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# WEIGHTS, ONE-SIDED OPERATORS, SINGULAR INTEGRALS AND ERGODIC THEOREMS

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## 1. Introduction

The aim of these lectures is to study weighted inequalities for one-sided operators. By a one-sided operator in the real line we mean an operator $T$ acting on measurable functions such that for all measurable functions $f$ the value of $Tf(x)$ depends only on the values of $f$ in $[x,\infty)$ or in $(-\infty,x]$. Examples of these operators are the following:

(1) The Hardy operator

$$Tf(x)=\int_{-\infty}^{x} f \quad \text{and its adjoint} \quad T^*f(x)=\int_{x}^{\infty} f.$$

(2) The Riemann-Liouville and the Weyl integral operators defined, for $0<\alpha<1$, by

$$I_\alpha^- f(x)=\int_{-\infty}^{x} \frac{f(s)}{(x-s)^{1-\alpha}}\,ds, \quad \text{and} \quad I_\alpha^+ f(x)=\int_{x}^{\infty} \frac{f(s)}{(s-x)^{1-\alpha}}\,ds.$$

(For these operators, one usually assumes that the support of $f$ is contained in $[0,\infty)$.)

(3) The one-sided Hardy-Littlewood maximal operators

$$M^- f(x)=\sup_{c<x}\frac{1}{x-c}\int_{c}^{x}|f| \quad \text{and} \quad M^+ f(x)=\sup_{c>x}\frac{1}{c-x}\int_{x}^{c}|f|.$$

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(4) The one-sided fractional maximal operators defined, for $0 < \beta \leq 1$, by

$$M_\beta^- f(x) = \sup_{c<x} \frac{1}{(x-c)^\beta} \int_c^x |f| \quad \text{and} \quad M_\beta^+ f(x) = \sup_{c>x} \frac{1}{(c-x)^\beta} \int_x^c |f|.$$

We are interested in obtaining characterizations of the pairs of nonnegative measurable functions $(u, v)$ such that the one-sided operators apply $L^p(v)$ into $L^q(u)$ or in weak-$L^q(u)$. In order to study these questions, it is interesting to begin with studying the operator $M^+$.

Which are the good weights for $M^+$? First, we remember the corresponding results for the two-sided Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{h,k>0} \frac{1}{h+k} \int_{x-k}^{x+h} |f| \quad .$$

B. Muckenhoupt proved [Mu] that the following theorems hold:

**Theorem A** [Mu]. *Let $u$ and $v$ be nonnegative measurable functions and let $1 \leq p < \infty$. The following statements are equivalent:*

(a) *There exists a constant $C > 0$ such that for all $\lambda > 0$ and every $f \in L^p(v)$*

$$\int_{\{x: Mf(x) > \lambda\}} u \leq \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |f|^p v.$$

(b) *$(u,v)$ satisfies $A_p$, i.e.,there exists a nonnegative real number $A$ such that*

$$\sup_{a<b} \left( \frac{1}{b-a} \int_a^b u \right) \left( \frac{1}{b-a} \int_a^b v^{-1/(p-1)} \right)^{p-1} = A \qquad \text{if} \quad p > 1,$$

$$Mu(x) \leq Av(x) \quad \text{a.e.} \quad \text{if} \quad p = 1.$$

**Theorem B** [Mu]. *Let $w$ be a nonnegative measurable function and let $1 < p < \infty$. The following statements are equivalent:*

(a) *There exists a constant $C > 0$ such that for every $f \in L^p(w)$*

$$\int_{-\infty}^{\infty} (Mf)^p w \le C \int_{-\infty}^{\infty} |f|^p w.$$

(b) *$w$, i.e. the pair $(w, w)$, satisfies $A_p$.*

Therefore, if we restrict ourselves to the single weight case, $u = v = w$, and keep in mind that $M^+ f \le Mf$, it is clear that if $w \in A_p$ then we have

$$\int_{\{x: M^+ f(x) > \lambda\}} w \le \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |f|^p w \quad \text{if } \ p \ge 1, \tag{1.1}$$

and

$$\int_{-\infty}^{\infty} (M^+ f)^p w \le C \int_{-\infty}^{\infty} |f|^p w \quad \text{if } \ p > 1. \tag{1.2}$$

The question is: are there more weights $w$ such that the above inequalities hold? If we connect this problem with ergodic theory we see easily that the answer is affirmative and that it has been known since long time ago.

Observe that $M^+$ is the ergodic maximal operator associated to the semigroup of operators $\{T_t : t \ge 0\}$ given by $T_t f(x) = f(x + t)$, i.e.,

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_0^h |T_t f(x)|\, dt.$$

Let us consider the real line with the measure $w(x)\, dx$. The Dunford-Schwartz theorem states (see [Kr] and Section 5 in this paper), roughly speaking, that if

$$\|T_t f\|_{L^1(w)} \le \|f\|_{L^1(w)} \quad \text{and} \quad \|T_t f\|_{L^\infty(w)} \le \|f\|_{L^\infty(w)}$$

for all $f$, then

$$\int_{\{x: M^+ f(x) > \lambda\}} w \le \frac{C_1}{\lambda} \int_{-\infty}^{\infty} |f| w \quad \text{for all } f \in L^1(w) \tag{1.3}$$

and

$$\int_{-\infty}^{\infty} (M^+ f)^p w \leq C_p \int_{-\infty}^{\infty} |f|^p w \quad \text{if } \ p > 1 \text{ and } f \in L^p(w) \tag{1.4}$$

hold with constants $C_1 = 1$ and $C_p = p/(p-1)$. But what do $\|T_t f\|_{L^1(w)} \leq \|f\|_{L^1(w)}$ and $\|T_t f\|_{L^\infty(w)} \leq \|f\|_{L^\infty(w)}$ mean? First, observe that if $w$ is positive then the inequality for the $L^\infty$-norm is always verified. Second, the inequality for the $L^1$-norm is equivalent to

$$\int |f(x)| w(x-t)\, dx \leq \int |f(x)| w(x)\, dx \quad \text{for all } t > 0 \text{ and all } f \in L^1(w),$$

which holds if $w$ is increasing. Therefore we see that if $w$ is positive and increasing then the Dunford-Schwartz theorem implies that $w$ is a good weight for $M^+$, more precisely, inequalities (1.3) and (1.4) hold. Since $w(x) = \mathrm{e}^x$ is increasing but it is not a weight in the $A_p$ classes we see that certainly the classes of functions $w$ for which inequalities (1.3) and (1.4) hold are wider than the $A_p$-classes.

Now the problem is to find a characterization of the weights for which the inequalities (1.3) and (1.4) hold. The same problems can be studied for the one-sided Hardy-Littlewood maximal function associated to a Borel measure $\mu$ which is finite on bounded intervals. For such a measure, we define

$$M_\mu^- f(x) = \sup_{c<x} \frac{1}{\mu(c,x]} \int_{(c,x]} |f|\, d\mu, \quad M_\mu^+ f(x) = \sup_{c>x} \frac{1}{\mu[x,c)} \int_{[x,c)} |f|\, d\mu,$$

where the quotients are understood as zero if $\mu(c,x] = 0$ or $\mu[x,c) = 0$. If $\mu$ is the Lebesgue measure then $M_\mu^+ = M^+$; the weights for this operator were studied by E. Sawyer [Sa]. If $\mu$ is a measure equivalent to the Lebesgue measure, the weights for $M_\mu^+$ were studied in [MOT]. The results of [Sa] and [MOT] were generalized in [An] where the following theorems were obtained:

**Theorem C** [An,Sa,MOT]. *Let $\mu$ be a Borel measure on $\mathbb{R}$ which is finite on bounded intervals. Let $u$ and $v$ be nonnegative measurable functions. If $1 \leq p < \infty$ then the following statements are equivalent:*

(a) *There exists a constant $C > 0$ such that for all $\lambda > 0$ and every $f \in L^p(v d\mu)$*

$$\int_{\{x : M_\mu^+ f(x) > \lambda\}} u\, d\mu \leq \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |f|^p v\, d\mu.$$

(b) *$(u,v)$ satisfies $A_p^+(\mu)$, i.e., there exists a nonnegative real number $A$ such that*

$$\sup_{a<b<c}\left(\frac{1}{\mu(a,c)}\int_{(a,b]} u\,d\mu\right)\left(\frac{1}{\mu(a,c)}\int_{[b,c)} v^{-1/p-1}\,d\mu\right)^{p-1} = A \quad \textit{if} \quad p>1,$$

$$M_\mu^- u(x) \le Av(x) \quad \mu\text{-a.e.} \quad \textit{if} \quad p=1.$$

*The constants $C$ and $A$ depend only on each one.*

**Theorem D** [An,Sa,MOT]**.** *Let $\mu$ be a Borel measure on $\mathbb{R}$ which is finite on bounded intervals. Let $w$ be a nonnegative measurable function and let $1<p<\infty$. Then the following statements are equivalent:*

(a) *There exists a constant $C>0$ such that for every $f\in L^p(wd\mu)$*

$$\int_{-\infty}^{\infty}(M_\mu^+ f)^p w d\mu \le C\int_{-\infty}^{\infty}|f|^p w d\mu.$$

(b) *$w$, i.e. the pair $(w,w)$, satisfies $A_p^+(\mu)$.*

*Remarks.*

(1) Analogous results are obtained for $M_\mu^-$ changing $A_p^+(\mu)$ by the obvious $A_p^-(\mu)$.
(2) If $\mu$ is the Lebesgue measure then we shall simply write $A_p^+$ and $A_p^-$, instead of $A_p^+(\mu)$ and $A_p^-(\mu)$.
(3) Weighted inequalities for $M^+$ in the setting of $L^{p,q}$ and Orlicz spaces have been studied in [O4], [O5] and [OP].

