimal selection for the* J*-method.*

Proof. Let $\Phi^K(a_0, a_1) = 0 = a_0(t) + a_1(t)$ with $(a_0, a_1) \in H^K(\bar{A})$. Then,

$$\Psi^K(a_0, a_1) = \int_0^\infty a_0(t)\, \frac{dt}{t},$$

and, since $\Phi_{\vartheta,p}\big(J(t, a_0(t))\big) \le \Phi_{\vartheta,p}\big(\|a_0(t)\|_0 + t\|a_1(t)\|_1\big) = \|(a_0, a_1)\|_H$, we have

$$\Psi^K(a_0, a_1) \in \bar{A}_{\vartheta,p;J} = \bar{A}_{\vartheta,p;K} \quad \text{and} \quad \|\Psi^K(a_0, a_1)\|_\Phi \le \|(a_0, a_1)\|_H.$$

Let now $a \in \bar{A}_\Phi$. We have to find $(b_0, b_1) \in H(\bar{A})$ such that

$$\Phi^K(b_0, b_1) = b_0(t) + b_1(t) \equiv 0 \quad \text{and} \quad \Psi^K(b_0, b_1) = \int_0^\infty b_0(t) \frac{dt}{t} = a.$$

This follows from the Fundamental Lemma of Interpolation Theory (cf. [BL]):

If we discretize, we have to show that there exists a sequence $(b_0^n, b_1^n) \in A_0 \times A_1$ such that

$$b_0^n + b_1^n = 0 \quad \text{and} \quad \sum_{n=-\infty}^{\infty} b_0^n = a.$$

Let $a = a_0^n + a_1^n$, with $\|a_0^n\|_0 + 2^n \|a_1^n\|_1 \le (1+\varepsilon) K(2^n, a)$. We have

$$\lim_{n \to -\infty} \|a_0^n\|_0 = 0 \quad \text{and} \quad \lim_{n \to \infty} \|a_1^n\|_1 = 0.$$

Write $b_0^n = a_0^n - a_0^{n-1}$ and $b_1^n = a_1^n - a_1^{n-1}$. Then $b_0^n + b_1^n = 0$ and

$$K\Big(1, a - \sum_{n=-N}^{M} b_0^n\Big) = K\big(1, a_0^{-N-1} + a_1^M\big) \to 0 \quad \text{as} \quad N, M \to \infty.$$

Hence $\sum_n b_0^n = a$. □

4.3. A big real interpolation method

If

$$Sf(t) := \int_0^t f(s) \frac{ds}{s} + t \int_t^\infty f(s) \frac{ds}{s^2} = \int_0^\infty f(s) \min\Big(1, \frac{t}{s}\Big) \frac{ds}{s},$$

the Calderón operator, we set

$$\sigma(\bar{A}) := \{x \in \Sigma(\bar{A}) : S(K(\cdot, x))(1) < \infty\}. \tag{15}$$

Let us prove that σ is an interpolation method by showing that it may be defined by a convenient interpolator.

For a given Banach couple $\bar{A}$, let $H(\bar{A})$ be the Banach space of all measurable functions

$$(x_0, x_1) : \mathbb{R}^+ \to A_0 \times A_1$$

such that $x_0(t) + x_1(t) = \Phi(x_0, x_1) \in \Sigma(\bar{A})$, constant, and

$$\|(x_0, x_1)\|_H := S(\alpha(x_0, x_1))(1) < \infty$$

(recall that $\alpha(x_0, x_1)(t) = \|x_0(t)\|_0 + t\|x_1(t)\|_1$).

Then $\Phi_{\bar{A}} = \Phi : H(\bar{A}) \to \Sigma(\bar{A})$ where $\Phi(x_0, x_1) = x_0(t) + x_1(t)$ since

$$\begin{aligned}\|\Phi(x_0, x_1)\|_\Sigma &= \|x_0(t) + x_1(t)\|_\Sigma \\ &\le 2\int_1^2 \alpha(x_0, x_1)(s)\,\frac{ds}{s} \\ &\le 2\|(x_0, x_1)\|_H.\end{aligned}$$

If $T \in \mathcal{L}(\bar{A}; \bar{B})$, we define $H(T)(x_0, x_1) := T \circ (x_0, x_1) = (Tx_0, Tx_1)$ and we obtain a bounded linear operator $H(T) : H(\bar{A}) \to H(\bar{B})$ such that $\|H(T)\| \le \|T\|$ and $T \circ \Phi_{\bar{A}} = \Phi_{\bar{B}} \circ H(T)$.

