Sergej K. Vodop'yanov

Boundary behaviour of differentiable functions and related topics

In: Miroslav Krbec and Alois Kufner and Bohumír Opic and Jiří Rákosník (eds.): Nonlinear Analysis, Function Spaces and Applications, Proceedings of the Spring School held in Roudnice nad Labem, 1990, Vol. 4. B. G. Teubner Verlagsgesellschaft, Leipzig, 1990. Teubner Texte zur Mathematik, Band 119. pp. 224--253.

Persistent URL: http://dml.cz/dmlcz/702445

BOUNDARY BEHAVIOUR OF DIFFERENTIABLE FUNCTIONS AND RELATED TOPICS

Sergej K. Vodop'yanov
Novosibirsk, USSR

A description of restrictions of differentiable functions to sets $F \subset \mathbb{R}^n$ is a classical topic in the function theory arising from papers by Lebesgue [1] and Whitney [2]. There is a numerous bibliography concerning the Sobolev spaces $W_p^{\ell}(\mathbb{R}^n)$, $\ell \in \mathbb{N}$, the Nikol'skii spaces $H_p^{\ell}(\mathbb{R}^n)$, the Besov spaces $B_{p,q}^{s}(\mathbb{R}^n)$ and the spaces of Bessel potentials or the Liouville spaces $L_p^{\ell}(\mathbb{R}^n)$, $\ell \in \mathbb{R}_+$. (Note, that $W_p^{\ell}(\mathbb{R}^n) = L_p^{\ell}(\mathbb{R}^n)$, $\ell \in \mathbb{N}$, and $H_p^{\ell}(\mathbb{R}^n) = B_{p,\infty}^{\ell}(\mathbb{R}^n)$, $\ell \in \mathbb{R}_+$.)

In this paper we study the boundary behaviour of differentiable functions from the Sobolev classes $W_p^{\ell}(\mathbb{R}^n)$ and the Nikol'skii classes $H_p^{\ell}(\mathbb{R}^n)$ for $p = \infty$. The domain of definition of the function classes in question is an arbitrary connected open set in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Without considering regularity properties of the boundary of the domain we need to introduce new concepts and a language for the description of the boundary behaviour. We establish that boundary values exist always but we shall understand them in a special sense.

Our method is based on a new equivalent normalization of the Sobolev and Nikol'skii spaces in domains which include geometrical characteristics of the domain in an explicit way. The geometry of the domain is determined by the modulus of continuity defining the function space. The inner geometry of the domain reflects the substance of the studied problem and represents suitable tools to its resolution.

The first application deals with traces (boundary values) of functions on the boundary of the domain of definition. The boundary can be obtained by means of the completion of the domain with respect to the corresponding metric. The elements of the function space are extended by continuity to the boundary and these traces belong to some function space. It is proved that such a characterization of traces is reversible.

The second application concerns some necessary and sufficient extendibility conditions for differentiable functions across the boundary of the domain of definition. They are formulated in terms of the equivalence of the corresponding metrics in the domain and in the surrounding space for $p = \infty$ and involve some conditions on the measure in spaces with integral metrics.

1. Geometric normalizations in spaces of differentiable functions

1.1. Sobolev and Nikol'skii spaces

For an arbitrary $\alpha\in(0,1]$ we define the inner α-metric $d_{\alpha,G}(x,y)$ on a domain $G\subset\mathbb{R}^n$ as follows

$$d_{\alpha,G}(x,y) = \inf_{\gamma} \sum_{i=1}^{m} |x_i-x_{i-1}|^{\alpha},$$

where the infimum is taken over all broken lines γ consisting of segments $[x_{i-1},x_i]\subset G$, $x_0 = x$, $x_m = y$. It is evident that the 1-metric coincides with the infimum of the lengths of rectifiable curves connecting the points $x,y\in G$, and therefore it is the inner metric of the domain in the commonly accepted sense (see [3]). We denote by G_α the metric space $(G,d_{\alpha,G})$.

The elements of the function space $\mathrm{Lip}(\ell,G_\alpha)$, $\ell\in\mathbb{R}_+ = \{x\colon\ x>0\}$, $\ell\in(k,k+1]$, $k = 0,1,2,\dots,$. are L_∞-functions $f\colon G\longrightarrow\mathbb{R}$, whose weak partial derivatives, denoted by D^jf, also belong to L_∞ for $|j|\le k$. The norm in $\mathrm{Lip}(\ell,G_\alpha)$ is defined by

$$\|f|\mathrm{Lip}(\ell,G_\alpha)\| = \sum_{|j|\le k}\left\{\|D^jf|L_\infty(G)\| + \sup \frac{|R_j(x,y)|}{\bar{d}_{\alpha,G}(x,y)^{\ell-|j|}}\right\},$$

where $\bar{d}_{\alpha,G}(x,y) = d_{\alpha,G}(x,y)^{1/\alpha}$, $\alpha = \ell-k$ and

$$D^jf(x) = \sum_{|j+s|\le k} \frac{D^{j+s}f(y)}{s!}(x-y)^s + R_j(x,y),\ 0\le|j|\le k.$$

We use the usual multi-index notation, $j = (j_1,j_2,\dots,j_n)$, $s = (s_1,s_2,\dots,s_n)$, $s! = s_1!s_2!\dots s_n!$, $|j| = j_1+j_2+\dots+j_n$ and $x^s = x_1^{s_1}x_2^{s_2}\dots x_n^{s_n}$.

Two function spaces coincide if the operator of embedding of one of the spaces into the other is a bounded isomorphism.

THEOREM 1.1. *Let* G *be an arbitrary domain in* $\mathbb{R}^n$. *The following function spaces coincide:*

$$\mathrm{Lip}(\ell, G_\alpha) = W^\ell_\infty(G) \text{ for } \ell\in\mathbb{N} \quad \text{and} \quad \mathrm{Lip}(\ell, G_\alpha) = H^\ell_\infty(G) \text{ for } \ell\notin\mathbb{N}.$$

Here the space $W^\ell_\infty(G)$ and $H^\ell_\infty(G)$ is a member of the scale of the Sobolev spaces $W^\ell_p(G)$ [4] and the Nikol'skii spaces $H^\ell_p(G)$ [5, 6], respectively. Let us recall that the elements of $W^\ell_p(G)$, $\ell\in\mathbb{N}$, $p\in[1,\infty]$ and $H^\ell_p(G)$, $\ell\in\mathbb{R}_+$, $\ell\in(k,k+1)$, $k = 0,1,2,\ldots$, $p\in[1,\infty]$ are L_p-functions f whose weak partial derivatives, denoted by D^jf, also belong to L_p for $|j|\le\ell$ and $|j|\le k$, respectively. The norm in $W^\ell_p(G)$ and in $H^\ell_p(G)$ is defined by

$$\|f|W^\ell_p(G)\| = \|f|L_p(G)\| + \|\nabla_\ell f|L_p(G)\|$$

or

$$\|f|H^\ell_p(G)\| = \|f|L_p(G)\| + \sup \frac{\|\Delta_h\nabla_k f|L_p(G)\|}{|h|^\alpha},$$

respectively, where $\nabla_\ell = \{D^jf:\ |j| = \ell\}$ and $\alpha = \ell-k$,

$$\Delta_h g(x) = \begin{cases} g(x+h) - g(x) & \text{if the segment } [x,x+h]\subset G, \\ 0 & \text{otherwise.} \end{cases}$$

The Nikol'skii spaces $H^\ell_p(G)$ belong to the scale of the Besov spaces $B^\ell_{p,q}(G)$, $q\in[1,\infty]$, as well, namely,

$$H^\ell_p(G) = B^\ell_{p,\infty}(G) \text{ [see 5, 6].}$$

P r o o f of Theorem 1.1. If $f\in\mathrm{Lip}(\ell,G)$, $\ell\in\mathbb{N}$, then any point $x\in G$ is the center of some ball B in which all the derivatives of the function f of the order ℓ-1 are bounded and satisfy the Lipschitz condition. This implies that the function f has bounded generalized derivatives of order ℓ. Thus the boundedness of the embedding $\mathrm{Lip}(\ell,G_\alpha)\longrightarrow W^\ell_\infty(G)$, $\ell\in\mathbb{N}$, is proved. The continuity of the embedding

$$\mathrm{Lip}(\ell,G_\alpha)\longrightarrow H^\ell_\infty(G),\ \ell\notin\mathbb{N},$$

is evident.

Now we shall prove the continuity of the converse embedding. Let us consider any function from of the Sobolev or the Nikol'skii space with the norm equal to 1. The estimates $|D^jf(x)|\le c$, $|j|\le\ell$, $x\in G$, are proved in [5].

Therefore it remains to establish the inequalities

$$(1.1)\qquad |R_j(x,y)| \le c_j \bar{d}_{\alpha,G}(x,y)^{\ell-|j|},$$

where $\alpha = \ell-k$, $k\in\mathbb{N}$, $k<\ell\le k+1$, $|j|\le k$ and $c_j = n^{(k-|j|)/2}$. We shall prove the estimate (1.1) by induction. If $|j| = k$, then $R_j(x,y) = D^jf(x) - D^j(y)$. We have the following estimate for the difference of the derivative D^jf, $|j| = k$, along a broken line consisting of segments $[x_{i-1},x_i]\subset G$, $i = 1,\dots,m$, $x_0 = x$, $x_m = y$:

$$(1.2)\qquad |D^jf(x) - D^jf(y)| \le \sum_{i=1}^{m} |D^jf(x_i) - D^jf(x_{i-1})| \le \sum_{i=1}^{m}|x_i - x_{i-1}|^{\alpha},$$

where $\alpha = 1$ if $f\in W^{\ell}_{\infty}(G)$, and $\alpha = \ell-k$ if $f\in H^{\ell}_{\infty}(G)$. Now minimizing the right hand side of (1.2), we obtain the estimate

$$(1.3)\qquad |R_j(x,y)| = |D^jf(x) - D^jf(y)| \le d_{\alpha,G}(x,y) = d_{\alpha,G}(x,y)^{(\ell-|j|)/\alpha},$$

where $|j| = k$ and α depends on the choice of the function space made in the above mentioned way.

The inequalities (1.3) constitute the basis for the induction. Let us suppose that the estimate (1.1) with $c_j = n^{(k-|j|)/2}$ is proved for all j such that $0\le p<|j|\le k$, $p = 0,1,\dots,k-1$. For any multi-index j, $|j| = p$, we shall estimate $R_j(x,y)$, where the points $x,y\in G$ are fixed. Since the value $R_j(y,y)$ in question is equal to 0, it is

$$(1.4)\qquad |R_j(x,y)| \le \sup\left\{|\nabla R_j(\xi,y)| \sum_i|a_i|:\ \xi\in G,\ d_{\alpha,G}(\xi,y) \le d_{\alpha,G}(x,y) + \varepsilon\right\},$$

where the segments a_j form a broken line with endpoints x and y, for which

$$\sum|a_i|^{\alpha} \le d_{\alpha,G}(x,y) + \varepsilon.$$

Since

$$|\nabla R_j(\xi,y)|^2 = \sum_{|j+k|=p+1} |R_{j+k}(\xi,y)|^2$$

and

$$\sum_i|a_i| \le \left[\sum_i|a_i|^{\alpha}\right]^{1/\alpha} \le \left[d_{\alpha,G}(x,y) + \varepsilon\right]^{1/\alpha}$$

by the reverse Minkowski inequality (see [7]), we obtain from (1.4) that

$$\begin{aligned}|R_j(x,y)| &\le n^{1/2}n^{(k-(p+1))/2}\ \bar{d}_{\alpha,G}(x,y)^{\ell-(p+1)}\ \bar{d}_{\alpha,G}(x,y)\\ &= n^{(k-p)/2}\ \bar{d}_{\alpha,G}(x,y)^{\ell-p}.\end{aligned}$$

This proves the inequality (1.1) for every j, $|j| = p$. $\square$

Another application concerns some necessary and sufficient condition for an extension of differentiable functions beyond the boundary. They are formulated in terms of equivalence of the corresponding metric in the domain and in the surrounding space for $p = \infty$ and they involve certain conditions on the measure in spaces with integral metrics.