The purpose of the next section is to present recent results [MT5] about weighted norm inequalities for general one-sided maximal operators which include the operators mentioned at the beginning of this introduction and others as the maximal operator associated to Cesàro averages of order $\alpha$, $0<\alpha\le 1$, which is defined by

$$M_{\alpha,\alpha}^+ f(x) = \sup_{c>x}\frac{1}{(c-x)^\alpha}\int_x^c \frac{|f(s)|}{(c-s)^{1-\alpha}}\,ds.$$

In order to study these general one-sided maximal operators we shall need some important properties of the weights belonging to the $A_p^+(\mu)$ classes;

their proofs will be given in Section 3. The following Section 4 is devoted to the content of [AFM], i.e., to the study of weighted inequalities for singular integrals associated to Calderón–Zygmund kernels with support in $(-\infty, 0)$. Finally, in Section 5 we go back to ergodic theory and obtain a general dominated ergodic theorem [MT1] using the theory of one-sided weights.

Throughout the paper, $C$ will denote a constant which may change from one line to another. If $p$ is a number between 1 and $\infty$, then $p'$ will denote its conjugate exponent. For any measurable function $g$ and any measurable set $E$, $g(E)$ and $|E|$ will stand for the integral of $g$ over $E$ and the Lebesgue measure of $E$, respectively. The weights $u, v$ and $w$ will be assumed positive and finite to avoid technical difficulties.

## 2. General One-Sided Maximal Operators

### I. Definitions and examples.

**Definition 2.1.** Let $f$ be a locally integrable function defined on $\mathbb{R}$, and let $\alpha, \beta$ be two real numbers such that $0 \leq \beta \leq \alpha \leq 1$. We define the maximal operators

$$M^+_{\alpha,\beta} f(x) = \sup_{c>x} \frac{1}{(c-x)^\beta} \int_x^c \frac{|f(s)|}{(c-s)^{1-\alpha}}\, ds,$$

and

$$N^+_{\alpha,\beta} f(x) = \sup_{c>x} \frac{1}{(c-x)^\beta} \int_x^c \frac{|f(s)|}{(s-x)^{1-\alpha}}\, ds.$$

Our aim is to study the good weights for these operators.

**Examples.**

(1) If $\alpha = \beta = 1$ the operator $M^+_{\alpha,\beta}$ is the one-sided Hardy-Littlewood maximal operator. The pairs of weights for which this operator is of weak or strong type are well known [Sa, MOT].

(2) If $\alpha = 1$ and $0 < \beta < 1$ then $M^+_{\alpha,\beta}$ is the fractional one-sided maximal operator. The pairs of weights $(u, v)$ for which this operator is bounded from $L^p(vdx)$ to $L^q(udx)$, $1 < p \leq q < \infty$, were characterized by Andersen and Sawyer in [AS] and by Martín-Reyes and de la Torre in [MT3].

(3) If $\alpha = \beta \neq 1$, then the operator $M^+_{\alpha,\beta}$ is the maximal operator associated to the Cesàro averages $C_\alpha$ and the weights have not been

studied. For Lebesgue measure it is known [JT] that it maps $L^p$ into itself if $p > 1/\alpha$. In the limit case $p = 1/\alpha$ it maps $L^{p,1}$ into $L^{p,\infty}$. In other words, it is of restricted weak type $(1/\alpha, 1/\alpha)$.

(4) For the operator $N^+_{\alpha,\beta}$ we have that when $\beta = 0 < \alpha < 1$, $N^+_{\alpha,\beta} = \int_x^\infty f(s)(s-x)^{\alpha-1}\,ds$. This is the Weyl fractional integral studied in [AS] (see also [KG] and [LT]), while if $\beta$ is positive the operator is equivalent to the fractional one-sided maximal operator $M^+_{0,1-\alpha+\beta}$ studied in [AS] and [MT3].

(5) If $\alpha = 1$, $\beta = 0$ then $M^+_{\alpha,\beta} f(x) = \int_x^\infty f(s)\,ds$ which is the adjoint of the Hardy operator. The weights for the Hardy operator have been studied in many papers. For example, we can cite here [Br], [AM] and the book by B. Opic and A. Kufner [OK].

Of course one could consider also the operators $M^-_{\alpha,\beta}$ and $N^-_{\alpha,\beta}$ defined by

$$M^-_{\alpha,\beta} f(x) = \sup_{c<x} \frac{1}{(x-c)^\beta} \int_c^x \frac{|f(s)|}{(s-c)^{1-\alpha}}\,ds,$$

and

$$N^-_{\alpha,\beta} f(x) = \sup_{c<x} \frac{1}{(x-c)^\beta} \int_c^x \frac{|f(s)|}{(x-s)^{1-\alpha}}\,ds.$$

Therefore, if $f$ is positive and has support in $\mathbb{R}^+$, $M^-_{1,0} f(x) = \int_0^x f(s)\,ds$ for $x > 0$.

We are going to study the pairs of weights for which the operators $M^+_{\alpha,\beta}$ and $N^+_{\alpha,\beta}$ are of weak type or restricted weak type. We shall also obtain the strong type characterization in the case of "equal" weights. In this way we obtain unified results for the examples considered above. A generalization of these results in the setting of $L^{p,q}$ spaces will appear in a forthcoming paper by María Dolores Sarrión.

**II. Weak type inequalities.** In this subsection we give the characterization of the pairs of weights for which the above operators $M^+_{\alpha,\beta}$ and $N^+_{\alpha,\beta}$ are of weak type.

**Theorem 2.2** [MT5]. *Let $0 \leq \beta \leq \alpha \leq 1$, $\alpha > 0$ and $1 \leq p \leq q$. If $p < q$ or $\alpha - \beta = 1/p - 1/q$ then the following statements are equivalent:*

(1) *There exists a constant $C$ such that*

$$u(\{x : M^+_{\alpha,\beta} f(x) > \lambda\}) \leq C\lambda^{-q} \left( \int |f|^p v \right)^{q/p}$$

*for all $\lambda > 0$ and all $f \in L^p(v)$.*

(2) *The pair $(u, v)$ satisfies $A^+_{p,q,\alpha,\beta}$, i.e., there exists a constant $C$ such that*

$$\left( \int_a^b u \right)^{1/q} \left( \int_b^c \frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}} \, ds \right)^{1/p'} \leq C(c-a)^\beta,$$

*for all numbers $a < b < c$, where, from now on, if $p = 1$ then $\left( \int_b^c v^{1-p'}(s)(c-s)^{(\alpha-1)p'} \, ds \right)^{1/p'}$ is understood as the essential supremum of $\{v^{-1}(s)(c-s)^{\alpha-1} : s \in (b,c)\}$ .*

**Theorem 2.3** [MT5]. *Let $0 \leq \beta \leq \alpha \leq 1$, $0 < \alpha$ and $1 \leq p \leq q$. If $p < q$ or $1/p - 1/q = \alpha - \beta$ then the following statements are equivalent:*

(1) *There exists a constant $C$ such that*

$$u(\{x : N^+_{\alpha,\beta} f(x) > \lambda\}) \leq C\lambda^{-q} \left( \int |f|^p v \right)^{q/p}$$

*for all $\lambda > 0$ and all $f \in L^p(v)$.*

(2) *The pair $(u, v)$ satisfies $B^+_{p,q,\alpha,\beta}$, i.e., there exists a constant $C$ such that*

$$\left( \int_a^b u \right)^{1/q} \left( \int_b^c \frac{v^{1-p'}(s)}{(s-a)^{(1-\alpha)p'}} \, ds \right)^{1/p'} \leq C(c-a)^\beta$$

*for all numbers $a < b < c$.*

We shall only prove Theorem 2.2 in the case $p = q > 1$ and $0 < \alpha = \beta \leq 1$.

*Proof of Theorem 2.2 for $p = q > 1$ and $\alpha = \beta$.* The proof of (1) $\Longrightarrow$ (2) is standard. For $a < b < c$ fixed, we consider the function

$$g(s) = v^{1-p'}(s)(c-s)^{(\alpha-1)(p'-1)} \chi_{(b,c)}(s)$$

and the number

$$\lambda = \frac{1}{(c-a)^{\alpha}} \int_{b}^{c} \frac{v^{1-p'}(s)}{(c-s)^{(1-\alpha)p'}}\, ds.$$

It is easy to see that

$$(a,b) \subset \{x : M^{+}_{\alpha,\alpha} g(x) > \lambda\}.$$

Then, applying (1), $A^{+}_{p,p,\alpha,\alpha}$ follows .