Observe that $\sigma(\bar{A})$ is the image space, $\bar{A}_\Phi = \Phi(H(\bar{A}))$, endowed with the quotient norm

$$\|x\|_\Phi := \inf_{x = x_0(t) + x_1(t)} \|(x_0, x_1)\|_H = S(K(\cdot, x))(1).$$

Indeed, obviously $S(K(\cdot, x))(1) \le \|x\|_\Phi$. On the other hand, if $x \in \sigma(\bar{A})$, we can consider $x = x_0(t) + x_1(t)$ such that $\alpha(x_0, x_1)(t) \le (1 + \varepsilon)K(t, x)$. Then $(x_0, x_1) \in H(\bar{A})$ and $\|x\|_\Phi \le S(\alpha(x_0, x_1))(1) \le (1 + \varepsilon)S(K(\cdot, x))(1)$.

Proposition 5. *If $0 < \vartheta < 1$ and $1 \le q \le \infty$, then $\bar{A}_{\vartheta,q} \hookrightarrow \sigma(\bar{A}) \hookrightarrow \Sigma(\bar{A})$.*

Proof. To verify the first inclusion, we observe that, if $\bar{x} = (x_0, x_1) \in H^K(\bar{A})$ and $\alpha(t) = \alpha(x_0, x_1)(t)$, an application of Hölder's inequality gives

$$\begin{aligned}\|\bar{x}\|_\Phi &= \int_0^1 \frac{\alpha(t)}{t^\vartheta}\, t^\vartheta\, \frac{dt}{t} + \int_1^\infty \frac{\alpha(t)}{t^\vartheta}\, t^{\vartheta-1}\, \frac{dt}{t} \\ &\le C\Big(\int_0^\infty \Big(\frac{\alpha(t)}{t^\vartheta}\Big)^q \frac{dt}{t}\Big)^{1/q} = C\|\bar{x}\|\end{aligned}$$

with $C = (1/\vartheta q')^{1/q'} + (1/((1-\vartheta)q'))^{1/q'}$, and so $\bar{A}_{\vartheta,q} \hookrightarrow \sigma(\bar{A})$. □

Let $\Psi_{\bar{A}} = \Psi : H(\bar{A}) \to \Sigma(\bar{A})$ be a second operator such that $T \circ \Psi_{\bar{A}} = \Psi_{\bar{B}} \circ H(T)$.

If for every $x \in \sigma(\bar{A})$ we choose an almost optimal decomposition for the K-functional, $h_x = (x_0, x_1)$, in the sense that

$$x_0(t) + x_1(t) = x \quad \text{and} \quad \alpha(x_0, x_1)(t) \le cK(t, x) \quad (c = c_{\bar{A}} \ge 1),$$

then $\|h_x\|_H \le c\|x\|_\Phi$. Thus, $x \mapsto h_x$ is an almost optimal selection that has an associated Ω-operator $\Omega(x) = \Psi(h_x)$ for the interpolation method σ.

The following lemma is an abstract commutator theorem with pointwise estimates.

Lemma 2. *Assume that* Ψ *satisfies the following condition: For every* $(x_0, x_1) \in H(\bar{A})$ *such that* $x_0 + x_1 = 0$*, there exists a measurable function*

$$(y_0, y_1) : \mathbb{R}^+ \to A_0 \times A_1$$

with the properties

$$y_0(t) + y_1(t) = \Psi(x_0, x_1) \text{ and } \alpha(y_0, y_1)(t) \leq cS(\alpha(x_0, x_1))(t) \text{ for all } t > 0,$$

where c *is a constant which does not depend on* (x_0, x_1)*.*

Then

$$K(t, [T, \Omega](x)) \leq C\|T\|S(K(\cdot, x))(t). \tag{16}$$

P r o o f. Let $x \in \sigma(\bar{A})$. Then $Tx \in H(\bar{B})$ and for the almost optimal decompositions $h_x \in H(\bar{A})$ and $h_{Tx} \in H(\bar{B})$ we have $\alpha(h_x)(t) \leq cK(t, x)$ and $\alpha(h_{Tx})(t) \leq cK(t, Tx) \leq c\|T\|K(t, x)$.