1.2. Zygmund spaces

The case of Sobolev and Nikol'skii spaces studied in Section 1.1 differs essentially from the case of the spaces H^{ℓ}_{∞}, $\ell\in\mathbb{N}$, considered here. If $\ell = 1$ then a function f belongs to $H^1_{\infty}(\mathbb{R}^n)$ if and only if it has finite norm

$$\|f\,|H^1_{\infty}(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{\substack{x\in\mathbb{R}^n \\ 0\neq h\in\mathbb{R}^n}} \frac{|\Delta^2_h f(x)|}{|h|}.$$

As usual, $\Delta^2_h f(x) = f(x-h) - 2f(x) + f(x+h)$. For an arbitrary $\ell\in\mathbb{N}$, we define

$$H^{\ell}_{\infty}(\mathbb{R}^n) = \left\{f:\ f\in H^{\ell-1}_{\infty}(\mathbb{R}^n),\ \|f\,|H^{\ell}_{\infty}(\mathbb{R}^n)\| = \|f\,|H^{\ell-1}_{\infty}(\mathbb{R}^n)\| + \sum_{|j|=\ell-1} \sup_{\substack{x\in\mathbb{R}^n \\ 0\neq h\in\mathbb{R}^n}} \frac{|\Delta^2_h D^j f|}{|h|} < \infty\right\}.$$

Here we discuss the natural definition of the Zygmund spaces in domains. Our definition is in fact a successive realization of the concept that the norm of a function in a domain has to take into account the inner geometry of the domain.

Let g be an open set in $\mathbb{R}^n$ and k be a natural number. We say that a jet $\{f_j:\ |j|\le k-1\}$ of functions defined on G belongs to the class $\Lambda^k(G)$ if there exist collections $\{f_{j,\nu}:\ |j|\le k\}$, $\nu\in\mathbb{N}$, of functions defined on G, and a constant $M>0$ such that for all points $x,y\in G$ the following conditions hold:

$$(1.5)\qquad |f_j(x) - f_{j,\nu}(x)| \le M2^{-\nu(k-|j|)},\quad \nu\ge 1,\ |j|\le k-1,$$

$$(1.6)\qquad |f_{j,\nu}(x) - f_{j,\mu}(x)| \le M2^{\mu-\nu},\quad \mu\ge\nu\ge 1,\ |j| = k,$$

$$(1.7)\qquad \left|f_{j,\nu}(x) - \sum_{|j+s|\le k} \frac{f_{j+s,\nu}(y)}{s!}(x-y)^s\right| \le M2^{\nu} d_{1,G}(x,y)^{k+1-|j|},\quad |j|\le k,\ \nu\ge 1,$$

(1.8) $$|f_{j,1}(x)| \le M, \quad |j|\le k.$$

The norm $\|f|\Lambda^k(G)\|$ of the jet $f = \{f_j: |j|\le k-1\}$ is equal to infimum of all M satisfying conditions (1.5) - (1.8) with some $\{f_{j,\nu}: |j|\le k\}$, $\nu\in\mathbb{N}$.

THEOREM 1.2 [8]. *For all* $k\in\mathbb{N}$ *the spaces* $H^k_\infty(\mathbb{R}^n)$ *and* $\Lambda^k(\mathbb{R}^n)$ *coincide.*

2. Boundary values of differentiable functions

2.1. Trace operator

Recall that $k = 0,1,2,\dots$ and $\ell\in(k,k+1]$. Let us consider the metric space $G_\alpha = (G,d_{\alpha,G})$, $\alpha\in(0,1]$. Let $\tilde G_\alpha$ be the completion of G_α with respect to the metric $d_{\alpha,G}$. It is easy to verify that any function of $\mathrm{Lip}(\ell,G_\alpha)$ extends by the continuity to the completion $\tilde G_\alpha$. Indeed, if $|j| = k$, then $|f^{(j)}(x) - f^{(j)}(y)| \le R_j(x,y) \le Md_{\alpha,G}(x,y)$, where $f^{(j)}(x) = D^jf(x)$. Hence we have an extension by continuity to $\tilde G$ of the functions $f^{(j)}(x)$, $|j| = k$. Let us now suppose that the functions $f^{(j)}(x) = D^jf(x)$, $p\le|j|\le k$, $p = 1,2,\dots,k$, are already extended to $\tilde G_\alpha$. The extension to $\tilde G_\alpha$ of the functions $f^{(j)}(x) = D^jf(x)$, $|j| = p-1$, follows from the expansion

$$f^{(j)}(x) - f^{(j)}(y) = \sum_{|j|<|j+s|\le k} \frac{f^{(j+s)}(y)}{s!}(x-y)^s + R_j(x,y)$$

and the inequalities $|R_j(x,y)| \le Md_{\alpha,G}(x,y)^{(\ell-(p-1))/\alpha}$, $|x-y|^\alpha \le d_{\alpha,G}(x,y)$, hold. As there exists a unique extension of the functions from $\mathrm{Lip}(\ell,G_\alpha)$ to $\tilde G_\alpha$ it is natural to introduce the function class $\mathrm{Lip}(\ell,\tilde G_\alpha)$, consisting of the jets $\{f^{(j)}(x): |j|\le k\}$ defined on $\tilde G_\alpha$ as the extension by continuity of collections from $\mathrm{Lip}(\ell,G_\alpha)$ with respect to the metric $d_{\alpha,G}(x,y)$.

In order to describe the class $\mathrm{Lip}(\ell,\tilde G_\alpha)$ in an invariant way, let us define the mapping $i_\alpha:\tilde G_\alpha \longrightarrow G$ as the extension by continuity of the identical mapping on G. This extension exists by the inequality $|i_\alpha(x) - i_\alpha(y)|^\alpha \le d_{\alpha,G}(x,y)$. The elements of the class $\mathrm{Lip}(\ell,\tilde G_\alpha)$ are the jets $\{f^{(j)}: |j|\le k\}$, consisting of the continuous functions defined on $\tilde G_\alpha$. The norm in the space $\mathrm{Lip}(\ell,\tilde G_\alpha)$ is

$$(2.1)\qquad \|f|\mathrm{Lip}(\ell,\tilde G_\alpha)\| = \sum_{|j|\le k}\left\{\sup|f^{(j)}(x)| + \sup\frac{|R_j(x,y)|}{\bar d_{\alpha,G}(x,y)^{\ell-|j|}}\right\},$$

where the supremum is taken over all points $x,y\in G_\alpha$, $\alpha = \ell-k$, and

$$f^{(j)}(x) = \sum_{|j+s|\le k}\frac{f^{(j+s)}(y)}{s!}(i_\alpha(x)-i_\alpha(y))^s + R_j(x,y).$$

It is easy to verify that $f^{(j)}(x) = D^j f(x)$ for $x\in G$, $|j|\le k$.

Thus we have proved the following assertion.

PROPOSITION 2.1. *Let* G *be an arbitrary domain in* $\mathbb{R}^n$. *Then there exists a natural isometry of the spaces* $\mathrm{Lip}(\ell,G_\alpha)$ *and* $\mathrm{Lip}(\ell,\tilde G_\alpha)$*: any element of* $\mathrm{Lip}(\ell,G_\alpha)$ *possesses a unique extension to* $\tilde G_\alpha$ *with respect to the inner* α*-metric* $d_{\alpha,G}(x,y)$ *which is an element of* $\mathrm{Lip}(\ell,\tilde G_\alpha)$.

Further, the set $\partial G_\alpha = \tilde G_\alpha\setminus G_\alpha$ will be called the *α-boundary* of the domain G. It is natural to consider the restrictions of the elements of $\mathrm{Lip}(\ell,\tilde G_\alpha)$ to ∂G_α as *boundary values* or *traces of the functions from* $\mathrm{Lip}(\ell,G_\alpha)$ *to the α-boundary of the domain* G. To give a rigorous definition of the concept of the trace we denote by $\mathrm{Lip}(\ell,\partial G_\alpha)$ the function space whose elements are jets $\{f^{(j)}: |j|\le k\}$, consisting of continuous functions defined on $(\partial G_\alpha, d_{\alpha,G})$ with the finite norm (2.1), where the supremum is taken over all points $x,y\in\partial G_\alpha$. The trace operator $tr_\ell:\mathrm{Lip}(\ell,G_\alpha)\longrightarrow\mathrm{Lip}(\ell,\partial G_\alpha)$ is defined as the superposition of the isometry $i:\mathrm{Lip}(\ell,G_\alpha)\longrightarrow\mathrm{Lip}(\ell,\tilde G_\alpha)$ given in Proposition 2.1 and the restriction of the elements of $\mathrm{Lip}(\ell,\tilde G_\alpha)$ to ∂G_α, which are contained in the class $\mathrm{Lip}(\ell,\partial G_\alpha)$.

Let us formulate the above as the following theorem.

THEOREM 2.1 [9]. *Let* G *be an arbitrary domain in* $\mathbb{R}^n$. *Then there exist the bounded trace operators*

$$tr_\ell: W^\ell_\infty(G)\longrightarrow \mathrm{Lip}(\ell,\partial G_1),\ \alpha = 1,\ \ell\in\mathbb{N},$$
$$tr_\ell: H^\ell_\infty(G)\longrightarrow \mathrm{Lip}(\ell,\partial G_\alpha),\ \alpha = \ell-k,\ \ell\in(k,k+1),\ k = 0,1,\dots,$$

defined by the continuity with respect to the metric $d_{\alpha,G}$.

2.2. Extension operator

The characterization of the trace of the functions from the Sobolev and Nikol'skii classes given by Theorem 2.1, are reversible.

<u>*THEOREM 2.2*</u> [9]. *Let* G *be an arbitrary domain in* $\mathbb{R}^n$. *Then there exist the linear bounded extension operators*

$$\mathrm{ext}_k : \mathrm{Lip}(\ell, \partial G) \longrightarrow W^{\ell}_{\infty}(G), \ k = \ell-1, \ \ell \in \mathbb{N},$$

$$\mathrm{ext}_k : \mathrm{Lip}(\ell, \partial G_\alpha) \longrightarrow H^{\ell}_{\infty}(G), \ \ell \in (k, k+1), \ k = 0, 1, \dots, \ \alpha = \ell - k,$$

such that $\mathrm{tr}_\ell \circ \mathrm{ext}_k$ *is the identical mapping.*

P r o o f. The mapping $i_\alpha : (G_\alpha, d_{\alpha,G}) \longrightarrow (G, |x-y|^\alpha)$ has the following property: If $U \subset G$ is a convex set, then $|i_\alpha(x) - i_\alpha(y)|^\alpha = d_{\alpha,G}(x,y)$ for every $x, y \in U$. Hence, we have the equality

$$d_{\alpha,G}(x, \partial G) = |i_\alpha(x) - \partial G|^\alpha, \ x \in G, \tag{2.2}$$

where

$$|i_\alpha(x) - \partial G|^\alpha = \inf \left\{ |x-y|^\alpha : \ y \in \partial G \right\}$$

and

$$d_{\alpha,G}(x, \partial G_\alpha) = \inf \left\{ d_{\alpha,G}(x,y) : \ y \in G_\alpha \right\}.$$

This relation implies that the space $G_\alpha = (G, d_{\alpha,G})$ can be decomposed into the "Whitney cubes" in the same way as it is done for the domain G with respect to the Euclidean metric in the Whitney extension theorem (see [10]).

Let G be an open set in $\mathbb{R}^n$. Then there exists a collection of closed cubes Q_k with sides parallel to the axes and with the following properties:

(a) $G = \cup Q_k$.