The implication (2) $\Longrightarrow$ (1) follows from the following proposition:

**Proposition 2.4** [MT5]. *Let $0 < \alpha \leq 1$ and $1 < p$. If the pair $(u, v)$ satisfies $A^{+}_{p,p,\alpha,\alpha}$ then there exists $C > 0$ such that for every measurable function $f$ and all real numbers $a$ the following inequality holds:*

$$M^{+}_{\alpha,\alpha} f(a) \leq C \left( M^{+}_{u}(|f|^{p} v u^{-1}) \right)^{1/p} (a),$$

*where $M^{+}_{u}$ stands for the one-sided Hardy-Littlewood maximal function associated to the Borel measure $u(x)dx$.*

*Proof of Proposition 2.4.* We may assume that $f$ is nonnegative. Let $b > a$. We define a sequence $x_0 = b > x_1 > x_2 > \cdots > a$ by the identity

$$\int_{a}^{x_{i+1}} u = \int_{x_{i+1}}^{x_i} u = \frac{1}{2} \int_{a}^{x_i} u.$$

On each interval $(x_{i+1}, x_i)$ we have:

$$\begin{aligned} \int_{x_{i+1}}^{x_i} \frac{f(s)}{(b-s)^{1-\alpha}}\, ds &= \int_{x_{i+1}}^{x_i} \frac{f(s)}{(x_i-s)^{1-\alpha}} \left( \frac{x_i - s}{b-s} \right)^{1-\alpha} ds \\ &\leq \left( \frac{x_i - x_{i+2}}{b - x_{i+2}} \right)^{1-\alpha} \int_{x_{i+1}}^{x_i} \frac{f(s)}{(x_i - s)^{1-\alpha}} v^{1/p}(s) v^{-1/p}(s)\, ds \end{aligned}$$

(we have used that the function $s \mapsto ((x_i - s)/(b-s))^{1-\alpha}$ is decreasing)

$$\leq \left(\frac{x_i - x_{i+2}}{b - x_{i+2}}\right)^{1-\alpha} \left(\int_{x_{i+1}}^{x_i} f^p v\right)^{1/p} \left(\int_{x_{i+1}}^{x_i} \frac{v^{-p'/p}(s)}{(x_i - s)^{(1-\alpha)p'}}\, ds\right)^{1/p'}$$

$$\leq C \left(\frac{x_i - x_{i+2}}{b - x_{i+2}}\right)^{1-\alpha} (x_i - x_{i+2})^{\alpha} \left(\int_{x_{i+1}}^{x_i} f^p v\right)^{1/p} \left(\int_a^{x_i} u\right)^{-1/p}$$

$$\leq C \frac{x_i - x_{i+2}}{(b - x_{i+2})^{1-\alpha}} \left(M_u^+(f^p v u^{-1})\right)^{1/p}(a).$$

Summing up in $i$, we get

$$\int_a^b \frac{f(s)}{(b-s)^{1-\alpha}}\, ds \leq C \left(M_u^+(f^p v u^{-1})\right)^{1/p}(a) \sum_i \frac{x_i - x_{i+2}}{(b - x_{i+2})^{1-\alpha}}$$

$$\leq C \left(M_u^+(f^p v u^{-1})\right)^{1/p}(a) \sum_i \frac{1}{(b - x_{i+2})^{1-\alpha}} \int_{x_{i+2}}^{x_i} ds$$

$$\leq C \left(M_u^+(f^p v u^{-1})\right)^{1/p}(a) \sum_i \int_{x_{i+2}}^{x_i} \frac{1}{(b-s)^{1-\alpha}}\, ds$$

$$\leq C \left(M_u^+(f^p v u^{-1})\right)^{1/p}(a) \int_a^b \frac{1}{(b-s)^{1-\alpha}}\, ds$$

$$= C \left(M_u^+(f^p v u^{-1})\right)^{1/p}(a)(b-a)^{\alpha}.$$

*Proof of* (2) $\Longrightarrow$ (1). Assume again that $f$ is a nonnegative function. By Proposition 2.4 we have that the set $\{x : M^+_{\alpha,\alpha} f(x) > \lambda\}$ is contained in $\{x : M_u^+(f^p v u^{-1})(x) > C\lambda^p\}$. Therefore, since $M_u^+$ is of weak type (1,1) with respect to the measure $u(x)\, dx$,

$$\int_{\{x: M^+_{\alpha,\alpha} f(x) > \lambda\}} u \leq \int_{\{x: M_u^+(f^p v u^{-1})(x) > C\lambda^p\}} u \leq \frac{C}{\lambda^p} \int_{-\infty}^{\infty} f^p v.$$

**III. Restricted weak type inequalities.** It is known that if $0 < \alpha < 1$ then the operator $M^+_{\alpha,\alpha}$ is not of weak type $(p,p)$ with respect to the Lebesgue measure when $p = 1/\alpha$, but it is of restricted weak type. In the following theorem we will characterize restricted weak type.

**Theorem 2.5** [MT5]. *Let $0 \le \beta \le \alpha \le 1$, $\alpha > 0$ and $1 \le p \le q$. If $p < q$ or $1/p - 1/q = \alpha - \beta$ then the following statements are equivalent:*

(1) *$M^+_{\alpha,\beta}$ is of restricted weak type, i.e., there exists $C$ such that*

$$\int\limits_{\{x: M^+_{\alpha,\beta}\chi_E(x) > \lambda\}} u \le \left(\frac{C}{\lambda^p}\int \chi_E v\right)^{q/p}$$

*for all $\lambda > 0$ and all measurable sets $E$.*

(2) *There exists $C$ such that*

$$\frac{\int\limits_E (c-s)^{\alpha-1}\,ds}{(c-a)^\beta} \le C\frac{(v(E))^{1/p}}{\left(\int\limits_a^b u\right)^{1/q}}$$

*for all numbers $a < b < c$ and all measurable sets $E \subset (b,c)$.*

*Proof of Theorem 2.5 in the case $p = q$ and $\alpha = \beta$.* We shall begin proving (1) $\Longrightarrow$ (2). If $a < b < c$, $E \subset (b,c)$ and $\lambda = (c-a)^{-\alpha}\int\limits_b^c \chi_E(s)(c-s)^{\alpha-1}\,ds$, then it is easy to see that

$$(a,b) \subset \{x : M^+_{\alpha,\alpha}\chi_E(x) > \lambda\},$$

and then, by (1),

$$\int\limits_a^b u \le \frac{C}{\lambda^p}\int \chi_E v = \frac{(c-a)^{\alpha p}\int\limits_E v}{\left(\int\limits_E (c-s)^{\alpha-1}\,ds\right)^p}.$$

For the converse, we take any given interval $(a,b)$ and define a sequence $\{x_k\}$ as in the proof of Proposition 2.4. It follows that if $E$ is a measurable set and $E_i = E \cap (x_{i+1}, x_i)$, then

$$\int_{x_{i+1}}^{x_i} \frac{\chi_E(s)}{(b-s)^{1-\alpha}}\, ds \le \left(\frac{x_i - x_{i+2}}{b - x_{i+2}}\right)^{1-\alpha} \int_{E_i} \frac{1}{(x_i - s)^{1-\alpha}}\, ds$$

$$\le C \frac{x_i - x_{i+2}}{(b - x_{i+2})^{1-\alpha}} \frac{\left(\int_{E_i} v\right)^{1/p}}{\left(\int_{x_{i+2}}^{x_{i+1}} u\right)^{1/p}}$$

$$\le C \frac{x_i - x_{i+2}}{(b - x_{i+2})^{1-\alpha}} \left(\frac{\int_a^{x_i} v\chi_{E_i}}{\int_a^{x_i} u}\right)^{1/p}$$

$$\le C \frac{x_i - x_{i+2}}{(b - x_{i+2})^{1-\alpha}} \left(M_u^+(v\chi_E u^{-1})\right)^{1/q}(a).$$

If we sum up in $i$ as in Proposition 2.4, we get

$$M^+_{\alpha,\alpha}\chi_E(a) \le C \left(M_u^+(\chi_E v u^{-1})\right)^{1/p}(a),$$

and restricted weak type follows as in Theorem (2.2).

**IV. Strong type inequalities: the case of equal weights.** If the weights $u$ and $v$ satisfy that $v^q = u^p$ then the condition $A^+_{p,q,\alpha,\beta}$ is sufficient for the strong type inequality.

**Theorem 2.6** [MT5]. *Let $0 \le \beta \le \alpha \le 1$, $\alpha > 0$ and $1 < p \le q$. If $1/p - 1/q = \alpha - \beta$ and $v = u^{p/q}$ then the following statements are equivalent:*

(1) *There exists a constant $C$ such that*

$$\left(\int \left(M^+_{\alpha,\beta} f\right)^q u\right)^{1/q} \le C \left(\int |f|^p v\right)^{1/p}$$

*for all $f \in L^p(v)$.*

(2) *The pair $(u, v)$ satisfies $A^+_{p,q,\alpha,\beta}$.*

*Proof of the case $p = q$ and $\alpha = \beta$.* In this proof we shall write $w = u = v$. In order to prove the theorem, we need to consider the maximal operators

$M^+_{\mu_d}$ associated to the measure $\mu_d = (d-s)^{\alpha-1}\chi_{(-\infty,d)}(s)\,ds$ given by a real number $d$. Observe that we have

$$M^+_{\mu_d} f(x) \le M^+_{\alpha,\alpha} f(x). \tag{2.7}$$

In order to see this, we consider $c \le d$ with $x < c$. Then

$$\begin{aligned}\int_x^c \frac{|f(s)|}{(d-s)^{1-\alpha}}\,ds &= \int_x^c \frac{|f(s)|}{(c-s)^{1-\alpha}} \frac{(c-s)^{1-\alpha}}{(d-s)^{1-\alpha}}\,ds \\ &\le \frac{(c-x)}{(d-x)^{1-\alpha}} M^+_{\alpha,\alpha} f(x) \le M^+_{\alpha,\alpha} f(x) \int_x^c \frac{1}{(d-s)^{1-\alpha}}\,ds,\end{aligned}$$

which implies inequality (2.7). Therefore, if $M^+_{\alpha,\alpha}$ is of weak type $(p,p)$ with respect to the measure $w(x)\,dx$, or equivalently, if $w$ satisfies $A^+_{p,p,\alpha,\alpha}$ then the maximal operators $M^+_{\mu_d}$ are of weak type $(p,p)$ with the same constant, i.e., there exists $C$ such that

$$w(\{x : M^+_{\mu_d} f(x) > \lambda\}) \le \frac{C}{\lambda^p} \int_{-\infty}^{\infty} |f|^p w,$$

for all $d$, all positive $\lambda$ and all measurable functions $f$. Now, by Theorem C, we have that $w(s)(d-s)^{1-\alpha}$ satisfies $A^+_p(\mu_d)$ with the same $A^+_p(\mu_d)$-constant, i.e., there exists $C > 0$ such that