Then

$$[T, \Omega]x = T\Psi h_x - \Psi h_{Tx} = \Psi_{\bar{B}}(H(T)h_x - h_{Tx}),$$

where $H(T)h_x - h_{Tx} \in H(\bar{B})$ and $\Phi(H(T)h_x - h_{Tx}) = 0$. Hence, there exists $(y_0(t), y_1(t))$ such that $y_0 + y_1 = \Psi(H(T)h_x - h_{Tx})$ and $\alpha(y_0, y_1)(t) \leq cS(\alpha(H(T)h_x - h_{Tx}))(t)$. Thus

$$K(t, [T, \Omega]x) \leq \alpha(y_0, y_1)(t) \leq cS(\alpha(H(T)h_x - h_{Tx}))(t).$$

To estimate the right-hand side, we observe that $\alpha(H(T)h_x - h_{Tx}) \leq 2c\|T\|K(t, x)$ and S is positive. □

If the above estimate (16) holds for some constant $C > 0$, for all $x \in \sigma(\bar{A})$ and all $T \in \mathcal{L}(\bar{A}; \bar{B})$, we say that Ω is *K-commuting*. In the terminology of R. A. DeVore, S. D. Riemenschneider and R. Sharpley (see [DRS]) this means that $[T, \Omega]$ is of generalized weak type $((1, 1), (\infty, \infty))$.

Since S is positive, from condition (16) we obtain

$$\begin{aligned} \|[T, \Omega]x\|_\sigma &= S(K(\cdot, [T, \Omega]x)(1) \\ &\leq C\|T\|S(K(\cdot, x))(1) \\ &\leq C\|T\| \, \|x\|_\sigma \end{aligned}$$

and $[T, \Omega] : \sigma(\bar{A}) \to \bar{B}$. We have also the following:

Proposition 6. *If Ω is K-commuting, then Ω is well defined on the spaces $\bar{A}_{\vartheta,q}$ and*

$$\|[T,\Omega](x)\|_{\vartheta,q} \le c\|T\|\,\|x\|_{\vartheta,q} \tag{17}$$

for all $T \in \mathcal{L}(\bar{A};\bar{B})$ with $c > 0$ independent of T.

P r o o f. By the Minkowski inequality and Hardy's inequalities for averages (cf. [BS]),

$$\begin{aligned}
&\|[T,T_\mu]f\|_{\bar{B}_{\theta,q}} \\
&\le C\|T\|\Big(\Big\|t^{-\theta}\int_0^t \frac{K(s,f;\bar{A})}{s}\,ds\Big\|_{L^q(dt/t)} + \Big\|t^{1-\theta}\int_t^\infty \frac{K(s,f;\bar{A})}{s}\,\frac{ds}{s}\Big\|_{L^q(dt/t)}\Big) \\
&\le \frac{C\|T\|}{\theta(1-\theta)}\,\|f\|_{\bar{A}_{\theta,q}}.
\end{aligned}$$

□

R e m a r k 10. Assume that $\bar{A}$ and $\bar{B}$ are the Gagliardo completions of $\bar{A}'$ and $\bar{B}'$, and that the condition of Lemma 2 holds for $\bar{A}$, so that $\Omega_{\bar{A}}$ is K-commuting. Then $\Omega_{\bar{A}'}$ is also K-commuting.

If $T : \bar{A}' \to \bar{B}'$, then also $T : \bar{A} \to \bar{B}$. In the proof of Lemma 2, $K(t,[T,\Omega];\bar{B}') = K(t,[T,\Omega];\bar{B}) \le c\alpha(y_0,y_1)$ and $K(t,x;\bar{B}) = K(t,x;\bar{B}')$.

R e m a r k 11. If $\bar{A}^r$ is a retract of $\bar{A}$ (i.e., $\mathrm{Id}_{\bar{A}^r} = \mathcal{P}\mathcal{J}$ with $\mathcal{P} : \bar{A} \to \bar{A}^r$ and $\mathcal{J} : \bar{A}^r \to \bar{A}$), then $\Omega_{\bar{A}^r} = \mathcal{P}\Omega_{\bar{A}}\mathcal{J}$.

To obtain concrete examples, we associate with every $\lambda \in L^\infty(\mathbb{R}^+)$ the operator $\Psi_{\bar{A}} : H(\bar{A}) \to \Sigma(\bar{A})$ such that

$$\Psi_{\bar{A}}(x_0,x_1) = \int_0^1 \lambda(t)x_0(t)\,\frac{dt}{t} + \int_1^\infty \lambda(t)x_1(t)\,\frac{dt}{t},$$

which is linear and bounded since

$$\begin{aligned}
\Big\|\int_0^1 \lambda(t)x_0(t)\,\frac{dt}{t}\Big\|_0 &\le \|\lambda\|_\infty \int_0^1 \|x_0(t)\|_0\,\frac{dt}{t} \\
&\le \|\lambda\|_\infty \int_0^1 \alpha(x_0,x_1)(t)\,\frac{dt}{t},
\end{aligned}$$

and similarly

$$\Big\|\int_1^\infty \lambda(t)x_1(t)\,dt\Big\|_1 \le \|\lambda\|_\infty \int_1^\infty \alpha(x_0,x_1)(t)\,\frac{dt}{t^2}.$$

Thus, $\|\Psi_{\bar{A}}(x_0,x_1)\|_\Sigma \le \|\lambda\|_\infty\|(x_0,x_1)\|_H$.