(b) The interiors of the cubes Q_k are mutually disjoint.

(c) The distance $d(Q_k, F)$ of Q_k to F satisfies

$$\mathrm{diam}\, Q_k \le d(Q_k, F) \le 4 \mathrm{diam}\, Q_k, \ k \in \mathbb{N}.$$

(d) If $Q_k \cap Q_\nu \neq \emptyset$, then

$$\frac{1}{4} \mathrm{diam}\, Q_k \le \mathrm{diam}\, Q_\nu \le 4\, \mathrm{diam}\, Q_k.$$

(e) Let ε be a fixed number satisfying $0<\varepsilon<1/4$, and let Q_k^* denote the cube which has the same center as Q_k but is expanded by the factor $1+\varepsilon$. Then each point in G is contained in at most N_0 cubes Q_k^*, where N_0 is a fixed number.

Furthermore, $Q_k^* \cap Q_\nu \neq \emptyset$ if and only if $Q_k \cap Q_\nu \neq \emptyset$.

In connection with the decomposition, we shall use the following notation:

x_k is the center of Q_k,

ℓ_k is the diameter of Q_k,

a_k is the length of the side of Q_k (thus, $\ell_k = \sqrt{n}a_k$).

Now, consider a partition of unity: Let ψ be a C^∞-function satisfying $0 \le \psi \le 1$, $\psi(x) = 1$ for $x \in Q$ and $\psi(x) = 0$ for $x \notin (1+\varepsilon)Q$, where Q denotes the cube centered at the origin with sides of length 1 parallel to the axes. Define ψ_k by $\psi_k(x) = \psi((x-x_k)/a_k)$, and φ_k by $\varphi_k(x) = \psi_k(x)/\sum\limits_k \psi_k(x)$, $x \in G$. Then $\varphi_k(x) = 0$ for $x \notin Q_k^*$, $\sum\limits_k \varphi_k(x) \equiv 1$ on G, and it is easy to show that for any multi-index j

$$|D^j \varphi_k(x)| \le A_j (\operatorname{diam} Q_k)^{-|j|}. \tag{2.3}$$

Using the equality (2.1) we can rewrite the property (c) in the following way:

$$\operatorname{diam} Q_k \le \bar{d}_{\alpha,G}(Q_k, \partial G) \le 4 \operatorname{diam} Q_k. \tag{c'}$$

Note that if $x \in Q_k^*$, then

$$\bar{d}_{\alpha,G}(x, \partial G_\alpha) \sim \operatorname{diam} Q_k \tag{2.4}$$

and

$$\bar{d}_{\alpha,G}(Q_k^*, \partial G_\alpha) \sim \operatorname{diam} Q_k. \tag{2.5}$$

As usual, the symbol $\sim$ denotes equivalent quantities. Furthermore, if $y \in \partial G_\alpha$ and $x \in Q_k^*$, then

$$d_{\alpha,G}(y, p_k) \le d_{\alpha,G}(y, x) + d_{\alpha,G}(x, p_k),$$

where $p_k \in \partial G_\alpha$ is a point for which $d_{\alpha,G}(Q_k, \partial G_\alpha) = d_{\alpha,G}(Q_k, p_k)$. Hence, according to (2.4) and (2.5), for $y \in \partial G_\alpha$ and $x \in Q_k^*$ we have

$$d_{\alpha,G}(y, p_k) \le c\, d_{\alpha,G}(y, x), \tag{2.6}$$

where c is a constant.

Let $\{f^{(j)} : |j| \le k\}$ be a collection of functions defined on ∂G_α. We define the value of an extension operator ext_k on the collection $\{f^{(j)}\}$ as follows:

$$\mathrm{ext}_k(\{f^{(j)}\}) = \begin{cases} f^{(0)}(x), & x \in \partial G_\alpha, \\ \sum\limits_i{}' P(x, p_i)\varphi_i(x), & x \in G, \end{cases} \tag{2.7}$$

where $P(x,y)$ is the polynomial giving the "Taylor expansion" of the function f with respect to the point $y\in\partial G_\alpha$, i.e.

$$P(x,y) = \sum_{|s|\le k} \frac{f^{(s)}(y)}{s!}(x-i_\alpha(y))^s, \quad x\in G, \; y\in\partial\tilde{G}_\alpha,$$

and $p_i\in\partial G_\alpha$ is the closest point to the cube Q_i. The symbol $\sum'$ indicates that the summation is taken over all cubes whose distance to ∂G_α does not exceed 1. It is easy to see that the value of the extension operator ext_k on the collection $\{f^{(j)}\}$ is a function $f = ext_k(\{f^{(j)}\})$ extending the function $f = f^{(0)}$ onto $\tilde{G}_\alpha$.

Assuming

$$P_j(x,y) = \sum_{|j+s|\le k} \frac{f^{(j+s)}(y)}{s!}(i_\alpha(x)-i_\alpha(y))^s, \quad x,y\in\tilde{G}_\alpha,$$

we have $f^{(j)}(x) - P_j(x,y) = R_j(x,y)$, $x,y\in\partial G_\alpha$. We denote $P_0(x,y) = P(x,y)$ and $R_0(x,y) = R(x,y)$.

LEMMA 2.1. Let $a,b\in\partial G_\alpha$ *and* $x\in G$. *Then*

$$P(x,b) - P(x,a) = \sum_{|s|\le k} \frac{R_s(b,a)}{s!}(x-i_\alpha(b))^s$$

or, in the general case,

$$P_j(x,b) - P_j(x,a) = \sum_{|j+s|\le k} \frac{R_{j+s}(b,a)}{s!}(x-i_\alpha(b))^s.$$

Using properties (a) - (e) for the points $x\in Q_k^*$, we have

(2.8) $$\delta(x) = |x-\partial G| = d_{\alpha,G}(x,\partial G_\alpha)^{1/\alpha} \sim d_{\alpha,G}(Q_k,\partial G_\alpha)^{1/\alpha} \sim d_{\alpha,G}(Q_k^*,\partial G_\alpha)^{1/\alpha}.$$

As in the Whitney type extension theorem [10] it is enough to consider points $x\in G$ such that $\delta(x)\le c$ where c is a positive constant. Further, we assume that $\|f\,|\,\mathrm{Lip}(\ell,\partial G_\alpha)\| = 1$.

Now, we shall prove the following inequalities:

(i) $|f(x) - P(x,a)| \le A\tilde{d}_{\alpha,G}(x,a)^\ell$, $x\in\tilde{G}_\alpha$, $a\in\partial G_\alpha$;

(ii) $|f^{(j)}(x) - P_j(x,a)| \le A\tilde{d}_{\alpha,G}(x,a)^{\ell-|j|}$, $x\in\tilde{G}_\alpha$, $a\in\partial G_\alpha$, $|j|\le k$;

(iii) $|f^{(j)}(x)| \le A$, $|j|\le k$;

(iv) $|f^{(j)}(x)| \le A\delta(x)^{\ell-k-1}$, $x\in G$, $|j| = k+1$.

We recall that $\tilde{d}_{\alpha,G}(x,y) = d_{\alpha,G}(x,y)^{1/\alpha}$.

Inequality (i) is valid for points $x\in\partial G_\alpha$ with the constant $A = 1$ because

$f\in \mathrm{Lip}(\ell,\partial G_\alpha)$. Supposing $x\in G$, $\delta(x)\le c$, we have

$$f(x) - P(x,a) = \sum_i (P(x,p_i) - P(x,a))\varphi_i(x).$$

Applying Lemma 2.1 we obtain

$$|f(x) - P(x,a)| \le \sum_{|s|\le k} \sum_i \tilde{d}_{\alpha,G}(p_i,a)^{\ell-|s|}\tilde{d}_{\alpha,G}(x,a)^{|s|},$$

where the inner sum is taken over all cubes Q_i such that $x\in Q_i^*$. According to (2.5) we have $\tilde{d}_{\alpha,G}(p_i,a) \le c\tilde{d}_{\alpha,G}(x,a)$. Hence the inequality (i) holds.

To prove (ii) we write

$$f^{(j)}(x) = \sum_i (\partial/\partial x)^j(P(x,p_i))\varphi_i(x) + \text{other summands}.$$

Since $(\partial/\partial x)^j P(x,p_i) = P_j(x,p_i)$, in the above mentioned way we obtain the result for the first summand.

As $\sum_i(\partial/\partial x)^j\varphi_i(x) = 0$, the summands of the type

$$P_{j-s}(x,p_i)(\partial/\partial x)^s\varphi_i(x), \tag{2.9}$$

$|s|>0$ and $s_i\le j_i$, $i\le 1,2,\dots,n$, are equal to

$$\sum_i (P_{j-s}(x,p_i) - P_{j-s}(x,a))(\partial/\partial x)^s\varphi_i(x). \tag{2.10}$$

Using (2.3) and (2.8), we prove the inequality (ii).

The estimate (iii) for points $x\in G$, $\delta(x)\le c$, follows from (ii).

If we differentiate the function $f(x)$ in $\{x\in G: \delta(x)\le c_1\}$, we obtain that $f^{(j)}(x)$ is equal to a sum of expressions of type (2.9). As $|j| = k+1$, then necessarily $|s|>0$ because $P_j(x,p_i) = 0$. Therefore, $f^{(j)}(x)$, $|j| = k+1$, is the sum of type (2.10), where $a\in\partial G_\alpha$ is the closest point to x. By Lemma 2.1, (2.3), (2.6) and (2.8), we obtain an upper bound for the sum of type (2.10):

$$A\tilde{d}_{\alpha,G}(p_i,a)^{\ell-|j|+|s|}\delta(x)^{-|s|} \le A'\delta(x)^{\ell-k-1}.$$

The inequality (iv) follows.

From (i) and (ii) we obtain

$$|R_j(x,y)| \le A\tilde{d}_{\alpha,G}(x,y)^{\ell-|j|}, \quad |j|\le k, \tag{2.11}$$

at points $x\in G$ and $y\in\partial G_\alpha$. Let us prove now that inequalities (2.11) are valid at $x,y\in G$. The main case is when the segment L connecting the points x and y, is contained in the domain G. First we suppose that the length of the segment L is less than the distance from the segment L to the boundary ∂G. By the Taylor formula and the use of (iv) we obtain the bound

$$|R_j(x,y)| \le C \sup_{z\in L} |\nabla_{k+1} f(z)|\,|x-y|^{k+1-|j|} \le C\, |x-y|^{k+1-|j|} \sup_{z\in L} \delta(z)^{\ell-k-1}.$$

Hence in this case $\delta(z)>|x-y|$, $z\in L$, and the result immediately follows:

$$|R_j(x,y)| \le c|x-y|^{\ell-|j|} \le c\tilde{d}_{\alpha,G}(x,y)^{\ell-|j|}.$$

Now we consider the case when at some point $z\in L$ we have $\delta(x)\le|y-x|$. Then there exist points $z'\in L$ and $y'\in\partial G_\alpha$ such that $d_{\alpha,G}(z',y') \le d_{\alpha,G}(x,y)$. Hence $d_{\alpha,G}(y',x) \le 2d_{\alpha,G}(y,x)$ and $d_{\alpha,G}(y',y) \le 2d_{\alpha,G}(x,y)$. According to Lemma 2.1 and the inequality (2.11) we have

$$|R_j(x,y)| = |f^{(j)}(x) - P_j(x,y) - P_j(x,y') + P_j(x,y')|$$

$$\le \left|R_j(x,y') + \sum_{|j+s|\le k} \frac{R_{j+s}(y',y)}{s!}(x-i_\alpha(y'))^s\right|$$

$$\le A\tilde{d}_{\alpha,G}(x,y')^{\ell-|j|} + \sum_{|j+s|\le k} \tilde{d}_{\alpha,G}(y,y')^{\ell-|j|-|s|}\tilde{d}_{\alpha,G}(x,y')^{|s|}$$

$$\le A\tilde{d}_{\alpha,G}(x,y)^{\ell-|j|}.$$

The last case concerns the situation when the points x and y cannot be connected by a segment contained in the domain G. The proof in this case is fully analogous to the previous one as long as there exist two points $z'\in G$ and $y\in\partial G_\alpha$ for which we have

$$d_{\alpha,G}(z',y') \le d_{\alpha,G}(x,y) \quad \text{and} \quad \max\{d_{\alpha,G}(z',x), d_{\alpha,G}(z',y)\} \le d_{\alpha,G}(x,y).$$