$$\begin{aligned}\sup_{a<b<c\le d} \left(\int_a^b w(s)\,ds\right)^{1/p} &\left(\int_b^c \frac{w^{1-p'}(s)}{(d-s)^{(1-\alpha)p'}}\,ds\right)^{1/p'} \\ &\le C \int_a^c \frac{1}{(d-s)^{1-\alpha}}\,ds\end{aligned} \tag{2.8}$$

for all real numbers $d$. Thus, we have seen that if $w$ satisfies $A^+_{p,p,\alpha,\alpha}$ then (2.8) holds. But the converse is also true, since if we put $c = d$ in (2.8) then we obtain $A^+_{p,p,\alpha,\alpha}$, and, therefore, we have that $w \in A^+_{p,p,\alpha,\alpha}$ if and only if $w(s)(d-s)^{1-\alpha} \in A^+_p(\mu_d)$ with the same $A^+_p(\mu_d)$-constant. Now, since $A^+_p(\mu_d) \implies A^+_{p-\varepsilon}(\mu_d)$ for some $\varepsilon > 0$ depending only on the $A^+_p(\mu_d)$-constant (see [MOT] and Section 3 in this paper), we get that if $w \in A^+_{p,p,\alpha,\alpha}$ then $w \in A^+_{p-\varepsilon,p-\varepsilon,\alpha,\alpha}$, and by Theorem 2.2 the operator $M^+_{\alpha,\alpha}$ maps $L^{p-\varepsilon}(w)$ into weak-$L^{p-\varepsilon}(w)$. Now, by interpolation, we obtain that $M^+_{\alpha,\alpha}$ applies $L^p(w)$ into $L^p(w)$. This proves (2) $\Longrightarrow$ (1). The converse implication follows from Theorem 2.2.

## 3. Some properties of the one-sided weights

Let $\mu$ be a continuous Borel measure finite on compact sets and $1 < p < \infty$. The implication $w \in A_p^+(\mu) \implies w \in A_{p-\varepsilon}^+(\mu)$ for some $\varepsilon > 0$ has been a key fact in the proof of the last theorem of Section 2. This section is aimed to provide a proof of this property based on the corresponding proof in [M] and on ideas of A. de la Torre. It is worth noting that the result does not hold for all noncontinuous measures; an example will be given at the end of this section.

Before stating and proving the results, it is convenient to consider the case in which $\mu$ is the Lebesgue measure and recall how the above implication is normally proved in the case of $A_p$ classes. First, it is seen that $v \in A_p$ implies that $v$ satisfies the following Reverse Hölder Inequality:

$$\left(\frac{1}{b-a}\int_a^b v^{1+\delta}\right)^{1/(1+\delta)} \leq \frac{C}{b-a}\int_a^b v$$

for some positive constants $C$ and $\delta$ independent of the numbers $a$ and $b$. Then, as a corollary, $v \in A_p$ and the Reverse Hölder Inequality give easily that $v \in A_{p-\varepsilon}$. But now, the Reverse Hölder Inequality does not hold for $A_p^+$ classes (consider, for instance, $v(x) = \exp x$). However, a substitute has been found in [M]: if $v \in A_p^+$ then there exist positive constants $C$ and $\delta$ such that for all $a$ and $b$

$$\int_a^b v^{1+\delta} \leq C(M^-(v\chi_{(a,b)})(b))^{\delta}\int_a^b v,$$

which implies

$$M^-(v^{1+\delta}\chi_{(a,b)})(b) \leq C(M^-(v\chi_{(a,b)})(b))^{1+\delta}.$$

This is what we have called Weak Reverse Hölder Inequality. This condition together with $v \in A_p^+$ gives $v \in A_{p-\varepsilon}^+$ in [M] but not so easily as in the classical case of Muckenhoupt's classes. After proving $A_p^+ \Rightarrow A_{p-\varepsilon}^+$ the following questions remained open: It is known that the Reverse Hölder Inequality is equivalent to the fact that the weight is in some $A_p$ class. Is this true for the Weak Reverse Hölder Inequality and $A_p^+$ classes? Moreover, is there a concept of $A_\infty^+$ weights, equivalent to the Weak Reverse Hölder Inequality, analogous to the concept of $A_\infty$ weights? The answers to these questions are affirmative. This has been obtained by L. Pick, A. de la Torre and the author [MPT].

Our first result in this section establishes that, for continuous measures $\mu$, $w$ satisfies the Weak Reverse Hölder Inequality if and only if $w$ belongs to $A_p^+(\mu)$ for some $p$; the third one is the required implication, while the fourth one introduces the $A_\infty^+(\mu)$ condition which will be useful in the next section. At the end of the section we give an example which proves that the implication $A_p^+(\mu) \implies A_{p-\varepsilon}^+(\mu)$ does not hold for all Borel measures $\mu$.

**Lemma 3.1** [MPT]**.** *Let $\mu$ be a continuous Borel measure on the real line, finite on bounded intervals. Let $w$ be a positive measurable function which is locally integrable with respect to $\mu$. The following statements are equivalent:*

(a) *The weight $w$ satisfies $A_p^+(\mu)$ for some $p \ge 1$.*

(b) *There exist positive constants $C$ and $\delta$ such that*

$$\frac{1}{\mu(a,b)}\int_a^b w^{1+\delta}\,d\mu \le \frac{C}{\mu(a,b)}\int_a^b w\,d\mu\,\left(M_\mu^-(w\chi_{(a,b)})(b)\right)^\delta$$

*for all numbers $a<b$ such that $\mu(a,b)>0$.*

(c) *There exist positive constants $C$ and $\delta$ such that*

$$M_{w\,d\mu}^-(w^\delta\chi_{(a,b)})(b) \le C\left(M_\mu^-(w\chi_{(a,b)})(b)\right)^\delta$$

*for all numbers $a<b$.*

(d) *There exists $p\ge 1$ such that $w^{-1}$ satisfies $A_p^-(w\,d\mu)$.*

*Proof.* We begin with proving (a)⇒(b). We may assume that $1<p$ and $w$ is bounded above. We first claim that for all $s$ with $0<s<1$ there exists $C$ such that

$$\frac{1}{\mu(a,b)}\int_a^b w\,d\mu \le C\left(\frac{1}{\mu(b,c)}\int_b^c w^s\,d\mu\right)^{1/s}$$

for all numbers $a<b<c$ such that $\mu(a,b)=\mu(b,c)>0$.

Proof of the claim. Let $r > 1$. Applying Hölder's inequality with exponents $r$ and $r' = r/(r-1)$ we obtain

$$1 = \left( \frac{1}{\mu(b,c)} \int_b^c w^{s/r} w^{-s/r} \, d\mu \right)^{r/s}$$

$$\leq \left( \frac{1}{\mu(b,c)} \int_b^c w^s \, d\mu \right)^{1/s} \left( \frac{1}{\mu(b,c)} \int_b^c w^{-sr'/r} \, d\mu \right)^{r/sr'}.$$

For fixed $s$, $0 < s < 1$, we choose $r$ such that $r/(sr') = p - 1$, i.e., $r = 1 + s(p-1)$. Then, since $w$ satisfies $A_p^+(\mu)$, we have

$$1 \leq C \left( \frac{1}{\mu(b,c)} \int_b^c w^s \, d\mu \right)^{1/s} \left( \int_a^b w \right)^{-1} \mu(a,c),$$

which, taking into account that $\mu(a,c) = 2\mu(a,b)$, is the claim that we wished to prove.

Now we shall use this inequality to prove that (b) holds. Let us fix the interval $I = (a,b)$, and let $A = M_\mu^-(w\chi_I)(b)$. For $\lambda > A$ we consider the set $O_\lambda = \{x \in I : M_\mu^-(w\chi_I)(x) > 2\lambda\}$. Then, since the measure is continuous, we have that there exists a countable disjoint family of intervals $I_i = (a_i, b_i)$ contained in $I$ such that $O_\lambda = \cup_i I_i$ and

$$\frac{1}{\mu(I_i)} \int_{I_i} w \, d\mu = 2\lambda.$$

Observe that

$$2\lambda\mu(I_i) = \int_{I_i} w \, d\mu \leq \int_{a_i}^b w \, d\mu \leq A\mu(a_i, b) < \lambda\mu(a_i, b).$$

Thus $2\mu(I_i) < \mu(a_i, b)$ and therefore for each $I_i$ there exists $I_i^+ = (b_i, c_i)$ contained in $I$ such that $\mu(I_i) = \mu(I_i^+)$ (here we are also using the continuity of the measure). Then

$$\int_{\{x \in I : w(x) > 2\lambda\}} w \, d\mu \leq \sum_i \int_{I_i} w \, d\mu = 2\lambda \sum_i \mu(I_i) \leq 2\lambda\mu(O_\lambda).$$

Now, observe that by the claim we have that $O_\lambda \subset \{x : M_\mu^+(w^s\chi_I)(x) > C\lambda^s\}$. Continuing our computation, we get, by the weak type (1,1) inequality of $M_\mu^+$ with respect to $\mu$, that

$$\int\limits_{\{x\in I: w(x)>2\lambda\}} w\, d\mu \leq C\lambda^{1-s} \int\limits_{\{x\in I: w^s(x)>C\lambda^s\}} w^s\, d\mu.$$

Multiplying by $\lambda^{\delta-1}$ and integrating from $A$ to $\infty$ we obtain

$$(*)\quad \int\limits_A^\infty \lambda^{\delta-1} \int\limits_{\{x\in I: w(x)>2\lambda\}} w\, d\mu\, d\lambda \leq C \int\limits_A^\infty \lambda^{\delta-s} \int\limits_{\{x\in I: w^s(x)>C\lambda^s\}} w^s\, d\mu\, d\lambda.$$

The left hand-side is equal to

$$\frac{1}{\delta} \int\limits_{\{x\in I: w(x)>2A\}} w \left( \left(\frac{w}{2}\right)^\delta - A^\delta \right) d\mu,$$

which is greater than or equal to

$$(**)\qquad \frac{1}{2^\delta \delta} \int\limits_I w^{1+\delta}\, d\mu - \frac{A^\delta}{\delta} \int\limits_I w\, d\mu.$$

The right hand-side of $(*)$ is less than or equal to

$$C \int\limits_I w^s \int\limits_0^{Cw} \lambda^{\delta-s}\, d\lambda\, d\mu \leq \frac{CC^{\delta-s+1}}{(\delta-s+1)} \int\limits_I w^{\delta+1}\, d\mu.$$

If we insert this inequality and $(**)$ into $(*)$, we get the desired inequality for $\delta$ small enough. In the last step we have used that $\int\limits_I w^{1+\delta}\, d\mu < \infty$ (since $w$ is bounded above).