Theorem 16. *For every* $\lambda \in L^\infty(\mathbb{R}^+)$*, the associated operator* $\Omega(x) := \Psi(h_x)$ *is* K*-commuting.*

Proof. Let $\bar{x} = (x_0, x_1) \in H(\bar{A})$ as in Lemma 2. Then $x_1(t) = -x_0(t) \in A_0 \cap A_1$ and

$$\Psi(\bar{x}) = \int_0^1 \lambda(t) x_0(t) \frac{dt}{t} - \int_1^\infty \lambda(t) x_0(t) \frac{dt}{t} = \int_0^\infty \tilde{\lambda}(s) x_0(s) \frac{ds}{s}$$

(we denote $\tilde{\lambda}(t) := \operatorname{sgn}(1-t)\lambda(t)$). Then, if $\Psi(\bar{x}) = y_0(t) + y_1(t)$ is an almost optimal decomposition for the K-functional,

$$\begin{aligned}
\alpha(y_0, y_1)(t) &\le cK(t, \Psi(\bar{x})) = cK\Big(t, \int_0^\infty \tilde{\lambda}(s) x_0(s) \frac{ds}{s}\Big) \\
&\le c \int_0^\infty K\big(t, \tilde{\lambda}(s) x_0(s)\big) \frac{ds}{s} \\
&\le c \int_0^\infty J\big(s, \tilde{\lambda}(s) x_0(s)\big) \min\Big(1, \frac{t}{s}\Big) \frac{ds}{s} \\
&= c \int_0^\infty |\lambda(s)| J(s, x_0(s)) \min\Big(1, \frac{t}{s}\Big) \frac{ds}{s} \\
&\le c \|\lambda\|_\infty S(\alpha(\bar{x}))(t).
\end{aligned}$$

For the last estimate observe that $J(s, x_0(s)) \le \alpha(\bar{x})(s)$. □

4.4. Almost optimal decomposition for approximation spaces

Let $\mathcal{V}$ be a Hausdorff topological linear space and X a Banach subspace of $\mathcal{V}$ with continuous embedding $X \hookrightarrow \mathcal{V}$.

Let us also consider a fixed *approximation family* A_t $(t > 0)$, i.e., a family of non-empty subsets of $\mathcal{V}$ with the following properties:

(a) $A_s \subset A_t$ if $s < t$,
(b) $-A_t = A_t$,
(c) $A_s + A_t \subset A_{s+t}$.

It is clear that $0 \in \bigcap_{t>0} A_t$ and that $A = \bigcup_{t>0} A_t$ is an Abelian group, that will be endowed with the (semi-)norm

$$\|x\|_A = \inf\{t > 0 : x \in A_t\}.$$

Then, as in [PS], we can define the *approximation spaces* $E_{p,q}$, similar to the Lorentz spaces $L^{p,q}$, of all elements $f \in A + X$ such that

$$\|f\|_{E_{p,q}} = \Big(\int_0^\infty \big[t^{1/p} E(f,t)\big]^q \frac{dt}{t}\Big)^{1/q} < \infty,$$

where $E(f,t) = \inf_{a\in A_t} \|f-a\|_X$. By f_t we denote an element in A_t such that

$$\|f-f_t\|_X \le cE(f,t) \tag{18}$$

with $c>1$ independent of $t>0$ and f.

A typical example (see [PS] or [Ni]) appears for $\mathcal{V}=L_0$, the space of all measurable functions on $\mathbb{R}^n$, $X=L^\infty$ and

$$A_t = \{f\in L_0 : \|f\|_0 = |\operatorname{supp} f| \le t\}$$

($|\operatorname{supp} f|$ denotes the measure of the support of f). In this case

$$E(f,t) = f^*(t),$$

the non-increasing rearrangement of f, $E_{p,q}=L^{p,q}$ and we have the Holmstedt formula for couples of Lorentz spaces,

$$\begin{aligned}&K(t^{1/p_0-1/p_1}, f; L^{p_0,q_0}, L^{p_1,q_1})\\&\quad\simeq \Big(\int_0^t \big[(s^{1/p_0} f^*(s)\big]^{q_0}\frac{ds}{s}\Big)^{1/q_0} + t^{1/p_0-1/p_1}\Big(\int_t^\infty \big[s^{1/p_1} f^*(s)\big]^{q_1}\frac{ds}{s}\Big)^{1/q_1},\end{aligned}$$

to estimate the K-functional.