Therefore, $d_{\alpha,G}(y',x) \le 2d_{\alpha,G}(x,y)$ and $d_{\alpha,G}(y',y) \le 2d_{\alpha,G}(x,y)$ what coincides with the above situation. Thus we have proved the theorem. □

2.3. Above mentioned questions in Zygmund spaces

Here we prove analogues of theorems 2.1 and 2.2 for the Zygmund spaces $\Lambda^k(G)$, $k\in\mathbb{N}$, G is an arbitrary domain in $\mathbb{R}^n$. Let $\{f_j: |j|\le k\}$, be an element of the space $\Lambda^k(G)$ and $\{f_{j,\nu}: |j|\le k\}$, $\nu\in\mathbb{N}$, collections of functions corresponding $\{f_j\}$ according to the definition of the space $\Lambda^k(G)$. From (1.5) and (1.8) for $|j|\le k-1$ we have

$$|f_{j,\nu}(x)| \le |f_{j,\nu}(x) - f_j(x)| + |f_j(x) - f_{j,1}(x)| + |f_{j,1}(x)| \le 2M,$$

and for $|j|=k$,

$$|f_{j,\nu}(x)| \le |f_{j,\nu}(x) - f_{j,1}(x)| + |f_{j,1}(x)|$$
$$\le \sum_{i=2}^{\nu} |f_{j,i}(x) - f_{j,i-1}(x)| + |f_{j,1}(x)| \le (\nu-1)2M + M \le 2M\nu.$$

Hence the following relations are valid for all $x \in G$ and $\nu \in \mathbb{N}$:

$$|f_{j,\nu}(x)| \le 2M, \quad |j| \le k-1, \tag{2.12}$$

$$|f_{j,\nu}(x)| \le 2M\nu, \quad |j| = k. \tag{2.13}$$

In the same way as in Section 2.1 we show that elements of the collection $\{f_{j,\nu}: |j| \le k\}$ for every fixed $\nu \in \mathbb{N}$ possess extensions onto $\tilde{G}_1$ by continuity with respect to the metric space $(G_1, d_{1,G})$.

It is natural to consider restrictions of extensions of the jets $\{f_{j,\nu}: |j| \le k\}$, $\nu \in \mathbb{N}$, to the 1-boundary ∂G_1 of the domain G as the boundary values of the element $\{f_j: |j| \le k\}$ from the space $\Lambda^k(G)$ on the 1-boundary of domain G. To give an exact sense to the concept of the trace let us define the function class $\Lambda^k(\partial G_1)$. Jets $\{f_j: |j| \le k-1\}$ of functions defined on the 1-boundary ∂G_1 are called elements of the class $\Lambda^k(\partial G_1)$ if there exist collections $\{f_{j,\nu}: |j| \le k\}$, $\nu \in \mathbb{N}$, of functions defined also on ∂G_1 such that the following conditions hold for $x, y \in \partial G_1$:

$$|f_j(x) - f_{j,\nu}(x)| \le M2^{-\nu(k-|j|)}, \quad \nu \ge 1, \quad |j| \le k-1, \tag{2.14}$$

$$|f_{j,\nu}(x) - f_{j,\mu}(x)| \le M2^{\mu-\nu}, \quad \mu \ge \nu \ge 1, \quad |j| = k, \tag{2.15}$$

$$|R^k_{j,\nu}(x,y)| \le M2^{\nu} d_{1,G}(x,y)^{k+1-|j|}, \quad \nu \ge 1, \quad |j| \le k, \tag{2.16}$$

$$|f_{j,1}(x)| \le M, \quad |j| \le k. \tag{2.17}$$

Here $R^k_{j,\nu}(x,y)$ are defined by

$$f_{j,\nu}(x) = \sum_{|j+s| \le k} \frac{f_{j+s,\nu}(y)}{s!}(i_1(x) - i_1(y))^s + R^k_{j,\nu}(x,y).$$

The norm $\|f|\Lambda^k(\partial G_1)\|$ of the jet $f = \{f_j: |j| \le k-1\}$ is defined as the infimum of all constants M such that the conditions (2.14) - (2.17) hold for some collection $\{f_{j,\nu}: |j| \le k\}$, $\nu \in \mathbb{N}$.

Let G be an arbitrary domain in $\mathbb{R}^n$. We define a trace operator by the following way: Elements of a jet $\{f_{j,\nu}: |j| \le k\}$, $\nu \in \mathbb{N}$, corresponding to a function $f = \{f_j\} \in \Lambda^k(G)$, extend to G by the continuity and then restrict to

∂G_1. As a result we have jets satisfying conditions (2.14) - (2.17) which form an element of the space $\Lambda^k(\partial G_1)$ because, according to (1.5) and (2.14),

$$\lim f_{j,\nu}(x) = f_j(x), \quad |j|\le k-1,$$

for any $x\in\partial G_1$. Hence we obtain the following result:

THEOREM 2.3. Let G be an arbitrary domain in $\mathbb{R}^n$. The trace $T_k f$ of $f\in\Lambda^k(G)$ is an element of the space $\Lambda^k(\partial G_1)$, and the trace operator

$$T_k : \Lambda^k(G) \longrightarrow \Lambda^k(\partial G_1), \quad k\in\mathbb{N},$$

is bounded with the norm not exceeding 1.

The characterization of traces of the Zygmund classes given by Theorem 2.3 is reversible.

THEOREM 2.4. Let G be an arbitrary domain in $\mathbb{R}^n$. Then there exists the bounded extension operator

$$E_k : \Lambda^k(\partial G_1) \longrightarrow \Lambda^k(G), \quad k\in\mathbb{N},$$

in the sense that $T_k \circ E_k$ is the identical mapping.

Theorems 2.3 and 2.4 have been proved jointly with Yu. Bojarskii and formulated in [11].

P r o o f. Let $\{Q_k\}$ be the above mentioned decomposition of the domain G into cubes and $\{\varphi_k\}$ be a corresponding partition of unity. Here it is more convenient for us to use a double numeration for the Whitney cubes. For $\nu\in\mathbb{N}$, cubes $Q_{\nu,i}$ are the cubes from the collection $\{Q_k\}$ such that $2^{-(\nu+1)} < \operatorname{diam} Q_{\nu,i} \le 2^{-\nu}$. Further, let $\{\varphi_{\nu,i}\}$ be the partition of unity corresponding to the cubes $\{Q_{\nu,i}\}$ and let $p_{\nu,i}\in\partial G_1$ be the closest point to the cube $Q_{\nu,i}$.

Let us consider an element $\{f_j : |j|\le k-1\}$, of the space $\Lambda^k(\partial G_1)$ and, according to the definition, a sequence of jets $\{f_{j,\nu} : |j|\le k\}$, $\nu\in\mathbb{N}$, corresponding to it. We set

$$P_\nu(x,y) = \sum_{|s|\le k} \frac{f_{s,\nu}(y)}{s!}(x-i_1(y))^s \quad \text{for } x\in G,\ y\in\partial G_1.$$

The extension operator E_k is defined on the jet $\{f_j\}\in\Lambda^k(\partial G_1)$ in the following way:

$$E_k(\{f_j\})(x) = \begin{cases} \sum\limits_{\mu=1}^{\infty} \sum\limits_{i} \varphi_{\mu,i}(x) P_\mu(x, p_{\mu,i}), & x \in G, \\ f(x), & x \in \partial G_1, \end{cases}$$

where for fixed $\mu \in \mathbb{N}$ the summation is taken over all i such that $2^{-(\mu+1)} < \operatorname{diam} Q_{\mu,i} \le 2^{-\mu}$. Let F(x) denote the function $E_k(\{f_j\})$. We define the approximating sequence $\{F_\nu : \nu \in \mathbb{N}\}$, corresponding to f, by

$$F_\nu(x) = \begin{cases} \sum\limits_{\mu=\nu+1}^{\infty} \sum\limits_{i} \varphi_{\mu,i}(x) P_\nu(x, p_{\mu,i}) + \sum\limits_{\mu=1}^{\nu} \sum\limits_{i} \varphi_{\mu,i}(x) P_\mu(x, p_{\mu,i}), & x \in G, \\ f_\nu(x), & x \in \partial G_1, \end{cases}$$

where the summation over i is the same as above.

Let $P_{j,\nu}(x,y)$, $|j| \le k$, $y \in \partial G_1$, $x \in G$, denote the polynomial $P_\nu(x,y)$ if $j = 0$ and the polynomial $D_x^j P_\nu(x,y)$ if $0 < |j| \le k$, respectively, where D_x denotes the differentiation with respect to x. In this notation, we have

$$P_{j,\nu}(x,y) = \sum_{|j+s| \le k} \frac{f_{j+s,\nu}(y)}{s!} (x - i_1(y))^s.$$

Further, we consider a function $\{f_j\}$ whose norm is equal to 1.

<u>*LEMMA 2.2.*</u> *Let* $a, b \in \partial G_1$, $x \in G$ *and let the conditions* $d_{1,G}(a,b) \le c_1 \rho$, $d_{1,G}(x,a) \le c_2 \rho$ *hold with some constants* c_1, c_2. *Then there exists a constant* $c > 0$ *such that*

$$|P_{j,\nu}(x,a) - P_{j,\nu}(x,b)| \le c 2^\nu \rho^{k+1-|j|}, \quad |j| \le k.$$

P r o o f. According to Lemma 2.1 and the inequality (2.16) we have

$$\left| P_{j,\nu}(x,a) - P_{j,\nu}(x,b) \right| = \left| \sum_{|j+s| \le k} \frac{R_{j+s,\nu}(a,b)}{s!} (x - i_1(a))^s \right|$$

$$\le c_1 c_2 2^\nu d_{1,G}(a,b)^{k+1-|j+s|} d_{1,G}(x,a)^{|s|} \le c \rho^{k+1-|j|} 2^\nu.$$

$\square$

<u>*LEMMA 2.3.*</u> *Let the points* $a, b \in \partial G_1$, $x \in G$, *satisfy the conditions* $d_{1,G}(a,b) \le 2^{-\nu}$, $d_{1,G}(x,b) \le 2^{-\mu}$ *with* $\mu \ge \nu$. *Then the inequality*

$$|P_{j,\nu}(x,a) - P_{j,\mu}(x,b)| \le c 2^{-\nu(k-|j|)}$$

holds for every j, $|j| \le k-1$.

P r o o f. Let us estimate the difference

$$|P_{j,\nu}(x,b) - P_{j,\mu}(x,b)| \le \sum_{|j+s|\le k} \frac{|f_{j+s,\nu}(b) - f_{j+s,\mu}(b)|}{s!}|x - i_1(b)|^{|s|}$$

Using the assumption of the lemma and (2.14) we estimate the summands on the right hand side with $|j|\le k-1$ from above by

$$c2^{-\nu(k+|j+s|)}2^{-\mu|s|} \le c2^{-\nu(k-|j|)}.$$

To estimate summands with $|j+s| = k$ we use (2.15) and

$$c2^{-\mu|s|}2^{\mu-\nu} \le c2^{-\nu|s|} = c2^{-\nu(k-|j|)}.$$

Applying Lemma 2.2 we obtain the result

$$|P_{j,\nu}(x,a) - P_{j,\mu}(x,b)| \le |P_{j,\nu}(x,a) - P_{j,\nu}(x,b)| + |P_{j,\nu}(x,b) - P_{j,\mu}(x,b)|$$

$$\le c2^{-\nu(k-|j|)}. \qquad \square$$

In the folowing lemmas, we shall show that the functions $F_j(x) = D^jF(x)$, $|j|\le k-1$, and the approximating sequence $\{F_{j,\nu}(x)\colon |j|\le k\}$, $\nu\in\mathbb{N}$, of functions $F_{j,\nu}(x) = D^jF_\nu(x)$, possess the properties (1.5) - (1.8).