Now, observe that (b)$\Rightarrow$(c) is immediate. In order to finish the proof of the lemma, it will suffice to establish (c)$\Rightarrow$(d), since (d)$\Rightarrow$(a) is proved as (a)$\Rightarrow$(d) changing the orientation of the real line and the roles of the measures $\mu$ and $w\, d\mu$.

Let $a < b < c$ and $x \in (b,c)$. If $\lambda = \left( \int\limits_a^c w\, d\mu \right)^{-1} \int\limits_a^b w^{1+\delta}\, d\mu$ then for every $x \in (b,c)$ we have

$$\begin{aligned}\lambda &\leq C \left( M_{w\, d\mu}^-(w^\delta \chi_{(a,x)}) \right)(x) \\ &\leq C \left( M_\mu^-(w\chi_{(a,x)}) \right)^\delta (x) \leq C \left( M_\mu^-(w\chi_{(a,c)}) \right)^\delta (x).\end{aligned}$$

Therefore, since $M_\mu^-$ is of weak type (1,1) with respect to the measure $\mu$,

$$\mu(b,c) \leq \mu(\{x : M_\mu^-(w\chi_{(a,c)})(x) \geq C\lambda^{1/\delta}\}) \leq \frac{C}{\lambda^{1/\delta}} \int_a^c w\, d\mu$$

which means that $w^{-1}$ satisfies $A_p^-(w\, d\mu)$ with $p = (1+\delta)/\delta$.

The following corollary is an immediate consequence of the implication (a)⇒(b) of Lemma 3.1.

**Corollary 3.2** [M]. *Let $\mu$ be a continuous Borel measure on the real line, finite on bounded intervals. If $w$ satisfies $A_p^+(\mu)$ then there exist $\delta > 0$ and $C > 0$ such that for every bounded interval $(a,b)$*

$$M_\mu^-\left(w^{1+\delta}\chi_{(a,b)}\right)(b) \leq C\left(M_\mu^-(w\chi_{(a,b)})(b)\right)^{1+\delta}.$$

**Theorem 3.3** [M]. *Let $\mu$ be a continuous Borel measure on the real line, finite on bounded intervals. If $1 < p < \infty$ and $w$ satisfies $A_p^+(\mu)$ then there exists $\varepsilon > 0$ such that $w$ satisfies $A_{p-\varepsilon}^+(\mu)$.*

*Proof of Theorem 3.3.* First, we observe that the $A_p^+(\mu)$-condition implies that $\sigma = w^{-1/(p-1)}$ is locally integrable (we are assuming that $w$ is positive). Second, we note that $w$ satisfies $A_p^+(\mu)$ if and only if $\sigma$ satisfies $A_{p'}^-(\mu)$. Then, by the analogue of the above corollary for $A_{p'}^-(\mu)$-classes, we have that there exist $\delta > 0$ and $C > 0$ such that for every bounded interval $(b,c)$

$$M_\mu^+\left(\sigma^{1+\delta}\chi_{(b,c)}\right)(b) \leq C\left(M_\mu^+(\sigma\chi_{(b,c)})(b)\right)^{1+\delta}.$$

Now we will show that $w$ satisfies $A_{p-\varepsilon}^+(\mu)$ where $p-\varepsilon = s = (p+\delta)/(1+\delta)$.

Fix $a < b < c$. Since $\sigma$ is locally integrable it follows from the above inequality that the same holds for $\sigma^{1+\delta}$. Therefore, there exists a finite decreasing sequence $x_0 = b > x_1 > \cdots > x_N \geq a = x_{N+1}$ such that

$$\int_{x_k}^c \sigma^{1+\delta}\, d\mu = 2^k \int_b^c \sigma^{1+\delta}\, d\mu \quad \text{if } k = 0, \ldots, N$$

and

$$\int_a^{x_N} \sigma^{1+\delta}\, d\mu < 2^N \int_b^c \sigma^{1+\delta}\, d\mu.$$

From this it follows easily that for every $k = 0, \dots, N$,

$$\int_{x_{k+1}}^{c} \sigma^{1+\delta}\, d\mu \le 2^{k+1} \int_{b}^{c} \sigma^{1+\delta}\, d\mu,$$

which will be useful later on. On the other hand,

$$\begin{aligned}
\int_a^b w\, d\mu \left( \frac{1}{\mu(a,c)} \int_b^c \sigma^{1+\delta}\, d\mu \right)^s &= \sum_{k=0}^{N} \frac{1}{2^{ks}} \int_{x_{k+1}}^{x_k} w\, d\mu \left( \frac{1}{\mu(a,c)} \int_{x_k}^{c} \sigma^{1+\delta}\, d\mu \right)^s \\
&\le \sum_{k=0}^{N} \frac{1}{2^{ks}} \int_{x_{k+1}}^{x_k} w(y) \left( M_\mu^+(\sigma^{1+\delta}\chi_{(y,c)})(y) \right)^s d\mu(y) \\
&\le \sum_{k=0}^{N} \frac{C}{2^{ks}} \int_{x_{k+1}}^{x_k} w(y) \left( M_\mu^+(\sigma\chi_{(x_{k+1},c)})(y) \right)^{p+\delta} d\mu(y).
\end{aligned}$$

Since $w$ satisfies $A_p^+(\mu)$ we know by Theorem C that $M_\mu^+$ applies $L^p(w\, d\mu)$ into weak-$L^p(w\, d\mu)$. Then, by Marcinkiewicz's interpolation theorem, $M_\mu^+$ applies $L^{p+\delta}(w\, d\mu)$ into $L^{p+\delta}(w\, d\mu)$. Thus

$$\begin{aligned}
\int_a^b w\, d\mu \left( \frac{1}{\mu(a,c)} \int_b^c \sigma^{1+\delta}\, d\mu \right)^s &\le C \sum_{k=0}^{N} \frac{1}{2^{ks}} \int_{x_{k+1}}^{c} \sigma^{1+\delta}\, d\mu \\
&\le C \sum_{k=0}^{N} \frac{2^{k+1}}{2^{ks}} \int_b^c \sigma^{1+\delta}\, d\mu \le C \int_b^c \sigma^{1+\delta}\, d\mu < \infty,
\end{aligned}$$

and therefore the proof of the theorem is finished.

**Theorem 3.4** [MPT]. *Let $1 \le p < \infty$. If $w \in A_p^+(\mu)$ then $w$ satisfies $A_\infty^+(\mu)$, i.e., there exist positive real numbers $C$ and $\delta$ such that*

$$\frac{\int_E w\, d\mu}{\int_a^c w\, d\mu} \le C \left( \frac{\mu(E)}{\mu(b,c)} \right)^\delta$$

*for all numbers $a < b < c$ and all subsets $E \subset (a,b)$.*

*Proof.* It follows from Lemma 3.1 that there exists $r > 1$ such that $w^{-1}$ satisfies $A_r^-(w\,d\mu)$. Then, for fixed $a < b < c$ and $E \subset (a,b)$,

$$\int_E w\,d\mu \le \left(\int_E w^{r/(r-1)}\,d\mu\right)^{(r-1)/r} (\mu(E))^{1/r} \le C\int_a^c w\,d\mu \left(\frac{\mu(E)}{\mu(b,c)}\right)^{1/r},$$

where we have used in the last inequality that $w^{-1}$ satisfies $A_r^-(w\,d\mu)$.

*Remark.* It can be proved that the fact that $w$ satisfies $A_\infty^+(\mu)$ is equivalent to each one of the statements of Lemma 3.1 (see [MPT]). From this it turns out that $w \in A_\infty^+(d\mu)$ if and only if $w^{-1} \in A_\infty^-(w\,d\mu)$ which means that there exist positive real numbers $C$ and $\delta$ such that

$$\frac{\mu(E)}{\mu(a,c)} \le C\left(\frac{\int_E w\,d\mu}{\int_a^b w\,d\mu}\right)^\delta$$

for all numbers $a < b < c$ and all subsets $E \subset (b,c)$ (the notations in this paper are different that the ones in [MPT]).

Theorem 3.4 is important in the study of one-sided sharp functions and one-sided $BMO$ spaces [MT4]. It is established in [MT4] the relation between $A_p^+$ weights and one-sided $BMO$ spaces and it is also obtained an inequality of John-Nirenberg type for one-sided sharp functions.

The proofs of the above theorems rely heavily on the fact that $\mu$ is a continuous measure. In fact, as we noticed in the introduction of this section, the theorems do not hold for all Borel measures. The following example shows that Theorem 3.3 does not hold for general Borel measures (this example was obtained jointly with A. de la Torre and María Dolores Sarrión and it is part of a joint work with P. Gurka and L. Pick).

