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ORLICZ - SOBOLEV SPACES AND NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

Jean-Pierre Gossez(*)

Introduction

These notes are concerned with the existence of solutions for variational boundary value problems for elliptic operators in divergence form:

$$(1) \qquad \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \nabla u, \ldots, \nabla^m u).$$

This question has been extensively studied since 1963 in the context of the theory of mappings of monotone type from a reflexive Banach space into its dual (see the works of BROWDER, LERAY-LIONS, BRÉZIS, ...). The condition of reflexivity impose that the A_{α}'s, at least for $|\alpha| = m$, have polynomial growth in u and its derivatives.

Our purpose here is to treat cases where the coefficients A_{α} do not necessarily have polynomial growth in u and its derivatives. To avoid technicalities, we will concentrate on the Dirichlet problem for the equation

$$(2) \qquad - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left| \phi \frac{\partial u}{\partial x_j} \right| = f$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, odd, strictly increasing, with $\phi(+\infty) = +\infty$. Equation (2) can be thought of as a simple nonlinear generalization of the Laplace equation. An existence and uniqueness theorem will be proved for this problem. We insist that <u>no growth assumptions</u> are made on ϕ , which could behave at infinity for instance as an exponential, or as a logarithm (this latter case turns out to be more delicate).

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*) A preliminary version of these notes was written while the author was visiting at the University of Brasilia.

The crucial points in the treatment of "rapidly or slowly increasing" A_α's are the following: (i) the Banach spaces in which the problems seem to be appropriately formulated - the Orlicz-Sobolev spaces - are generally not reflexive nor separable, (ii) the corresponding mappings of monotone type are not bounded nor everywhere defined and do not generally satisfy a global a priori bound (and consequently are not coercive). It is in fact a bit surprising that for an equation such as (2), with the Dirichlet boundary conditions, a bound on the right hand side f does not always imply a bound on the corresponding solutions u ; this phenomena occurs for instance if ϕ behaves at infinity as a logarithm. We will see that a more sophisticated kind of a priori bound holds; the notion of a locally bounded mapping introduced by ROCKAFELLAR [20] in monotone operator theory finds an application here.

Our existence results are derived from abstract surjectivity theorems for mappings of monotone type which are not everywhere defined, unbounded, noncoercive, ... and which operate in complementary systems. These are quadruples of Banach spaces related to each other in roughly the same way as conjugate Orlicz spaces.

There are three chapters. Chapter I lists briefly some definitions and well known from Orlicz spaces theory. With the possible exception of the section 1.2 (approximation of functions in L_M), the material is classical and can be found e. g. in [13] or in [14]. Chapter 2 is concerned with Orlicz-Sobolev spaces, i. e. Sobolev spaces built from Orlicz spaces. Duality is studied in the first two sections, and section 2.3 deals with the trace of a function in $W^1L_M(\Omega)$. A generalized Poincaré's inequality is proved in section 2.4. Other results on Orlicz-Sobolev spaces can be found in DONALDSON-TRUDINGER [5] (imbedding theorems), LACROIX [15] (trace spaces), DONALDSON [4] (inhomogeneous spaces), ...; see also the references in [14]. Chapter 3 contains the treatment of the Dirichlet problem for equation (1). One section is devoted to the study of the Nemyckii operator $u(x) \mapsto \phi(u(x))$ because the properties of this simple nonlinear operator are quite revealing of the difficulties one has to deal with when studying (1) in full generality. Chapter 2 and 3 are based on [8], [9], [10], where other results and detailed references can be found. Earlier works on the subject are [3], [7].

For simplicity, we have always assumed the open subset Ω of $\mathbb{R}^N$ bounded. This assumption will generally not be repeated explicitely. However most of the results presented here can be suitably extended to the case of unbounded Ω, see [17].