LEMMA 2.4. *If* $x\in\partial G_1$, *then*

$$\lim_{\substack{d_{1,G}(x,y)\to 0\\ y\in G_1}} F_{j,\nu}(y) = f_{j,\nu}(x),\ |j|\le k,\ \nu\in\mathbb{N}.$$

If $x\in\tilde{G}_1$, *then*

(2.18) $$|F_j(x) - F_{j,\nu}(x)| \le c2^{-\nu(k-|j|)},\ |j|\le k-1,\ \nu\in\mathbb{N}.$$

P r o o f. If $x\in G$, $d_{1,G}(x,\partial G_1) > 2^{-\nu}$, then $F(x) \equiv F_\nu(x)$ by the construction. Let $x\in G$ be a point such that $2^{-(\tau+1)} < d_{1,G}(x,\partial G_1) \le 2^{-\tau}$, where $\tau\ge\nu$. Then the values $F_j(x)$ and $F_{j,\nu}(x)$ are equal to a finite sum whose summands have the form $D^\ell\varphi_{\mu,i}(x)P_{m,\mu}(x,p_{\mu,i})$ and $D^\ell\varphi_{\mu,i}(x)P_{m,\nu}(x,p_{\mu,i})$, $m+\ell = j$, respectively, and also the cubes participating in the sum satisfy $c_12^{-(\tau+1)} \le \operatorname{diam} Q_{\mu,i} \le c_22^{-\tau}$, where the constants c_1 and c_2 do not depend on x. The number of the summands is bounded by a constant which is independent of x. Now, using Lemma 2.3 and the fact that $\sum_{\mu,i}\varphi_{\mu,i}(x) = 1$ in the domain G, we obtain the estimate

$$|F_j(x) - F_{j,\nu}(x)| \le \sum_{m+\ell=j}\sum_{\mu,i}|D^\ell\varphi_{\mu,i}(x)||P_{m,\mu}(x,p_{\mu,i}) - P_{m,\nu}(x,p_{\mu,i})|$$

$$\le c\sum_{m+\ell=j}2^{\nu|\ell|}2^{-\nu(k-|m|)} \le c2^{-\nu(k-|j|)}.$$

If $x \in \partial G_1$ and $|j| = 0$, then the estimate (2.18) reduces to (2.14). To show (2.18) for $|j|>0$ we prove that $F_{j,\nu}(y) \to f_{j,\nu}(x)$ as $d_{1,G}(y,x) \to 0$. Let us write the difference $F_{j,\nu}(y) - f_{j,\nu}(x)$ in the form

$$\sum_{\mu,i} \varphi_{\mu,i}(y)(f_{j,\nu}(p_{\mu,i}) - f_{j,\nu}(x)) + \text{other summands.}$$

The estimate (2.6) yields $d_{1,G}(x,p_{\mu,i}) \le c d_{1,G}(x,y)$. The function $f_{j,\nu}$ is continuous on ∂G_1 and so the explicitly expressed summands tend to zero as $d_{1,G}(x,y) \to 0$. Since $\sum_{m,i} D^{\alpha}\varphi_{\mu,i}(y) \equiv 0$ for $y \in G$ the other summands can be written in the form

$$\sum_{\mu,i} D^{\alpha}\varphi_{\mu,i}(y)(f_{m,\nu}(p_{\mu,i}) - f_{m,\nu}(x)) + \sum_{0<|m+s|\le k} \frac{f_{m+s,\nu}(p_{\mu,i})}{s!}(y - i_1(p_{\mu,i}))^s,$$

where $\ell+m = j$, $|\ell|>0$. Using the estimates

$$|y - i_1(p_{\mu,i})| \le d_{1,G}(y,p_{\mu,i}) \le c d_{1,G}(x,y)$$

and an analogue of (2.12) for functions of $\Lambda^k(\partial G_1)$ we obtain that in this case the summands tend to 0 as $d_{1,G}(x,y) \to 0$, as well. $\square$

LEMMA 2.5. Let $|j| = k$ *and* $x \in \tilde{G}$. *Then*

$$|F_{j,\mu}(x) - F_{j,\nu}(x)| \le c 2^{\mu-\nu}, \quad \mu \ge \nu \ge 1. \tag{2.19}$$

The proof is completely analogous to that of Lemma 2.4. The difference $F_{j,\mu}(x) - F_{j,\nu}(x)$ is estimated by a finite sum of terms among which there are expressions of the form $\varphi_{\mu,i}(f_{j,\mu}(p_{\mu,i}) - f_{j,\nu}(p_{\nu,i}))$. We can estimate them by (2.15). In the proof of (2.19) for $x \in \partial G_1$ we use an analogue of (2.13) for functions from $\Lambda^k(\partial G_1)$. $\square$

LEMMA 2.6. There exists a constant $A>0$ *such that the inequality*

$$|F_{j,1}(x)| \le A$$

holds for every $x \in \tilde{G}$ *and* $|j| \le k$.

P r o o f. Since the number ν in the expression for $F_1(x)$ is equal to 1 we can write the functions $F_{j,1}(x)$ in the form

$$\begin{aligned} F_{j,1}(x) = {} & \sum_{\mu,i} \varphi_{\mu,i}(x) P_{j,1}(x,p_{\mu,i}) \\ & + \sum_{\substack{\ell+m=j \\ |\ell|>0}} D^{\ell}\varphi_{\mu,i}(x)(P_{m,1}(x,p_{\mu,i}) - P_{m,1}(x,b)). \end{aligned} \tag{2.20}$$

The first sum is estimated by

$$|f_{j,1}(p_{\mu,i})| + \sum_{0<|j+s|\le k} \frac{|f_{j+s,1}(p_{\mu,i})|}{s!}|x-i_1(p_{\mu,i})|^s \le c + c2^{-\mu(k-|j|)} \le A$$

where we used 2.17. To evaluate the second sum we use Lemma 2.2. Note that

$$c2^{\mu|\ell|}2^{-\mu(k-|m|)} = c2^{-\mu(k-|j|)} \le c. \qquad \square$$

LEMMA 2.7. Let $x\in G$ *and* $|j| = k+1$. *Then*

$$|F_{j,\nu}(x)| \le c2^{\nu}, \quad \nu\in\mathbb{N}.$$

P r o o f. Since $P_{m,\nu}(x,p_{\mu,i}) \equiv 0$ for $|m|>k$ we can write (cf. (2.20))

$$F_{j,\nu}(x) = \sum_{\mu>\nu}\sum_{i}\sum_{\substack{\ell+m=j\\|\ell|>0}} D^{\ell}\varphi_{\mu,i}(x)(P_{m,\nu}(x,p_{\mu,i}) - P_{m,\nu}(x,b))$$

$$+ \sum_{\mu=1}^{\nu}\sum_{i}\sum_{\substack{\ell+m=j\\|\ell|>0}} D^{\ell}\varphi_{\mu,i}(x)(P_{m,\mu}(x,p_{\mu,i}) - P_{m,\mu}(x,b))$$

where $b\in\partial G_1$ is the closest point to x. Estimating the summands on the right hand side by Lemma 2.2 we obtain the upper bound $c\rho^{-|\ell|}2^{\nu}\rho^{k+1-|m|} = c2^{\nu}$ for the first summand and $c\rho^{-|\ell|}2^{\mu}\rho^{k+1-|m|} \le c2^{\nu}$ for the second one. $\square$

LEMMA 2.8. Let $x\in G$ *and* $a\in\partial G_1$. *Then*

$$|F_{j,\nu}(x) - P_{j,\nu}(x,a)| \le c2^{\nu}d_{1,G}(x,a)^{k+1-|j|}, \quad |j|\le k.$$

P r o o f. Using the definition of functions F_ν and Lemma 2.2 we obtain

$$|F_{j,\nu}(x) - P_{j,\nu}(x,a)| = \Big|\sum_{\mu,i}\varphi_{\mu,i}(x)(P_{j,\nu}(x,p_{\mu,i}) - P_{j,\nu}(x,a))$$

$$+ \sum_{\substack{\ell+m=j\\|\ell|>0}}\sum_{\mu,i} D^{\ell}\varphi_{\mu,i}(x)(P_{m,\nu}(x,p_{\mu,i}) - P_{m,\nu}(x,b))\Big|$$

$$\le c2^{\nu}d_{1,G}(x,a)^{k+1-|j|},$$

where $b\in\partial G_1$ is the closest point to x. $\square$

LEMMA 2.9. Let $x,y\in G$ *and let* $R^k_{j,\nu}(x,y)$ *be the remainders in the expansion of the function* $F_{j,\nu}(x) = \sum_{|j+s|\le k}\frac{F_{j+s,\nu}(y)}{s!}(i_1(x)-i_1(y))^s + R^k_{j,\nu}(x,y)$. *Then*

$$|R^k_{j,\nu}(x,y)| \le c2^{\nu}d_{1,G}(x,y)^{k+1-|j|}, \quad |j|\le k.$$

P r o o f. The case when $x\in\partial G$, $y\in\partial G_1$, reduces to (2.16) as $F_{j,\nu}(y) = f_{j,\nu}(y)$ (see Lemma 2.4). The case $x\in G$, $y\in\partial G_1$ is considered in Lemma 2.8.

It remains the case $x,y\in G$. As in Theorem 2.2 we need to study three cases of the mutual disposition of points x,y with respect to the boundary ∂G. The main situation is when the segment L connecting the points x and y is contained in the domain G. First let the length $|x-y|$ of L be less than the Euclidean distance from L to the boundary ∂G. By the Taylor formula and Lemma 2.7 we obtain

$$|R^k_{j,\nu}(x,y)| \le c \sup_{z\in L} |\nabla_{k+1}F_\nu(z)|\,|x-y|^{k+1-|j|} \le c2^\nu d_{1,G}(x,y)^{k+1-|j|}.$$

Now, let us consider the second case when $d_{1,G}(z,\partial G_1) \le |x-y|$ for some point $z\in L$. Then there exist points $z'\in L$ and $y'\in\partial G_1$ such that for any $\nu\in\mathbb{N}$ we have $d_{1,G}(z',y') \le d_{1,G}(x,y)$. According to Lemmas 2.2 and 2.8 we obtain the following bound for the remainders $R^k_{j,\nu}(x,y)$:

$$\begin{aligned}|R^k_{j,\nu}(x,y)| &= |F_{j,\nu}(x) - P_{j,\nu}(x,y) - P_{j,\nu}(x,y') + P_{j,\nu}(x,y')| \\ &\le c2^\nu d_{1,G}(x,y')^{k+1-|j|} + c2^\nu d_{1,G}(x,y')^{k+1-|j|} \\ &\le c2^\nu d_{1,G}(x,y)^{k+1-|j|}.\end{aligned}$$

When it is not possible to connect the points x, y by a segment $L\subset G$ we can find points $z'\in G$ and $y'\in\partial G_1$ such that

$$d_{1,G}(z',y') \le d_{1,G}(x,y), \qquad \max\{d_{1,G}(z',x), d_{1,G}(z'y)\} \le d_{1,G}(x,y).$$