**Example.** Let $\mu = \sum_{n=1}^\infty n^n \delta_n$ where $\delta_n$ is the Dirac measure at the point $n$. Let $w$ defined $\mu$-a.e. by $w(n) = 2^n/n^n$, $n \in \mathbb{N}$. We claim that $w \in A_1^+(\mu)$. In order to see this, we have to prove that there exists $C$ such that

$$M_\mu^- w(n) \le Cw(n) \quad \text{for all natural numbers } n.$$

In order to prove this inequality, it suffices to establish that if $m$ and $n$ are natural numbers with $m \le n$ then

$$\frac{\sum_{j=m}^n 2^j}{\sum_{j=m}^n j^j} \le Cw(n),$$

which follows easily since

$$\frac{\sum_{j=m}^{n} 2^j}{\sum_{j=m}^{n} j^j} \leq \frac{\sum_{j=m}^{n} 2^j}{n^n} = \frac{2^{n+1}-2^m}{n^n} \leq 2\frac{2^n}{n^n}.$$

Once we have seen that $w \in A_1^+(\mu)$, it is very easy to obtain that $w^{-1} \in A_2^-(\mu)$. However, $w^{-1} \notin A_p^-(\mu)$ for all $p < 2$. In order to prove it, let us consider $1 < p < 2$ and let $p'$ be its conjugate exponent. If $n \geq 2$ is a natural number then we have

$$\begin{aligned}\frac{1}{\mu(n-1,n+2)} &\left(\int_{[n+1,n+2)} w^{-1}d\mu\right)^{1/p} \left(\int_{(n-1,n+1]} w^{p'-1}d\mu\right)^{1/p'} \\ &\geq \frac{1}{2(n+1)^{n+1}} \left(\frac{(n+1)^{2(n+1)}}{2^{n+1}}\right)^{1/p} \left(\frac{2^n}{n^n}\right)^{1/p} n^{n/p'} \\ &= 2^{-1-1/p} \left(\frac{n+1}{n}\right)^{n/p} \left(\frac{n}{n+1}\right)^{n/p'} (n+1)^{1/p-1/p'}.\end{aligned}$$

These inequalities show that $w^{-1} \notin A_p^-(\mu)$, for $p < 2$, since the last term tends to $\infty$ as $n \to \infty$.

The weights belonging to $A_p^+(\mu)$ classes for a continuous measure $\mu$ have other properties that, however, do not hold for the weights in $A_p^+(\mu)$ for a general Borel measure finite on bounded intervals. For instance, for general Borel measures, we have:

(1) $w \in A_1^+(\mu) \not\Rightarrow$ there exists $\delta > 0$ such that $w^{1+\delta} \in A_1^+(\mu)$.
(2) $w \in A_1^+(\mu) \not\Rightarrow$ there exist $\delta > 0$, and functions $f$ and $k$ with $M_\mu^- f < \infty$ and $k, k^{-1} \in L^\infty(d\mu)$ such that $w = k(M_\mu^- f)^\delta$.

However the result analogous to Peter Jones' factorization theorem holds for all Borel measures which are finite on bounded intervals.

**Theorem 3.5** [An, Sa, MOT]. *Let $\mu$ be a Borel measure on the real line, finite on bounded intervals and let $1 \leq p < \infty$. A weight $w$ is in $A_p^+(\mu)$ if and only if there exist $w_1 \in A_1^+(\mu)$ and $w_2 \in A_1^-(\mu)$ such that $w = w_1 w_2^{1-p}$.*

## 4. One-Sided Weights And Singular Integrals On The Real Line

We shall say that a function $k$ in $L^1_{\mathrm{loc}}(\mathbb{R} - \{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(4.1) there exists a finite constant $B_1$ such that

$$\left| \int_{\varepsilon<|x|<N} k(x)\,dx \right| \le B_1 \quad \text{for all } \varepsilon \text{ and all } N, \text{ with } 0<\varepsilon<N,$$

and furthermore $\lim_{\varepsilon\to 0^+} \int_{\varepsilon<|x|<1} k(x)\,dx$ exists,

(4.2) there exists a finite constant $B_2$ such that

$$|k(x)| \le \frac{B_2}{|x|}, \quad \text{for all } x \neq 0,$$

(4.3) there exists a finite constant $B_3$ such that

$$|k(x-y)-k(x)| \le B_3|y||x|^{-2} \quad \text{for all } x \text{ and } y \text{ with } |x|>2|y|>0.$$

Associated to $k$ we consider the maximal operator

$$T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|,$$

with

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x-y)f(y)\,dy.$$

and the singular integral

$$Tf(x) = P.V. \int k(x-y)f(y)\,dy = \lim_{\varepsilon\to 0^+} T_\varepsilon f(x).$$

It is a well known result (see [CF] and[GR]) that if $w$ satisfies $A_p$ then

$$\begin{aligned}\int_{-\infty}^{\infty} |Tf(x)|^p w(x)\,dx &\le \int_{-\infty}^{\infty} |T^*f(x)|^p w(x)\,dx \\ &\le C \int_{-\infty}^{\infty} |f(x)|^p w(x)\,dx, \quad \text{if } 1<p<\infty,\end{aligned}$$

and

$$w(\{|Tf(x)|>\lambda\})\le w(\{T^*f(x)>\lambda\})\le \frac{C}{\lambda}\int_{-\infty}^{\infty}|f(x)|w(x)\,dx,\quad \text{if } p=1,$$

where the constant $C$ is independent of $f$ and $\lambda$. If we consider the Hilbert transform

$$Hf(x)=P.V.\int\frac{f(y)}{x-y}\,dy,$$

i.e., the singular integral associated to the kernel $k(x)=1/x$, then the conditions $A_p$ are necessary for the above inequalities to hold [HMW].

Our aim in this section is to determine singular integrals in the real line (one-sided singular integrals) which map $L^p(w)$ into $L^p(w)$ (or weak $L^p(w)$) for $A_p^+$ weights. This leads us to consider one-sided truncation of Calderón-Zygmund singular kernels.

Observe that the symmetry properties of the Hilbert kernel $k(x)=1/x$ produce the necessary cancellation properties of a singular integral, so that, no one-sided truncation of $1/x$ is expected to produce a (one-sided) singular integral. Nevertheless, the class of general singular Calderón-Zygmund kernels supported in a half-line is non trivial. For instance

$$k(x)=\frac{1}{x}\cdot\frac{\sin(\log|x|)}{\log|x|}\cdot\chi_{(-\infty,0)}(x)$$

is a Calderón-Zygmund kernel. It turns out that $A_p^+$ weights are good weights for the singular integral associated to a Calderón-Zygmund kernel with support in $(-\infty,0)$. The results that we shall present in this section have been obtained jointly with H. Aimar and L. Forzani [AFM].

**Theorem 4.4** [AFM]. *Let $k$ be a singular integral kernel satisfying* (4.1), (4.2) *and* (4.3) *with support in $\mathbb{R}^-=(-\infty,0)$. Then*

(a) *given a weight $w$ in $A_\infty^+=A_\infty^+(\mu)$ ($\mu$ =Lebesgue measure) there exists a constant $C_p$ depending only on $B_1$, $B_2$, $B_3$, $p$ and the constant in the condition $A_\infty^+$, such that*

$$\int_{-\infty}^{\infty}|T^*f(x)|^p w(x)\,dx\le C_p\int_{-\infty}^{\infty}|M^+f(x)|^p w(x)\,dx,\quad 1<p<\infty,$$

*and*

$$\sup_{\lambda>0}\lambda^p w(\{T^*f(x)>\lambda\})\le C_p\sup_{\lambda>0}\lambda^p w(\{M^+f(x)>\lambda\}),\quad 1\le p<\infty,$$

*for all* $f \in L^p(w)$*,*

(b) *given a weight* $w \in A_p^+$ *with* $1 < p < \infty$ *there exists a constant* $C$ *depending only on* $B_1$*,* $B_2$*,* $B_3$*,* $p$ *and the constant in the condition* $A_p^+$*, such that*

$$\int_{-\infty}^{\infty} |T^* f(x)|^p w(x)\, dx \le C \int_{-\infty}^{\infty} |f(x)|^p w(x)\, dx,$$

*for all* $f \in L^p(w)$*,*

(c) *given a weight* $w \in A_1^+$ *there exists a constant* $C$ *depending only on* $B_1$*,* $B_2$*,* $B_3$ *and the constant in the condition* $A_1^+$ *such that*

$$w(\{T^* f(x) > \lambda\}) \le \frac{C}{\lambda} \int_{-\infty}^{\infty} |f(x)| w(x)\, dx$$

*for all* $f \in L^1(w)$ *and all* $\lambda > 0$*.*

*Remarks.*

(1) An analoguous result holds for $A_p^-$ weights, $1 \le p \le \infty$, and singular integrals associated to Calderón-Zygmund kernels with support in $(0, \infty)$.

(2) Consider for all $\lambda > 0$ the dilation of the kernel $k$ given by

$$k_\lambda(x) = \lambda k(\lambda x).$$

It is clear that if $k$ is a Calderón-Zygmund kernel then $k_\lambda$ is also a Calderón-Zygmund kernel with the same constants $B_1$, $B_2$ and $B_3$ as $k$. If $T_\lambda^*$ are the maximal singular integrals associated to the dilations $k_\lambda$ then $T_\lambda^*$ are uniformly bounded from $L^p(w)$ into $L^p(w)$ if $w$ satisfies $A_p^+$, $1 < p < \infty$, and from $L^1(w)$ into weak-$L^1(w)$ if $w$ satisfies $A_1^+$. The next theorem is a kind of converse of this remark and includes a two-sided version.