A similar result holds for couples of approximation spaces and gives an estimate for the K-functional:

Theorem 17. *If $(E_{p_0,q_0}, E_{p_1,q_1})$ is a couple of approximation spaces and $p_0<p_1$, then*

$$K(t^{1/p_0-1/p_1}, f; E_{p_0,q_0}, E_{p_1,q_1}) \simeq \|f_t\|_{E_{p_0,q_0}} + t^{1/p_0-1/p_1}\|f-f_t\|_{E_{p_1,q_1}}.$$

Proof. Let $\delta = 1/p_0 - 1/p_1$. It is known (cf. [Ni]) that

$$\begin{aligned}&K(t^\delta, f; E_{p_0,q_0}, E_{p_1,q_1})\\&\quad\simeq \Big(\int_0^t \big[s^{1/p_0} E(f,s)\big]^{q_0}\frac{ds}{s}\Big)^{1/q_0} + t^\delta\Big(\int_t^\infty \big[s^{1/p_1} E(f,s)\big]^{q_1}\frac{ds}{s}\Big)^{1/q_1}.\end{aligned} \tag{19}$$

Let f_t be as in (18). Then we have $E(f_t,s)=0$ when $s>t$, and $E(f_t,s)\le 2cE(f,s)$ when $s\le t$ since $\|f_t-f_s\|\le cE(f,t)+cE(f,s)\le 2cE(f,s)$. Hence,

$$\begin{aligned}\|f_t\|_{E_{p_0,q_0}} &= \Big(\int_0^t \big[s^{1/p_0} E(f_t,s)\big]^{q_0}\frac{ds}{s}\Big)^{1/q_0}\\&\le 2c\Big(\int_0^t \big[s^{1/p_0} E(f,s)\big]^{q_0}\frac{ds}{s}\Big)^{1/q_0}.\end{aligned} \tag{20}$$

On the other hand,

$$t^\delta \|f - f_t\|_{E_{p_1,q_1}} = t^\delta \Big(\int_0^\infty \big[s^{1/p_1} E(f - f_t, s) \big]^{q_1} \frac{ds}{s} \Big)^{1/q_1} = I_1 + I_2$$

with

$$I_1 = t^\delta \Big(\int_0^{2t} \big[s^{1/p_1} E(f - f_t, s) \big]^{q_1} \frac{ds}{s} \Big)^{1/q_1}$$

and

$$I_2 = t^\delta \Big(\int_{2t}^\infty \big[s^{1/p_1} E(f - f_t, s) \big]^{q_1} \frac{ds}{s} \Big)^{1/q_1}.$$

From $E(f - f_t, s) \le \|f - f_t\|_X$ we obtain the estimate

$$\begin{aligned} I_1 &\le (p_1/q_1)^{1/q_1} t^\delta \|f - f_t\|_X 2^{1/p_1} t^{1/p_1} \\ &\le c(p_1/q_1)^{1/q_1} 2^{1/p_1} t^{1/p_0} E(f, t) \\ &\le c(p_1/q_1)^{1/q_1} 2^{1/p_1} \Big(\int_0^t \big[s^{1/p_0} E(f, s) \big]^{q_0} \frac{ds}{s} \Big)^{1/q_0}. \end{aligned}$$

Since $E(f_t, s/2) = 0$ when $s \ge 2t$, from $E(f - f_t, s) \le E(f, s/2) + E(f_t, s/2)$ we have

$$\begin{aligned} I_2 &\le t^\delta \Big(\int_{2t}^\infty \big[s^{1/p_1} E(f, s/2) \big]^{q_1} \frac{ds}{s} \Big)^{1/q_1} \\ &= t^\delta \Big(\int_t^\infty \big[(2s)^{1/p_1} E(f, s) \big]^{q_1} \frac{ds}{s} \Big)^{1/q_1}. \end{aligned}$$

By combining these estimates, (20) and (19), we obtain

$$\|f_t\|_{E_{p_0,q_0}} + t^\delta \|f - f_t\|_{E_{p_1,q_1}} \le CK(t^\delta, f; E_{p_0,q_0}, E_{p_1,q_1}).$$

Obviously, $K(t^\delta, f; E_{p_0,q_0}, E_{p_1,q_1}) \le \|f_t\|_{E_{p_0,q_0}} + t^\delta \|f - f_t\|_{E_{p_1,q_1}}$. □

4.5. A commutator for Fourier multipliers on Besov spaces

The Besov space $B_X^{\sigma,q}$ (or $B_X^{\sigma,q}(\mathbb{R})$), always with $0 < \sigma < \infty$ and $1 \le q < \infty$, is the approximation space

$$B_X^{\sigma,q} := \Big\{ f \in X : \|f\|_{\sigma,q} = \Big(\int_0^\infty \big[r^\sigma E(r, f) \big]^q \frac{dr}{r} \Big)^{1/q} < \infty \Big\}$$

with

$$E(r,f) := d_X(f, V(r)) = \inf_{g\in V(r)} \|f-g\|_X,$$

where $V(0) = 0$ and $V(r) = \{g \in \mathcal{S}' : \operatorname{supp} \widehat{g} \subset [-r, r]\}$.