Therefore,

$$d_{1,G}(y',x) \le 2d_{1,G}(x,y), \qquad d_{1,G}(y',y) \le 2d_{1,G}(x,y),$$

which coincides with the above considered situation. Thus, for the jet $\{F_j:\ |j|\le k-1\}$ and for the approximating sequence of the jets $\{F_{j,\nu}:\ |j|\le k\}$, $\nu\in\mathbb{N}$, the conditions (1.5) - (1.8) hold. Note, that (1.5) is Lemma 2.4, (1.6) is Lemma 2.5, (1.7) is Lemma 2.9. It means that the jet $\{F_j:\ |j|\le k-1\}$ belongs to the space $\Lambda^k(G)$. From Lemma 2.4 we have also that $T_k\circ E_k$ is the identical operator. Theorem 2.4 is proved. $\square$

3. Extension of differentiable functions

An inner α-metric $d_{\alpha,G}(x,y)$, $\alpha\in(0,1]$, in a domain $G\subset\mathbb{R}^n$ is called *locally equivalent* to the α-metric $d_\alpha(x,y) = |x-y|^\alpha$ if there exist numbers $r>0$ and $M>0$

such that for all points $x,y\in G$ with $|x-y|<r$ the inequality

$$\tilde{d}_{\alpha,G}(x,y) \le M|x-y| \tag{3.1}$$

is valid. Let us indicate by

$$d_{\alpha,G}(x,y) \underset{\mathrm{loc}}{\sim} d_{\alpha}(x,y)$$

that the metrics $d_{\alpha,G}(x,y)$ and $d_{\alpha}(x,y)$ are locally equivalent in the domain G, and let $M_{\alpha}(r)$ be the least constant for which the inequality (3.1) holds under the condition $|x-y|<r$, $x,y\in G$. Let us also set

$$M_{\alpha} = \lim_{r\to 0} M_{\alpha}(r), \tag{3.2}$$

$$r_{\alpha} = \sup \{r:\ M_{\alpha}(r)<\infty\}. \tag{3.3}$$

It follows at once from the definition that the inner α-metric $d_{\alpha,G}(x,y)$ is locally equivalent to the α-metric $d_{\alpha}(x,y)$ in the domain G if and only if the value M_{α} is finite.

Let us consider two seminormed spaces $F(G)$ and $N(\mathbb{R}^n)$ of functions defined on a domain $G\subset\mathbb{R}^n$ and on the Euclidean space $\mathbb{R}^n$, respectively. The mapping $\mathrm{ext}:F(G)\longrightarrow N(\mathbb{R}^n)$ is called *the extension operator* if $\mathrm{ext}(f)|_G = f$ for all functions $f\in F(G)$ and

$$\|\mathrm{ext}\| = \sup_{f\in F(G)} \left\{ \|\mathrm{ext}(f)\,|N(\mathbb{R}^n)\|\ \|f\,|F(G)\|^{-1}:\ f\in F(G)\right\} < \infty.$$

<u>*THEOREM 3.1.*</u> *Let G be a domain in $\mathbb{R}^n$ and let $\ell\in\mathbb{R}_+$ and $k=0,1,\ldots$ be such that $k<\ell\le k+1$. Put $\alpha = \ell-k$. If the inner α-metric $d_{\alpha,G}(x,y)$ is locally equivalent to the α-metric $d_{\alpha}(x,y) = |x-y|^{\alpha}$ in G, then there exist linear bounded extension operators*

$$\mathrm{ext}_k: W^{\ell}_{\infty}(G)\longrightarrow W^{\ell}_{\infty}(\mathbb{R}^n) \ \textit{for}\ \ell\in\mathbb{N}$$

and

$$\mathrm{ext}_k: H^{\ell}_{\infty}(G)\longrightarrow H^{\ell}_{\infty}(\mathbb{R}^n) \ \textit{for}\ \ell\notin\mathbb{N}.$$

The norm of the extension operator satisfies the estimate

$$\|\mathrm{ext}_k\| \le \gamma[\max\{M_{\alpha}, 1/r_{\alpha}\}]^{\ell},$$

where the constant γ does not depend on the domain G and the numbers M_{α}, r_{α} are defined by (3.2) - (3.3).

P r o o f. Let ℓ, k and α satisfy the assumptions of the lemma. According to Theorem 1.1 and Proposition 2.1 we have the coincidence of spaces

$$W^{\ell}_{\infty}(G) = \mathrm{Lip}(\ell,\tilde{G}_\alpha) \text{ for } \ell\in\mathbb{N} \quad\text{and}\quad H^{\ell}_{\infty}(G) = \mathrm{Lip}(\ell,\tilde{G}_\alpha) \text{ for } \ell\notin\mathbb{N}.$$

Therefore, assuming that $d_{\alpha,G}(x,y) \underset{\mathrm{loc}}{\sim} d_\alpha(x,y)$ in the domain G, it suffices to prove the existence of a bounded extension operator

$$\mathrm{ext}_k:\mathrm{Lip}(\ell,\tilde{G}_\alpha)\longrightarrow\mathrm{Lip}(\ell,\mathbb{R}^n)$$

such that $\mathrm{ext}_k(f)|_G = f|_G$ for an arbitrary function $f\in\mathrm{Lip}(\ell,\tilde{G}_\alpha)$.

To prove this we have to introduce the space $\mathrm{Lip}(\ell,F)$, where F is a closed set in $\mathbb{R}^n$ (see [10]). The elements of $\mathrm{Lip}(\ell,F)$ are jets $\{f^{(j)}: |j|\le k\}$ consisting of bounded functions such that $f^{(0)} = f$ and

$$f^{(j)}(x) = \sum_{|j+s|\le k} \frac{f^{(j+s)}(y)}{s!}(x-y)^s + R_j(x,y) \tag{3.4}$$

where

$$|f^{(j)}(x)| \le M, \quad |R_j(x,y)| \le M|x-y|^{\ell-|j|}, \quad x,y\in F, \ |j|\le k. \tag{3.5}$$

The norm of an element of $\mathrm{Lip}(\ell,F)$ is equal to the smallest value M for which the inequalities (3.5) hold. It is easy to verify that $f^{(j)}(x) = D^jf(x)$ in $G = \mathrm{Int}F$.

According to a Whitney type extension theorem (see [10]) there exists a linear bounded extension operator

$$\mathrm{ext}_k:\mathrm{Lip}(\ell,F)\longrightarrow\mathrm{Lip}(\ell,\mathbb{R}^n),$$

whose norm does not depend on the closed set $F\subset\mathbb{R}^n$. To prove Theorem 3.1, it is sufficient to establish the embedding

$$i:\mathrm{Lip}(\ell,\tilde{G}_\alpha)\longrightarrow\mathrm{Lip}(\ell,\bar{G}) \tag{3.6}$$

such that for each function $f\in\mathrm{Lip}(\ell,\tilde{G}_\alpha)$ the relation $(if)(x) = f(x)$, $x\in G$, is valid. The desired extension operator will be the superposition of mappings in the diagram

$$\mathrm{Lip}(\ell,\tilde{G}_\alpha)\xrightarrow{\ i\ }\mathrm{Lip}(\ell,\bar{G})\xrightarrow{\ \overline{\mathrm{ext}}_k\ }\mathrm{Lip}(\ell,\mathbb{R}^n).$$

Thus, $\mathrm{ext}_k = (\overline{\mathrm{ext}}_k\circ i):\mathrm{Lip}(\ell,\tilde{G}_\alpha)\longrightarrow\mathrm{Lip}(\ell,\mathbb{R}^n)$.

To find an upper estimate for the norm of the operator ext_k it is sufficient to find the contribution of the geometry of the space G_α to the norm of the operator i, since

$$\|\mathrm{ext}_k\circ i\| \le \|i\|\ \|\overline{\mathrm{ext}}_k\| \tag{3.7}$$

and the norm of operator $\overline{\mathrm{ext}}_k$ does not depend on the domain G.

Now let us prove the boundedness of the embedding (3.6). Let $\|f|\mathrm{Lip}(\ell,\tilde{G}_\alpha)\|$ be equal to 1. Since $|f^{(j)}(x)| = |D^j f(x)| \le 1$, $x\in G$, $|j|\le k$, it remains to prove the inequalities

$$|R_j(x,y)| \le C|x-y|^{\ell-|j|},\ x,y\in G. \tag{3.8}$$

Indeed, it follows from (3.8) that the functions $f^{(j)}$ extend by continuity to G, the extended functions satisfy the inequalities (3.5) for all points $x,y\in G$, and the embedding (3.6) is bounded.

To prove (3.8) we denote by r_m the largest number such that for points $x,y\in G$, $0<|x-y|<r_m$, we have the inequality

$$\tilde{d}_{\alpha,G}(x,y) \le mM_\alpha|x-y|,\ m\in N, \tag{3.9}$$

Since $|f^{(j)}(x)|\le 1$, the estimate

$$|R_j(x,y)| \le [C(n,k)/r_m^{(\ell-|j|)}]|x-y|^{\ell-|j|} \tag{3.10}$$

for $|j|\le k$, in $x,y\in G$, $|x-y|\ge r_m$, follows from the expansion (3.4). If $x,y\in G$, $|x-y|<r_m$, then we use (3.9) to obtain the estimate

$$|R_j(x,y)| \le \tilde{d}_{\alpha,G}(x,y)^{(\ell-|j|)} \le (mM_\alpha)^{\ell}|x-y|^{\ell-|j|}. \tag{3.11}$$

The inequalities (3.10) and (3.11) together yield the estimate (3.8).

It follows that the embedding (3.6) is continuous. If $|j| = 0$, the upper bound of the embedding is obtained from (3.10) and (3.11). As to the norm of the operator $\mathrm{ext}_k = \overline{\mathrm{ext}}_k\circ i$, we have the following estimate:

$$\|\mathrm{ext}_k\| \le \gamma\ \max\{mM_\alpha, 1/r_m\}^{\ell}, \tag{3.12}$$

where the constant γ depends only on n and ℓ. To finish the proof of Theorem 3.1 it remains to minimize the inequality (3.12) over all r_m such that

$$\tilde{d}_{\alpha,G}(x,y) \le mM_\alpha|x-y|,\ m\in\mathbb{N},\ x,y\in G,\ 0<|x-y|^{\alpha}<r_m.$$

In the case of the Sobolev spaces (α=1) Theorem 3.1 was proved by another method in [12].

There exists a relation between the α-metrics resulting from the reverse Minkowski inequality (see [7]): Given any numbers α and β, $0\le\alpha\le\beta\le 1$, the inequalities

$$d_{1,G}(x,y) \le d_{\beta,G}(x,y)^{1/\beta} \le d_{\alpha,G}(x,y)^{1/\alpha}, \tag{3.13}$$

hold for every $x,y\in G$. It follows that the relation $d_{\alpha,G}(x,y) \underset{\mathrm{loc}}{\sim} d_\alpha(x,y)$ is stronger than the relation $d_{\beta,G}(x,y) \underset{\mathrm{loc}}{\sim} d_\beta(x,y)$. For given numbers α and β,

$0<\alpha<\beta\le 1$, we can construct an example of a domain $G\subset\mathbb{R}^2$ such that $d_{\beta,G}(x,y) \underset{loc}{\sim} d_\beta(x,y)$, but $d_{\alpha,G}(x,y) \underset{loc}{\sim} d_\alpha(x,y)$ does not hold.