**Theorem 4.5** [AFM]**.** *Let* $k$ *be a singular integral kernel satisfying* (4.1), (4.2) *and* (4.3)*. For each* $\lambda > 0$ *let* $T_\lambda^*$ *denote the maximal operator with kernel* $k_\lambda$ *and let* $1 \le p < \infty$*. Let* $w$ *be a positive measurable function and assume that the operators* $T_\lambda^*$ *are uniformly bounded from* $L^p(w)$ *into weak-*$L^p(w)$*.*

(a) *If there exists* $x_0 < 0$ *such that* $k(x_0) \ne 0$ *then* $w \in A_p^+$*.*

(b) *If there exists* $x_1 > 0$ *such that* $k(x_1) \ne 0$ *then* $w \in A_p^-$*.*

(c) *If there exist* $x_0 < 0 < x_1$ *such that* $k(x_0) \ne 0 \ne k(x_1)$ *then* $w \in A_p$*.*

*Remarks.*

(1) Theorems 4.4 and 4.9 hold also for the singular integral

$$Tf(x) = \lim_{\varepsilon \to 0^+} T_\varepsilon f(x).$$

The proofs for $T$ are similar to the corresponding one for $T^*$ or follow easily from the theorem for $T^*$.

(2) Since the Hilbert kernel $k(x) = 1/x$ coincides with its dilations, statement (c) of Theorem 4.5 gives the necessary part of the theorem by Hunt, Muckenhoupt and Wheeden [HMW] which characterizes the good weights for the Hilbert transform.

Theorem 4.4 is an easy consequence of Sawyer's results for $M^+$ [Sa] and of the next lemma which is itself an extension, to the one-sided setting, of the good-$\lambda$ inequality of Coifman and Fefferman [CF]. The proof of this lemma shows the way in which one uses $A_\infty^+$ weights to prove weighted distribution function inequalities and, in particular, how to overcome the essential obstacles which appear when one uses the techniques of the $A_p$ weights theory in the one-sided setting.

**Lemma 4.6** [AFM]. *Let $k$ be a singular integral kernel satisfying* (4.1), (4.2) *and* (4.3) *with support in $\mathbb{R}^- = (-\infty, 0)$. Let $w$ be a weight in $A_\infty^+$. Then there exist constants $C$ and $\gamma_0$ such that for every $0 < \gamma \le \gamma_0$ the inequality*

$$\begin{aligned} w(\{x \in \mathbb{R} : T^*f(x) > 2\lambda, M^+f(x) < \gamma\lambda\}) \\ \le C\gamma^\delta w(\{x \in \mathbb{R} : T^*f(x) > \lambda\}), \end{aligned} \tag{4.7}$$

*holds for all $f \in L^1$ and for every positive $\lambda$, with $\delta$ the exponent in the condition $A_\infty^+$.*

*Sketch of the proof of Lemma 4.6.* Since the set $\{T^*f > \lambda\}$ is open and has finite measure for $f$ in $L^1$, it can be written as a disjoint countable union of open intervals. Let $I_j = (a, b)$ be such an interval. It is enough to prove that there exist $C$ and $\gamma_0$ such that

$$w(\{x \in I_j : T^*f(x) > 2\lambda, M^+f(x) < \gamma\lambda\}) \le C\gamma^\delta w(I_j), \tag{4.8}$$

for every $0 < \gamma \le \gamma_0$ and every $\lambda > 0$.

If we follow the proof of the classical case (see [CF]), we would prove at this point that if $E_j = \{x \in I_j : T^*f(x) > 2\lambda, M^+f(x) < \gamma\lambda\}$ then

$$|E_j| \le C\gamma|I_j|.$$

This inequality and the $A_\infty^+$ condition imply that

$$w(E_j) \leq C\gamma^\delta w(a,c),$$

where $c - b = b - a$. However, now we are in trouble because the intervals $(a,c)$ generated from the intervals $(a,b)$ are not necessarily pairwise disjoint, and therefore we are not able of summing in $j$ and obtaining $w(\{x \in \mathbb{R} : T^* f(x) > \lambda\})$.

In order to avoid the difficulty explained in the above paragraph, we proceed as follows: first, let us take the sequence defined by $x_0 = a$ and $x_k - x_{k-1} = b - x_k$ for every $k \geq 1$. Second, we establish (see [AFM]) that

$$|E_k| \leq C\gamma(x_{k+2} - x_{k+1}),$$

where $E_k = \{x \in (x_k, x_{k+1}) : T^* f(x) > 2\lambda, M^+ f(x) < \gamma\lambda\}$. Then we apply the $A_\infty^+$-condition and we get

$$w(E_k) \leq C\gamma^\delta w(x_{k+2} - x_k).$$

Adding up in $k$, and keeping in mind that the intervals $(x_{k+2} - x_k)$ are almost disjoint we get inequality 4.8.

## 5. Ergodic Theorems

We shall begin by introducing one of the problems studied in the ergodic theory. Let $(X, \mathcal{M}, \nu)$ be a measure space and let $\tau : X \to X$ be a measure preserving transformation, i.e.,

(1) $\tau^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{M}$.
(2) $\nu\left(\tau^{-1}(E)\right) = \nu(E)$ for all $E \in \mathcal{M}$.

Let us consider for a measurable function $f$ the operator

$$Tf(x) = f(\tau x),$$

and the averages associated to $T$ defined by

$$A_n f(x) = \frac{1}{n+1} \sum_{i=0}^{n} T^i f(x) \quad \text{for all } n \in \mathbb{N}.$$

The ergodic theory studies the convergence in some sense of the sequence of the averages $\{A_n f\}$. More precisely, we are interested in the a.e. convergence of the sequence of the averages associated to a function which belongs to some $L^p(\nu)$, $1 \le p < \infty$. In order to study this problem we can proceed, as usual, by considering the maximal operator

$$M_T f(x) = \sup_{n \in \mathbb{N}} |A_n f(x)|$$

and proving that

(1) $M_T$ is of weak type (1,1) and of strong type $(p,p)$, $1 < p < \infty$,
(2) the sequence $A_n f$ converges a.e. for all $f$ in a class which is dense in $L^p(\nu)$.

In this way, one obtains the following theorem:

**Theorem** [B,W]. *Let $(X, \mathcal{M}, \nu)$ be a measure space and let $\tau : X \to X$ be a measure preserving transformation. Then*

(i) $\nu(\{x : M_T f(x) > \lambda\}) \le \dfrac{1}{\lambda} \int_X |f| d\nu$ *for all $f \in L^1(\nu)$ and all $\lambda > 0$.*
(ii) $\int_X |M_T f|^p d\nu \le \dfrac{p}{p-1} \int_X |f|^p d\nu, \quad 1 < p < \infty$, *for all $f \in L^p(\nu)$.*
(iii) *$A_n f$ converges a.e. for all $f \in L^p(\nu)$, $1 \le p < \infty$.*

Once this theorem has been established, one can think of generalizations of it. Taking into account that if $\tau$ is a measure preserving transformation then $T$ is a contraction (in fact an isometry) in $L^1(\nu)$ and in $L^\infty(\nu)$ we see that the following theorem is a generalization of the previous one:

**Theorem** [DS]. *Let $T$ be a linear operator in $L^1(\nu)$ which is an $L^1(\nu) - L^\infty(\nu)$-contraction, i.e.,*

$$\|T\|_1 = \sup\{\|Tf\|_1 : f \in L^1(\nu), \|f\|_1 \le 1\} \le 1$$

*and*

$$\|T\|_{1,\infty} = \sup\{\|Tf\|_1 : f \in L^1(\nu) \cap L^\infty(\nu), \|f\|_\infty \le 1\} \le 1.$$

*Then* (i), (ii) *and* (iii) *hold.*

We observe that until now the operator $T$ is defined on $L^p(\nu)$ for all $p$, $1 \le p < \infty$. In the next generalization we consider linear operators which are contractions on some fixed $L^p(\nu)$ space, but the operator has to be positive in the sense that $f \ge 0$ a.e. implies $Tf \ge 0$ a.e. The following result is due to A. Ionescu-Tulcea [I] for isometries and to M. Akcoglu [A] in the general case (the proof in [A] uses the result for isometries).

**Theorem** [A]. *Let $1 < p < \infty$ and let $T$ be a linear positive contraction in $L^p(\nu)$, i.e.,*

$$\|T\|_p = \sup\{\|Tf\|_p : f \in L^p(\nu), \|f\|_p \leq 1\} \leq 1.$$

*Then*

(a) $\int\limits_X |M_T f|^p d\nu \leq \dfrac{p}{p-1} \int\limits_X |f|^p d\nu$ *for all* $f \in L^p(\nu)$,

(b) $A_n f$ *converges a.e. for all* $f \in L^p(\nu)$.

The next natural generalization is to consider positive linear operators defined on some $L^p(\nu)$ such that

$$\sup_{n\geq 0} \|T^n\|_p = \sup\{\|T^n f\|_p : f \in L^p(\nu), \|f\|_p \leq 1, n \in \mathbb{N}\} < \infty.$$

These operators will be called power bounded operators. Therefore, the question is: Does Akcoglu's Theorem hold for positive power bounded operators?

The next step is to consider positive mean bounded linear operators, i.e., positive linear operators such that

$$\sup_{n\geq 0} \|A_n\|_p = \sup\{\|A_n f\|_p : f \in L^p(\nu), \|f\|_p \leq 1, n \in \mathbb{N}\} < \infty.$$

Of course, that $T$ is a positive mean bounded linear operator, is the less we can ask for the operator. However, what is really interesting is that, as A. Brunel and R. Emilion [BE] proved, if Ackoglu's Theorem can be extended to positive power bounded linear operators then it can also be extended to positive mean bounded linear operators. A. Brunel used this reduction to prove the following theorem:

**Theorem** [Bru]. *Let $1 < p < \infty$ and let $T : L^p(\nu) \to L^p(\nu)$ be a positive mean bounded linear operator. Then*

(a) *there exists $C > 0$ such that* $\int\limits_X |M_T f|^p d\nu \leq C \int\limits_X |f|^p d\nu$ *for all* $f \in L^p(\nu)$,

(b) $A_n f$ *converges a.e. for all* $f \in L^p(\nu)$.