If $0 < \theta < 1$ and $1 \le q < \infty$, it is known that

$$(B_X^{\sigma_0,q_0}, B_X^{\tilde{\sigma}_0,q_1})_{\theta,q} = B_X^{\sigma,q} \quad (\sigma = (1-\theta)\sigma_0 + \theta\tilde{\sigma}_0)$$

and

$$(X, B_X^{\sigma,r})_{\theta,q} = B_X^{\theta\sigma,q} \tag{21}$$

with equivalent norms (we refer to [BL], [BS], [DL] and [Pe] for general properties of Besov spaces).

To prove a commutator theorem for $\Omega = T_\mu$ on Besov spaces, we need to select the admissible symbols $\mu : \mathbb{R} \to \mathbb{C}$. Let $\delta > 1$ and consider the partition $\Delta(\delta) = \{\Delta_j(\delta)\}_{j\in\mathbb{N}}$ of $\mathbb{R}$ defined by

$$\Delta_j(\delta) = \begin{cases} (-\delta^j, -\delta^{j-1}] \cup [\delta^{j-1}, \delta^j), & \text{if} \quad j > 0 \\ (-1, 1), & \text{if} \quad j = 0 \end{cases}$$

and $\bar{\Delta}_j(\delta) = [-\delta^j, -\delta^{j-1}] \cup [\delta^{j-1}, \delta^j]$ ($\bar{\Delta}_0(\delta) = [-1, 1]$). Then, μ is said to be *admissible* if

$$V(\mu) := \sup_{j\ge 0} \mathrm{VAR}_{\bar{\Delta}_j(\delta)}(\mu) < \infty, \tag{22}$$

where $\mathrm{VAR}_{\bar{\Delta}_j(\delta)}(\mu)$ is the total variation of μ over the closed set $\bar{\Delta}_j(\delta)$,

$$\mathrm{VAR}_{\bar{\Delta}_j(\delta)}(\mu) := \int_{\bar{\Delta}_j(\delta)} |d\mu| = \sup_\pi \sum |\mu(t_k) - \mu(t_{k-1})|$$

with the supremum taken over all partitions π of $\bar{\Delta}_j(\delta)$.

An example of unbounded admissible multiplier is $\log^+ |x|$.

Proposition 7. *Let X be a rearrangement invariant space with the Boyd indices satisfying $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$, and μ a* bounded *admissible multiplier. Then*

$$T_\mu : X \to X$$

with $\|T_\mu\| \le c_X \max(V(\mu), \|\mu\|_\infty)$.

Proof. If $X = L^p$, $1 < p < \infty$, this is the Marcinkiewicz multiplier theorem (cf. [EG] or [St1]). In the general case take $1/p < \underline{\alpha}_X \le \overline{\alpha}_X < 1/q$ with $1 < p, q < \infty$. Then

$$T_\mu : L^p \to L^p \text{ and } T_\mu : L^q \to L^q$$

and, by interpolation, $T_\mu : X \to X$. □

E x a m p l e 6. If X is a rearrangement invariant space with Boyd indices satisfying $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$, then the family of Fourier multipliers

$$P_t := T_{\chi_{[-t,t]}} \quad (t > 0)$$

is uniformly bounded on X ($C := \sup_{t>0} \|P_t\|_{X,X} < \infty$) and $P_t f \in V(t)$.

It is well known that $\|P_t\|$ does not depend on $t > 0$ and $\|P_t\| \leq \|H\|$, where $H : X \to X$ is the Hilbert transform; in fact, if $X = L^p$ and $1 < p < \infty$, then $\|P_t\| = \|H\|$ (cf. [CL]). Example 6 allows us to use Theorem 4 of [CKM] to describe the K-functional for pairs of Besov spaces.