The method of the extension of differentiable functions explained above can also be applied to other spaces. First, we consider the Zygmund space $\Lambda^k(G)$, $k\in\mathbb{N}$. Before formulation of an exact result we adopt the extension method for the spaces $\Lambda^k(G)$. The first step is to extend the functions $f\in\Lambda^k(G)$ onto $\tilde{G}_1$ so that they belong to the space $\Lambda^k(\tilde{G}_1)$. The class $\Lambda^k(\tilde{G}_1)$ consists of the jets $\{f_j: |j|\le k-1\}$ and their approximating sequences $\{f_{j,\nu}: |j|\le k\}$, $\nu\in\mathbb{N}$, such that the conditions (1.5) - (1.8) hold at all points $x,y\in\tilde{G}_1$. Indeed in Section 2.2 we have proved the following

<u>PROPOSITION 3.1.</u> *Let $\{f_j\}$, $|j|\le k-1$ be a collection of functions from the class $\Lambda^k(G)$. Then every f_j can be extended in a unique way onto $\tilde{G}_1$, the extended functions belong to the space $\Lambda^k(\tilde{G}_1)$ and preserve the norms.*

Now we shall find conditions for the existence of an extension operator

$$\mathrm{Ext}_k : \Lambda^k(\tilde{G}_1) \longrightarrow \Lambda^k(\mathbb{R}^n) \tag{3.14}$$

such that the equality $\mathrm{Ext}_k(\{f_j\})(x) = f_0(x)$, $x\in G$, holds for each function from the jet $\{f_j: |j|\le k-1\}$.

To prove (3.14) we need the space $\Lambda^k(F)$, $k\in\mathbb{N}$, where F is a closed set in $\mathbb{R}^n$. Jets $\{f_j: |j|\le k-1\}$ of functions defined on the set F are called elements of the class $\Lambda^k(F)$ if there exist collections $\{f_{j,\nu}: |j|\le k\}$, $\nu\in\mathbb{N}$, of functions defined on F, such that the conditions

$$|f_j(x) - f_{j,\nu}(x)| \le M2^{-\nu(k-|j|)}, \quad \nu\ge 1, \quad |j|\le k-1, \tag{3.15}$$

$$|f_{j,\nu}(x) - f_{j,\mu}(x)| \le M2^{\mu-\nu}, \quad \mu\ge\nu\ge 1, \quad |j|=k, \tag{3.16}$$

$$\left| f_{j,\nu}(x) - \sum_{|j+s|\le k} \frac{f_{j+s,\nu}(y)}{s!}(x-y)^s \right| \le M2^{-\tau(k-|j|)}, \quad |x-y|\le a2^{-\tau}, \quad \nu\ge 1, \quad |j|\le k, \quad \tau\ge\nu\ge 1; \tag{3.17}$$

$$|f_{j,1}(x)| \le M, \quad |j|\le k \tag{3.18}$$

hold for every $x,y\in F$. The norm $\|f|\Lambda^k(F)\|$ of the jet $\{f_j: |j|\le k-1\}$ is equal to the smallest constant M such that the conditions (3.15) - (3.18) hold.

THEOREM 3.2. Let F *be a closed subset of* $\mathbb{R}^n$. *Then every element* $f = \{f_j: |j| \le k-1\}$ *of the space* $\Lambda^k(F)$ *can be extended to a function* $\mathrm{Ext}_k(f) \in \Lambda^k(\mathbb{R}^n)$. *The function* $\mathrm{Ext}_k(f)$ *is an extension of the jet* $f = \{f_j: |j| \le k-1\}$ *such that for every* $|j| \le k-1$ *the restriction of the partial derivative* $D^j\mathrm{Ext}_k(f)$ *to the set* F *is equal to* f_j. *Moreover, the extension operator* $\mathrm{Ext}_k: \Lambda^k(F) \longrightarrow \Lambda^k(\mathbb{R}^n)$ *is bounded and it may be chosen in such way that the function* $\mathrm{Ext}_k(f)$ *is smooth outside of* F.

P r o o f of the Theorem 3.2 is completely analogous to that of Theorem 2.4. $\square$

REMARK 3.1. In [8] the authors define the space $\Lambda^k(F)$ in a different way setting $\tau = \nu$ in the formula corresponding to (3.17). Nevertheless, both classes, if considered on Euclidean space $\mathbb{R}^n$, coincide with the Zygmund space $H^k_\infty(\mathbb{R}^n)$. Our proof of Theorem 3.2 is based on the Whitney method, as well, but we use another construction of the extension operator $\mathrm{ext}: \Lambda^k(F) \longrightarrow H^k_\infty(\mathbb{R}^n)$.

To prove the existence of the operator (3.14) it is sufficient to establish the embedding

$$i: \Lambda^k(\tilde{G}_1) \longrightarrow \Lambda^k(\bar{G}) \tag{3.19}$$

such that the relation $(if)(x) = f(x)$, $x \in G$, holds for every function $f \in \Lambda^k(\tilde{G}_1)$. Then the desired operator is obtained as the superposition of operators in the following diagram

$$\Lambda^k(\tilde{G}_1) \xrightarrow{\ i\ } \Lambda^k(\bar{G}) \xrightarrow{\ \overline{\mathrm{Ext}}_k\ } \Lambda^k(\mathbb{R}^n).$$

Indeed, $\mathrm{Ext}_k = (\overline{\mathrm{Ext}}_k \circ i): \Lambda^k(\tilde{G}_1) \longrightarrow \Lambda^k(\mathbb{R}^n)$.

To find an upper bound for the norm of the operator Ext_k it suffices to know how the norm of the operator i depends on the geometry of the domain G. The boundedness of the operator (3.19) may be obtained by the geometric restriction to the domain G. To see its character it suffices to compare conditions (1.5) - (1.8) and (3.15) - (3.18), in particular, (1.7) and (3.17). We claim that

$$\lim_{\nu\to\infty} \sup \left\{ d_{1,G}(x,y)/|x-y| :\ x,y \in G,\ 0<|x-y| \le 2^{-\nu} \right\} = M_0 < \infty. \tag{3.20}$$

Let $\nu_0 \in \mathbb{N}$ be a number such that the condition

$$d_{1,G}(x,y) \le 2M_0|x-y| \tag{3.21}$$

holds for $|x-y| \le 2^{-\nu}$, $\nu \ge \nu_0$.

Let a jet $\{f_j: |j|\le k-1\}$ from the class $\Lambda^k(\tilde{G}_1)$ have the norm equal to 1 and let $\{f_{j,\nu}: |j|\le k\}$, $\nu\in\mathbb{N}$, be the corresponding sequence of jets such that the conditions (1.5) - (1.8) hold with the constant $M = 2$. The condition (3.21) implies that

$$d_{1,G}(x,y) \le 2M_0|x-y| \le 2M_0 2^{-\tau}$$

for every $x,y\in G$, $|x-y| \le 2^{-\tau}$, and $\tau\ge\nu\ge\nu_0$. We take the least natural number N such that $2^{N+\nu_0}a \ge 2M_0$ where a is the number from (3.17). Let us consider the sequence of jets $\{g_{j,\nu}: |j|\le k\}$, $\nu\in\mathbb{N}$, of functions defined on the domain G such that $g_{j,\nu} = f_{j,\nu-N-\nu_0}$ for $\nu>N+\nu_0$ and $g_{j,\nu} = f_{j,1}$ for $\nu = 1,2,\dots,N+\nu_0$. Then for the jet $\{f_j: |j|\le k-1\}$ and the corresponding sequence of jets $\{g_{j,\nu}: |j|\le k\}$, $\nu\in\mathbb{N}$, conditions (3.15) - (3.18) hold with a constant M depending on k, n, M_0 and ν_0 only. The validity of the conditions (3.15) - (3.18) at the inner points of the domain G implies that there exist extensions by the continuity to the closure $\bar{G}$ of both the functions from the jet $\{f_j: |j|\le k-1\}$ and of the functions from the sequence of the jets $\{f_{j,\nu}: |j|\le k\}$, $\nu\in\mathbb{N}$. Therefore, the extended jets belong to the space $\Lambda^k(G)$.

Thus, we proved the following assertion which is due to Yu. Bojarskii and the author.

THEOREM 3.3. *Let G be a domain in $\mathbb{R}^n$ and let the condition (3.20) hold. Then for any $k\in\mathbb{N}$ there exists a bounded extension operator*

$$\mathrm{Ext}_k: \Lambda^k(G) \longrightarrow H^k_\infty(R^n).$$

The idea of the geometric approach to the spaces of Hölder functions defined on a domain G can be also applied to spaces with the integral metric.

Let ℓ be a real number, $\ell\in(k,k+1)$, $k = 0,1,\dots$, $\alpha = \ell-k$ and $1 \le p,q \le \infty$. We say that a function f belongs to $\Lambda^\ell_{p,q}(G)$ if f has generalized derivatives of the order k and the finite norm

$$\|f|\Lambda^\ell_{p,q}(G)\| = \sum_{|j|\le k} \|D^j f|L_p(G)\| + \sum_{|j|\le k}\left\{\sum_{\nu=0}^{\infty} 2^{\nu(\ell-|j|)q}\left[\iint\limits_{\substack{x,y\in G\\ d_{\alpha,G}(x,y)\le 2^{-\nu}}} \frac{|R_j(x,y)|^p}{d_\alpha(x,y)^n}\,dxdy\right]^{q/p}\right\}^{1/q}, \tag{3.22}$$

where the remainders $R_j(x,y)$ are defined in Section 1.1. For $q = \infty$ the norm is modified in the usual way. We note that $\Lambda^{\ell}_{\infty,\infty}(G) = \mathrm{Lip}(\ell, G_\alpha)$. If G is an (ε,δ)-domain [13] then the space $\Lambda^{\ell}_{p,q}(G)$ coincides with the Besov space $B^{\ell}_{p,q}(G)$ (see the theorem below).

<u>*THEOREM 3.4.*</u> *Let* G *be a domain in* $\mathbb{R}^n$ *such that the following conditions hold:*

(i) $\bar{d}_{\alpha,G}(x,y) \underset{\mathrm{loc}}{\sim} |x-y|$;

(ii) *there exist numbers* $c, \rho_0 > 0$ *such that*

$$|B(x,\rho)\cap G| \geq c\,|B(x,\rho)|,\ x\in G,\ 0<\rho<\rho_0.$$

Then there exists a bounded linear extension operator

$$\mathrm{ext}_\ell : \Lambda^{\ell}_{p,q}(G) \longrightarrow B^{\ell}_{p,q}(\mathbb{R}^n). \tag{3.23}$$

In particular, if G *is an* (ε,δ)*-domain, then the spaces* $\Lambda^{\ell}_{p,q}(G)$ *and* $B^{\ell}_{p,q}(G)$ *coincide.*

P r o o f. We indicate the main steps of the proof. The condition (i) of the theorem permits to pass from the norm (3.22) to a similar one with $|x-y|$ in place of $\bar{d}_{\alpha,G}(x,y)$. The condition (ii) implies that the measure of the boundary ∂G is equal to 0 and so at the end we pass to the space $\Lambda^{\ell}_{p,q}(\bar{G})$. In [14] the Whitney type extension theorem is proved establishing the existence of the extension operator $\overline{\mathrm{ext}}_\ell : \Lambda^{\ell}_{p,q}(\bar{G}) \longrightarrow B^{\ell}_{p,q}(\mathbb{R}^n)$. The corresponding diagram has the form

$$\Lambda^{\ell}_{p,q}(G) \longrightarrow \Lambda^{\ell}_{p,q}(\bar{G}) \xrightarrow{\overline{\mathrm{ext}}_\ell} B^{\ell}_{p,q}(\mathbb{R}^n).$$

The superposition of the operators in the diagram is the desired extension operator.