Chapter 1. Orlicz spaces

1.1 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^n$, with Lebesgue measure dx, and let M be an *N-function*, i. e. a real valued continuous, convex, even function of $t \in \mathbb{R}$ with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to +\infty$ as $t \to +\infty$. The *Orlicz class* $\mathscr{L}_M(\Omega)$ is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))dx < \infty$. The *Orlicz space* $L_M(\Omega)$ is defined as the linear hull of $\mathscr{L}_M(\Omega)$. $L_M(\Omega)$ is a Banach space with respect to the *Luxemburg norm*

$$||u||_{(M)} = \inf \{\lambda > 0; \int_\Omega M(u/\lambda)dx \leqq 1\} .$$

One has $L_M(\Omega) = \mathscr{L}_M(\Omega)$ if and only if M satisfies the Δ_2 *condition*, i. e. there exist k and t_0 such that $M(2t) \leqq kM(t)$ for $t \geqq t_0$.

The closure in $L_M(\Omega)$ of the bounded measurable functions is denoted by $E_M(\Omega)$. One has $E_M(\Omega) \subset \mathscr{L}_M(\Omega)$; moreover $E_M(\Omega) = L_M(\Omega)$ if and only if M satisfies the Δ_2 condition. The space $E_M(\Omega)$ is separable, but $L_M(\Omega)$ is separable if and only if M satisfies the Δ_2 condition.

The *conjugate* function

$$\overline{M}(t) = \sup \{ts - M(s);\ s \in \mathbb{R}\}$$

of an N-function M is still an N-function, and one has $\overline{\overline{M}} = M$. Young's inequality follows from this definition: $st \leq M(s) + \overline{M}(t)$ for $s, t \in \mathbb{R}$, and one also has a Hölder's type inequality: if $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega u\, v\, dx \leq 2||u||_{(M)}\ ||v||_{(\overline{M})}.$$

Thus $\int_\Omega uvdx$ is a well defined continuous bilinear form on $L_M \times L_{\overline{M}}$. The dual of E_M can be identified by means of this form to $L_{\overline{M}}$; the norm on $L_{\overline{M}}$ dual to $||\ ||_{(M)}$ on E_M is called the *Orlicz norm* and denoted by $||\ ||_{(\overline{M})}$; it is equivalent to $||\ ||_{(\overline{M})}$: $||\ ||_{(\overline{M})} \leq ||\ ||_{(\overline{M})} \leq 2||\ ||_{(\overline{M})}$. The norm on L_M dual to $||\ ||_{\overline{M}}$ on $E_{\overline{M}}$ turns out to be $||\ ||_{(M)}$, and one has a stronger form of Hölder's inequality:

$$\int_\Omega u\, v\, dx \leq ||u||_{(M)}\ ||v||_{(\overline{M})}$$

for $u \in L_M$ and $v \in L_{\overline{M}}$. Finally the space L_M is reflexive if and only if both M and $\overline{M}$ satisfy the Δ_2 condition.

To conclude this section, we remark that the four spaces

$$(L_M,\ E_M;\ L_{\overline{M}},\ E_{\overline{M}})$$

constitue and example of a *complementary system*.

Let Y and Z be two real Banach spaces, with $<\ ,\ >$ a continuous regular bilinear form on $Y \times Z$ and let Y_0 and Z_0 be closed subspaces of Y and Z respectively. Then $(Y, Y_0; Z, Z_0)$ is called a complementary system if, by means of $<\ ,\ >$, Y_0^* can be identified to Z and Z_0^* to Y.

1.2. Approximation properties in L_M

Approximation results for functions in E_M are well known. The following simple approximation property of functions in L_M will be used later. We denote by u_y the translated function of u : $u_y(x) = u(x - y)$ and by u_ε the regularized function: $u_\varepsilon = u * \rho_\varepsilon$.