Proposition 8. *Let X be a rearrangement invariant space with Boyd indices satisfying $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$ and assume that $\varrho := \sigma_0 - \tilde{\sigma}_0 > 0$. Then*

$$K(t^\varrho, f; B_X^{\sigma_0,q_0}, B_X^{\tilde{\sigma}_0,\tilde{q}_0}) \simeq \|P_t f\|_{\sigma_0,q_0} + t^\varrho \|f - P_t f\|_{\tilde{\sigma}_0,\tilde{q}_0}.$$

P r o o f. By Proposition 7,

$$\|P_t f\|_X \leq c_X \max(2, \|\mu\|_\infty)\|f\|_X \quad (f \in X,\ t > 0)$$

since, for every $f \in X$ and $t > 0$, $P_t f \in V(t)$ is such that $\|f - P_t f\|_X \leq CE(t,f)$ with some constant $C > 0$, and thus, if $g_t = P_t g_t \in V(t)$ is such that $\|f - g_t\|_X \leq 2d_X(f, V(t))$, we have

$$\|f - P_t f\|_X \leq \|f - g_t\|_X + \|P_t g_t - P_t f\|_X \leq C d_X(f, V(t)).$$

Then Theorem 17 applies and

$$K(t^\varrho, f; \bar{B}) \simeq \|P_t f\|_{\sigma_0,q_0} + t^\varrho \|f - P_t f\|_{\tilde{\sigma}_0,\tilde{q}_0} \quad (f \in \Sigma(\bar{B})),$$

where $\bar{B} = (B_X^{\sigma_0,q_0}, B_X^{\tilde{\sigma}_0,\tilde{q}_0})$. □

Theorem 18. *Assume that $1 \leq q, q_0, q_1, \tilde{q}_0, \tilde{q}_1 < \infty$, $0 < \theta < 1$, $\sigma_0 > \sigma_1 > 0$, $\tilde{\sigma}_0 > \tilde{\sigma}_1 > 0$, $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$ and $\tilde{\sigma} = (1-\theta)\tilde{\sigma}_0 + \theta\tilde{\sigma}_1$. If μ is an "admissible multiplier", then T_μ is K-commuting, so that*

$$[T, T_\mu] : B_X^{\sigma_0,q_0} \to B_X^{\tilde{\sigma}_0,\tilde{q}_0}$$

whenever $T : (B_X^{\sigma_0,q_0}, B_X^{\sigma_1,q_1}) \to (B_X^{\tilde{\sigma}_0,\tilde{q}_0}, B_X^{\tilde{\sigma}_1,\tilde{q}_1})$.

Let us summarize the proof (we refer to [CM] for the details).

A first simplification is obtained by considering dyadic multipliers,

$$\mu = \{\mu_n\}_{n\geq 0} := \sum_{n=0}^{\infty} \mu_n \chi_{\Delta_n(2)},$$

a constant function on every $\Delta_n(2)$. In this case the admissibility condition (22) is

$$V(\mu) = \sup_{n\geq 0} |\mu_n - \mu_{n-1}| < \infty \quad (\mu_{-1} := 0).$$

We associate with every admissible multiplier μ the admissible dyadic multiplier $\mu^{(d)} = \{\mu_n\}$ defined by

$$\mu_n = \begin{cases} \mu(2^{n-1}), & \text{if } n \geq 1 \\ 0, & \text{if } n = 0, \end{cases}$$

and so we may assume that μ is a dyadic multiplier with $\delta = 2$,

$$\mu = \{\mu_n\}_{n\geq 0} = \sum_{n=0}^{\infty} \mu_n \chi_{\Delta_n(2)}.$$

In this case,

$$T_\mu f = \sum_{k=1}^{\infty} \mu_k (P_{2^k} f - P_{2^{k-1}} f)$$

and, denoting

$$\lambda_0 = \mu_1 - \mu_0 = \mu_1,\ \lambda_1 = \mu_2 - \mu_1,\ \dots,\ \lambda_k = \mu_{k+1} - \mu_k,\ \dots,$$

we obtain a bounded sequence $\lambda = \{\lambda_n\} \in \ell^\infty$ and we can consider

$$\begin{aligned} T_\mu f &= \sum_{n=1}^{\infty} \Big(\sum_{j=0}^{n} \lambda_j \Big) (P_{2^n} f - P_{2^{n-1}} f) \\ &= \sum_{j=0}^{\infty} \lambda_j \sum_{n>j} (P_{2^n} f - P_{2^{n-1}} f) \\ &= \sum_{j=0}^{\infty} \lambda_j (f - P_{2^j} f), \end{aligned}$$

where the series is convergent, and

$$T_\mu : \sigma(\bar{A}) \to \Sigma(\bar{A}) \quad \text{if} \quad \bar{A} = (B_X^{\sigma_0, q_0}; B_X^{\tilde{\sigma}_0, \tilde{q}_0}). \tag{23}$$

Similarly, $T_\mu : \sigma(\bar{B}) \to \Sigma(\bar{B})$, $\bar{B} = (B_Y^{\sigma_1, q_1}; B_Y^{\tilde{\sigma}_1, \tilde{q}_1})$.