To prove the last assertion we note that an (ε,δ)-domain satisfies both conditions (i) [15] and (ii) [16]. Therefore there exists not only the extension operator (3.23) but also the extension operator $\mathrm{ext} : B^{\ell}_{p,q}(G) \longrightarrow B^{\ell}_{p,q}(\mathbb{R}^n)$ [16]. Thus, if both the restriction operators $\mathrm{rest} : B^{\ell}_{p,q}(\mathbb{R}^n) \longrightarrow \Lambda^{\ell}_{p,q}(G)$ and $\mathrm{rest} : B^{\ell}_{p,q}(\mathbb{R}^n) \longrightarrow B^{\ell}_{p,q}(G)$ are bounded, then the spaces $\Lambda^{\ell}_{p,q}(G)$ and $B^{\ell}_{p,q}(G)$ coincide. $\square$

4. Necessary extension conditions for differentiable functions

4.1. Metric extension conditions

THEOREM 4.1 [9]. *If there exists an extension operator*

$$\mathrm{ext}_1 : W_p^1(G) \longrightarrow W_p^1(\mathbb{R}^n),\ \ p>n\ (\alpha = 1)$$

or

$$\mathrm{ext}_\ell : H_p^\ell(G) \longrightarrow H_p^\ell(R^n),\ \ \ell p>n\ (\alpha = \ell < 1),$$

then the inner α-metric $d_{\alpha,G}(x,y)$ *is locally equivalent to the α-metric* $d_\alpha(x,y) = |x-y|^\alpha$ *in the domain* G *and*

$$\|\mathrm{ext}_\alpha\| \ge KM_\alpha^{\ell-n/p}$$

where K *does not depend on the domain* G.

In the case $p = \infty$, the related topics can be found in [12, 15].

In the same way we prove the following

PROPOSITION 4.1. *Let* G *be a domain in* $\mathbb{R}^n$. *If there exists an extension operator*

$$\mathrm{ext} : H_p^1(G) \longrightarrow H_p^1(\mathbb{R}^n),\ \ n<p\le\infty,$$

then the inner metric $d_{1,G}(x,y)$ *is locally equivalent to the Euclidean metric* $|x-y|$ *in* G.

4.2. Regularity condition

It is well known that the space $W_p^\ell(\mathbb{R}^n)$ is embedded not only into the space of continuous functions but also into the Hölder space. In what follows we need only two embeddings:

$$\text{(4.1)}\qquad i : W_p^\ell(\mathbb{R}^n) \longrightarrow H_\infty^{\ell-n/p}(\mathbb{R}^n),\qquad i : B_{p,q}^\ell \longrightarrow H_\infty^{\ell-n/p}(\mathbb{R}^n).$$

With the help of the embeddings (4.1) we prove

PROPOSITION 4.2. *Let* $x,y\in\mathbb{R}^n$ *and let* $\ell p>n$. *Let* $f\in W_p^\ell(\mathbb{R}^n)$ *be a function such that* $f(x) = 1$, $f(y) = 0$. *Moreover, if* $\ell-n/p > 1$, *let* f *be equal to* 1 *in some neighbourhood of the point* x. *Then the estimate*

$$\frac{1}{|x-y|^{\ell-n/p}} \le \gamma\|f\,|W_p^\ell(\mathbb{R}^n)\|$$

is valid, where the constant γ is independent of x *and* f. *The analogous result holds for functions* $f\in B^{\ell}_{p,q}(\mathbb{R}^n)$.

P r o o f. The boundedness of the embeddings (4.1) yields the estimate

$$\|f|H_{\infty}^{\ell-n/p}(\mathbb{R}^n)\| \le \|i\| \ \|f|W_p^{\ell}(\mathbb{R}^n)\|. \tag{4.2}$$

If ℓ-n/p is a fractional number, then

$$\frac{1}{|x-y|^{\ell-n/p}} \le \frac{|R_0(x,y)|}{|x-y|^{\ell-n/p}} \le \|f|H_{\infty}^{\ell-n/p}(\mathbb{R}^n)\|,$$

since $|R_0(x,y)| = |f(y) - f(x)| = 1$ by the assumption.

If ℓ-n/p = m+1 is an integer then the left hand side of (4.2) is estimated from below by

$$\frac{1}{|x-y|^{m+1}} \le \sup_{\substack{h,z\in\mathbb{R}^n \\ h\ne0}} \frac{|\Delta_h^2\nabla_m g(z)|}{|h|} \le \|f|H_{\infty}^{m+1}(\mathbb{R}^n)\|,$$

where $g(z) = f((x-z)/|x-z|)$. Let us note that

$$\inf_{g} \sup_{\substack{h,z\in\mathbb{R}^n \\ h\ne0}} \frac{|\Delta_h^2\nabla_m g(z)|}{|h|} \ge \beta > 0,$$

where the infimum is taken over all functions equal to 1 in some neighbourhood of the origin and vanishing at some point of the unit sphere S(0,1). The proof is finished. □

A domain G in $\mathbb{R}^n$ is said to be *regular at a point* $x\in G$ if there exist positive constants $\delta = \delta(x)$ and $\gamma = \gamma(x)$ such that for any ball B(x,r), $0<r<\delta(x)$, the inequality

$$|B(x,r)\cap G| \ge \gamma|B(x,r)| \tag{4.3}$$

holds. The domain G is said to be *regular* if it is regular at every point $x\in G$.

THEOREM 4.2. Let G *be a domain in* $\mathbb{R}^n$ *and let* p>n. *If the inequality*

$$\sup_{\substack{u,v\in G, \\ u\ne v}} \frac{|f(u)-f(v)|}{|u-v|^{1-n/p}} \le c\|f|W_p^1(G)\| \tag{4.4}$$

holds for every function $f\in W_p^1(G)$, *then the domain* G *is regular.*

P r o o f. Let $\varphi:\mathbb{R}\longrightarrow\mathbb{R}^n$ be a smooth function supported in the unit ball, and such that $0\le\varphi(x)\le1$ for every $x\in\mathbb{R}^n$ and φ is equal to 1 in some neighbourhood of the origin. Let us fix points $x\in\overline{G}$, $y\in G$ and put $r = |x-y|$ and

$f(z) = \varphi((z-x)/r)$. Then $f(z) = 1$ in a neighbourhood of x, $f(y) = 0$ and $|\nabla f| \le a/r$. The inequality (4.4) yields

$$\frac{|B(x,r)|^{1/p}}{r} = \frac{\gamma}{|x-y|^{1-n/p}} \le c(\|f|L_p(G)\| + ar^{-1}\|\chi_{B(x,r)}|L_p(G)\|)$$

$$\le c(|B(x,r)\cap G|^{1/p} + ar^{-1}|B(x,r)\cap G|^{1/p}).$$

Comparing the first and the last term we obtain

$$|B(x,r)\cap G| \ge c^{-p}(a+r)^{-p}|B(x,r)|.$$

Thus for the domain G the inequality (4.3) is satisfied with the constants $\gamma = c^{-p}(a+1)^{-p}$ and $\delta=1$ independent of x. □

The last theorem has a generalization for the space W_p^ℓ. The formulation and the method of the proof is obvious from the following assertion.

THEOREM 4.3. Let G be a domain in $\mathbb{R}^n$ and let $n/\ell < p < \infty$. If there exists the bounded extension operator $\mathrm{ext}: W_p^\ell(G) \longrightarrow W_p^\ell(\mathbb{R}^n)$, then the domain G is regular.

P r o o f. Let $x \in \bar{G}$, $y \in G$ and let f be the function from the previous proof corresponding to points x and y. Then $f(z) = 1$ in some neighbourhood of x, $f(y) = 0$ and $|\nabla_\ell f| \le b/r^\ell$. Using the inequality

$$\|\mathrm{ext}(f)|W_p^\ell(\mathbb{R}^n)\| \le \|\mathrm{ext}\| \; \|f|W_p^\ell(G)\|$$

and Proposition 4.2, we obtain

$$\frac{|B(x,r)|^{1/p}}{r^\ell} = \frac{\gamma}{|x-y|^{\ell-n/p}} \le c\|\mathrm{ext}(f)|W_p^\ell(\mathbb{R}^n)\|$$

$$\le c\|\mathrm{ext}\|[|B(x,r)\cap G|^{1/p} + br^{-\ell}|B(x,r)\cap G|^{1/p}].$$

It follows that

$$|B(x,r)\cap G| \ge c^{-p}(b+r^\ell)^{-p}|B(x,r)|.$$

Thus, we have proved the regularity of the domain G, namely (4.3) holds with the constants $\gamma = c^{-p}(b+1)^{-p}$ and $\delta = 1$ independent of the point $x \in \bar{G}$. □

An analogous result is valid for the Nikol'skii-Besov spaces.

THEOREM 4.4. Let G be a domain in $\mathbb{R}^n$. Suppose that $\ell p>n$ if $1\le p<\infty$, $1\le\theta\le\infty$, and $\ell p\ge n$ if $1\le p<\infty$, $\theta = 1$. If there exists a bounded extension operator $\mathrm{ext}: B_{p,\theta}^\ell(G) \longrightarrow B_{p,\theta}^\ell(\mathbb{R}^n)$, then the domain G is regular.

REFERENCES

[1] I. M. Tumakov: *Henri Leon Lebesgue* (1875-1941) (Russian). Nauka, Moscow, 1975.

[2] H. Whitney: *Analytic extension of differentiable functions defined in closed sets*. Trans. Amer. Math. Soc. **36** (1934), 63-89.

[3] A. D. Aleksandrov: *Inner geometry of convex surfaces* (Russian). Gostekhizdat, Moscow, 1948. (German translation of the enlarged version: Die innere Geometrie der konvexen Flächen. Akademie-Verlag, Berlin, 1955.)

[4] S. L. Sobolev: *Some applications of functional analysis in mathematical physics*. (Russian) Izdat. Leningrad. Gos. Univ., Leningrad, 1950. (English translation: Amer. Math. Soc., Providence, R. I., 1963.)

[5] S. M. Nikol'skii: *Approximation of functions of several variables and imbedding theorems* (Russian). Nauka, Moscow, 1975. (English translation: Springer Verlag, Berlin - Heidelberg - New York, 1975.)

[6] O. V. Besov, V. P. Il'in, S. M. Nikol'skii: *Integral representation of functions and embedding theorems* (Russian). Nauka, Moscow, 1975. (English translation: Vol. 1-2, John Wiley & Sons, New York, 1979.)

[7] G. H. Hardy, J. E. Littlewood, G. Polya: *Inequalities*. Cambridge University Press, Cambridge, 1934.

[8] A. Jonsson, H. Wallin: *The traces to closed sets of functions in* $\mathbb{R}^n$ *with second difference of order* $O(h)$. J. Approx. Theory **26** (1979), 159-184.

[9] S. K. Vodop'yanov: *Equivalent normalizations of Sobolev and Nikol'skii spaces in domains. Boundary values and extension.* In: Function spaces and applications. Proc. US-Swedish Seminar, Lund 1986. Lecture Notes in Mathematics **1302**, Springer, Berlin - New York, 1988, 397-409.

[10] E. M. Stein: *Singular integrals and differentiability properties of functions*. Princeton University Press, Princeton, 1970.

[11] Yu. I. Bojarskii, S. K. Vodop'yanov: *Boundary values of functions of Zygmund classes* (Russian). Proc. School on Approx. Theory. Lutsk, September 1989. Kiev, 1989.

[12] V. N. Konovalov: *Descriptions of traces of certain classes of functions of several variables* (Russian). Akad. Nauk Ukrain. SSR Inst. Mat. Preprint 1984, no. 21, 64 pp.

[13] P. W. Jones: *Quasiconformal mappings and extendibility of functions in Sobolev spaces*. Acta Math. **147** (1981), 71-88.

[14] A. Jonsson, H. Wallin: *Function spaces on subsets of* $\mathbb{R}^n$. Math. Rep. **2** (1984), no. 1, Harwood Academic Publ., Chur, 1984.

[15] F. W. Gehring, O. Martio: *Quasiexternal distance domains and extension of quasiconformal mappings*. J. Analyse Math. **45** (1985), 181-206.

[16] P. A. Shvartsman: *Local approximations of functions and embedding theorems* (Russian). Preprint, Yaroslavl', 1983, Manuscript No. 2025-83, deponed VINITI, 1983.