Several years before, A. de la Torre and the author proved this theorem, but assuming that the operator $T$ is invertible and its inverse is a positive operator. The precise result is the following:

**Theorem 5.1** [MT1]. *Let $1 < p < \infty$ and let $T : L^p(\nu) \to L^p(\nu)$ be a positive invertible mean bounded linear operator with positive inverse. Then the following statements hold:*

(a) *There exists $C > 0$, depending only on $p$ and $\sup_{n\geq 0}\|A_n\|_p$, such that*

$$\int_X |M_T f|^p d\nu \leq C \int_X |f|^p d\nu \quad \textit{for all} \quad f \in L^p(\nu),$$

(b) *$A_n f$ converges a.e. for all $f \in L^p(\nu)$.*

Of course this theorem is included in Brunel's result with the only difference that in Brunel's theorem the constant $C$ is not only a function of $p$ and $\sup_{n\geq 0}\|A_n\|_p$.

In what follows we shall sketch the proof of Theorem 5.1.

*Sketch of the proof of Theorem 5.1.* First we must say that the key ideas for proving this theorem are the following:

(1) The theory of one-sided weights (although there are no weights in the statement of the theorem).
(2) Arguments of transference.
(3) The method of J.L. Rubio de Francia (see [R] and [CJR]) in order to factorize weights.
(4) The fact that $T$ is a positive linear operator with positive inverse implies that the operator $T$ and its powers separate supports ($fg = 0 \implies TfTg = 0$) and then [K] there exist positive functions $h_i$ such that

$$\int_X |f|^p d\nu = \int_X |T^i f|^p h_i d\nu$$

for all $i \in \mathbb{Z}$ and all $f \in L^{p'}(d\nu)$. Moreover, for all positive $f \in L^{p'}(d\nu)$,

$$h_i = (T^{-i})^* f (T^i f^{p'-1})^{1-p},$$

where $(T^i)^*$ denotes the adjoint of $T^i$. These functions $h_i$ will be the weights in the proof of the theorem.

In order to give the idea of the proof we need to introduce some notations: Let $\mu$ be the measure on the real line defined by $\sum_{n=0}^{\infty} \delta_n$ where $\delta_n$ is the

Dirac delta at the point $n$. For a measurable function $f$ and for all $x \in X$ the function $f_x$ on the real line is defined $\mu$-a.e. as $f_x(n) = T^n f(x)$ for nonnegative integers $n$.

For fixed $L$, let us consider the truncated ergodic maximal operator defined by

$$M_{T,L}f(x) = \sup_{n \leq L} |A_n f(x)|.$$

In order to prove (a) it suffices to obtain

$$\int_X |M_{T,L}f|^p d\nu \leq C \int_X |f|^p d\nu \quad \text{for all nonnegative } f \in L^p(\nu)$$

with a constant independent of $L$.

Let $f$ be a nonnegative measurable function and let $N$ be a positive integer. Then, by the properties of the functions $h_i$ we have

$$\int_X |M_{T,L}f|^p d\nu = \frac{1}{N+1} \int_X \sum_{i=0}^{N} |T^i(M_{T,L}f)|^p(x) h_i(x) d\nu(x).$$

Now, the fact that the operators $T^i$ separate supports gives

$$T^i(M_{T,L}f)(x) \leq M_{T,L}(T^i f)(x) \leq M_\mu^+(f_x \chi_{[0,N+L]})(i).$$

Therefore

$$\begin{aligned} \sum_{i=0}^{N} |T^i(M_{T,L}f)|^p(x) h_i(x) &\leq \sum_{i=0}^{\infty} |M_\mu^+(f_x \chi_{[0,N+L]})(i)|^p h_i(x) \\ &= \int_{-\infty}^{\infty} |M_\mu^+(f_x \chi_{[0,N+L]})(i)|^p h_i(x) d\mu(i), \end{aligned}$$

where we notice that the functions $i \mapsto h_i(x)$ are defined $\mu$-a.e. in $\mathbb{R}$. Thus, if

(∗) the functions $i \mapsto h_i(x)$ satisfy $A_p^+(\mu)$ for almost all $x \in X$ and with the same $A_p^+(\mu)$-constant then we obtain for almost all $x \in X$

$$\sum_{i=0}^{N} |T^i(M_{T,L}f)|^p(x)h_i(x) \le C \int_{-\infty}^{\infty} (f_x\chi_{[0,N+L]}(i))^p h_i(x)\, d\mu(i)$$
$$= C \sum_{i=0}^{N+L} (T^i f(x))^p h_i(x).$$

Finally, under the assumption $(*)$, we get

$$\int_X |M_{T,L}f|^p d\nu \le \frac{C}{N+1} \sum_{i=0}^{N+L} \int_X (T^i f(x))^p h_i(x)\, d\nu(x)$$
$$= C\frac{N+L+1}{N+1} \int_X (f(x))^p\, d\nu(x).$$

If we let $N$ tend to $\infty$ we obtain the inequality we wished to prove. Therefore (a) follows if we prove that $(*)$ is implied by the assumptions on $T$ (we have used until here the key ideas (1), (2) and (4)).

We may notice at this point that in the harmonic analysis it is easy to prove that the weights must satisfy the corresponding necessary condition. However, in the ergodic theory we need to work harder, since we have to obtain $(*)$ which is a condition on the orbits (on the integers) from the fact that $T$ is a mean bounded linear operator which is a condition on the measure space.

In order to prove $(*)$, we remind that $h_i = (T^{-i})^* f(T^i f^{p'-1})^{1-p}$ for all positive $f \in L^{p'}(\nu)$. Therefore, keeping in mind the factorization theorem (see Section 3), we have that if there exists $f \in L^{p'}(\nu)$, $f > 0$, such that the functions $i \to (T^{-i})^* f(x)$ and $i \to T^i f^{p'-1}(x)$, defined $\mu$-a.e., satisfy $A_1^+(\mu)$ and $A_1^-(\mu)$ respectively, for almost all $x \in X$, with a uniform constant, then the statement $(*)$ holds (in fact, it is not necessary that they satisfy $A_1^+(\mu)$ and $A_1^-(\mu)$ but weaker conditions). Now the problem is: How to choose such a function $f$? Here is where the Rubio de Francia's idea comes.

For fixed $n$, let us consider the sublinear operator

$$S_n f = (A_n|f|^{p'})^{1/p'} + (A_n^*|f|^p)^{1/p},$$

where $A_n^*$ stands for the adjoint of $A_n$. It is clear that the operators $S_n$ apply $L^{p,p'}(\nu)$ into $L^{p,p'}(\nu)$ and their norms are uniformly bounded. Let $K$ be a positive number such that

$$\|S_n\|_{p,p'} \leq K$$

for all $n$. Let us choose a positive function $g \in L^{pp'}(\nu)$ and define

$$w_n = \sum_{i=0}^{\infty} \frac{1}{(2K)^i} S_n^i g.$$

It is not difficult to see that $w_n \in L^{p,p'}(\nu)$ and $S_n w_n \leq 2K w_n$ a.e. From the last property we obtain

$$A_n w_n^{p'} \leq (2K)^{p'} w_n^{p'} \quad \text{a.e.} \quad \text{and} \quad A_n^* w_n^p \leq (2K)^p w_n^p \quad \text{a.e.}$$

Now, these inequalities and the positivity of $T$ and $T^{-1}$ give

$$\frac{1}{n+1} \sum_{i=0}^{n} T^{i+j} w_n^{p'} \leq (2K)^{p'} T^j w_n^{p'} \quad \text{a.e.}$$

and

$$\frac{1}{n+1} \sum_{i=0}^{n} (T^{-i-j})^* w_n^p \leq (2K)^p (T^{-n-j})^* w_n^p \quad \text{a.e.}$$

These two properties almost mean that, for $f = w_n^p$, $i \mapsto (T^{-i})^* f(x)$ and $i \mapsto T^i f^{p'-1}(x)$, defined $\mu$-a.e., satisfy $A_1^+(\mu)$ and $A_1^-(\mu)$. The only problem is that the function $f$ depends on $n$. However, these two properties and the relation $h_i = (T^{-i})^* f (T^i f^{p'-1})^{1-p}$ are enough to obtain that $(*)$ holds. With this we have finished the sketch of the proof of (a) (details can be found in [MT1]).

The statement (b) of the theorem follows from (a) and the fact that the functions of the type $h + f - Tf$ with $h$ invariant and $f$ simple are dense in $L^p(\nu)$ (see [MT1]). The a.e. convergence is clear for the invariant functions and it follows for the functions $f - Tf$ with $f$ simple from the fact that $n^{-1}T^n f$ converges a.e. to 0 for all characteristic functions of sets of finite measure.

*Final remarks.*

(1) Similar results to Theorem 5.1 for the ergodic Hilbert transform and the ergodic power function have been obtained in [S1] and [MO].

(2) The ergodic maximal operator and the convergence of the ergodic averages in weighted $L^{p,q}$ and Orlicz spaces have been studied by using the one-sided weights in [O1], [O3], [O4] and [O5] (see also [G]).

(3) The theory of weights has been useful to conjecture the answer to some problems in the ergodic theory (see [MT2], [S2], [S3], [GM], [FMT] and [O2]) although the results of the theory of weights are not strictly necessary in the proofs of the theorems of these papers.

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