Now, given $T \in \mathcal{L}(\bar{A}; \bar{B})$ and $f \in \sigma(\bar{A})$,

$$\begin{aligned}[T, T_\mu]f &= \sum_{j=0}^{\infty} \lambda_j (Tf - TP_{2^j} f) - \sum_{j=0}^{\infty} \lambda_j (Tf - P_{2^j} Tf) \\ &= \sum_{\Theta} \lambda_j (P_{2^j} Tf - TP_{2^j} f) + \sum_{\mathbb{N}\setminus\Theta} \lambda_j (Tf - TP_{2^j} f) \\ &\quad - \lambda_j (Tf - P_{2^j} Tf),\end{aligned}$$

where $\Theta = \{j \in \mathbb{N} : 2^{\varrho(j+1)} < t\}$. By Proposition 8,

$$K(2^{\varrho j}, Tf; \bar{B}) \simeq \|P_{2^j} Tf\|_{\sigma_1, q_1} + 2^{\varrho j} \|Tf - P_{2^j} Tf\|_{\tilde{\sigma}_1, \tilde{q}_1} \tag{24}$$

with

$$\begin{aligned}\|P_{2^j} Tf\|_{\sigma_1, q_1} &\lesssim K(2^{\varrho j}, Tf; \bar{B}) \le \|T\| K(2^{\varrho j}, f; \bar{A}), \\ \|TP_{2^j} f\|_{\sigma_1, q_1} &\le \|T\| \, \|P_{2^j} f\|_{\sigma_0, q_0} \lesssim \|T\| K(2^{\varrho j}, f; \bar{A})\end{aligned}$$

and we obtain

$$\Big\| \sum_{\Theta} \lambda_j (P_{2^j} Tf - TP_{2^j} f) \Big\|_{\sigma_1, q_1} \lesssim \frac{\|\lambda\|_\infty \|T\|}{\varrho \log 2} \int_0^t K(x, f; \bar{A}) \frac{dx}{x}.$$

Also, it follows from

$$\|Tf - P_{2^j} Tf\|_{\tilde{\sigma}_1, \tilde{q}_1} \lesssim \frac{K(2^{\varrho j}, Tf; \bar{B})}{2^{\varrho j}} \le \|T\| \frac{K(2^{\varrho j}, f; \bar{A})}{2^{\varrho j}}$$

and

$$\|Tf - TP_{2^j} f\|_{\tilde{\sigma}_1, \tilde{q}_1} \le \|T\| \, \|f - P_{2^j} f\|_{\tilde{\sigma}_0, \tilde{q}_0} \lesssim \|T\| \frac{K(2^{\varrho j}, f; \bar{A})}{2^{\varrho j}}$$

that

$$\begin{aligned}&\Big\| \sum_{\mathbb{N}\setminus\Theta} \lambda_j (Tf - TP_{2^j} f) - \lambda_j (Tf - P_{2^j} Tf) \Big\|_{\tilde{\sigma}_1, \tilde{q}_1} \\ &\qquad\qquad \le \frac{2^{2\varrho} \|\lambda\|_\infty \|T\|}{\varrho \log 2} \int_t^\infty \frac{K(x, f; \bar{A})}{x} \frac{dx}{x}.\end{aligned}$$

Summarizing, we have

$$\begin{aligned}K(t,[T,T_\mu]f;\bar{B}) &\le \Big\| \sum_{\Theta} \lambda_j (P_{2^j}Tf - TP_{2^j}f)\Big\|_{\sigma_1,q_1} \\ &\quad + t\Big\| \sum_{\mathbb{N}\setminus\Theta} \lambda_j (Tf - TP_{2^j}f) - \lambda_j(Tf - P_{2^j}Tf)\Big\|_{\tilde\sigma_1,\tilde q_1} \\ &\le c\Big(\int_0^t K(x,f;\bar{A})\frac{dx}{x} + t\int_t^\infty \frac{K(x,f;\bar{A})}{x}\frac{dx}{x}\Big) \\ &= cS(K(\cdot,f;\bar{A}))(t),\end{aligned}$$

and T_μ is K-commuting, as in Proposition 6. Here $\bar{A}_{\theta,q} = B_X^{\sigma,q}$ and $\bar{B}_{\theta,q} = B_Y^{\tilde\sigma,q}$. □

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