PROPOSITION. *Let* $u \in L_M(\Omega)$, *with compact support in* Ω. *Then* $u_y \to u$ *for* $\sigma(L_M, L_{\overline{M}})$ *as* $|y| \to 0$ *and* $u_\varepsilon \to u$ *for* $\sigma(L_M, L_{\overline{M}})$ *as* $\varepsilon \to 0$.

LEMMA. *Suppose that* $v_n \in \mathcal{L}_M$, $v_n \to v$ *a. e. and* $M(v_n) \leq f_n$ *where* $f_n \to f$ *in* L^1. *Then* $v \in \mathcal{L}_M$ *and* $v_n \to v$ *for* $\sigma(L_M, L_{\overline{M}})$.

PROOF. The fact that $v \in \mathcal{L}_M$ follows from

$$\int_\Omega M(v_n(x))dx \leq \int_\Omega f_n(x)dx \leq \text{constant},$$

by using Fatou's lemma. Take now $w \in \mathcal{L}_{\overline{M}}$. We have $v_n w \to vw$ a. e. and

$$v_n w \leq M(v_n) + \overline{M}(w) \leq f_n + \overline{M}(w) .$$

By the Vitali convergence theorem, this implies that $v_n w \to vw$ in $L^1(\Omega)$. Since $L_{\overline{M}}$ is the vector space generated by $\mathcal{L}_{\overline{M}}$, we deduce that $v_n \to v$ for $\sigma(L_M, L_{\overline{M}})$. Q. E. D.

PROOF OF THE PROPOSITION. It clearly suffices to prove the proposition for u in $\mathcal{L}_M(\Omega)$. Consider the case of u_ε. We know that $u_\varepsilon \to u$ in L^1 , so that, taking a subsequence if necessary, we can assume $u_\varepsilon \to u$ a. e. On the other hand $M(u_\varepsilon) \leq (M(u))_\varepsilon$ by Jensen's inequality. But $(M(u))_\varepsilon \to M(u)$ in L^1 since $u \in \mathcal{L}_M$. So the lemma can be applied, which gives $u_\varepsilon \to u$ for $\sigma(L_M, L_{\overline{M}})$. Similar proof for u_y . Q. E. D.

Chapter 2. Orlicz-Sobolev spaces

2.1 Complementary systems

We want to describe a method by which, given an abstract complementary system, one can build new ones. This will be applied in the next section to the case of Orlicz-Sobolev spaces.

Let $(Y, Y_0; Z, Z_0)$ be a complementary system and let E be a closed subspace of Y . We wish to build a new complementary system $(E, E_0; F, F_0)$ starting with E in the upper left corner. Write $E_0 = E \cap Y_0$, $F = Z/E_0^\perp$ and $F_0 = \{z + E_0^\perp \in Z/E_0^\perp;\ z \in Z_0\} \subset F$, where $\perp$ denotes the orthogonal in the duality (Y, Z), i. e.

$$E_0^\perp = \{z \in Z;\ \langle y, z\rangle = 0 \ \text{ for all } \ y \in E_0\}.$$

The pairing $\langle , \rangle_{Y,Z}$ between Y and Z induces a pairing $\langle , \rangle_{E,F}$ between E and F:

$$\langle y, z + E_0^{\perp}\rangle_{E,F} = \langle y, z\rangle_{Y,Z} \quad \text{for} \quad y \in E,\ z \in Z,$$

if and only if $E \subset (E_0^{\perp})^{\perp}$, which, by the bipolar theorem, means that E_0 is $\sigma(Y, Z)$ dense in E. The pairing between E and F obtained in this way is continuous, regular, and the dual of E_0 can be identified to $E_0^*/E_0^{\perp} \equiv Z/E_0^{\perp} = F$.

LEMMA. *Assume that* E_0 *is* $\sigma(Y, Z)$ *dense in* E. *Then the necessary and sufficient condition for* $(E, E_0; F, F_0)$ *above to be a complementary system is that* E *be* $\sigma(Y, Z_0)$ *closed in* Y.

It should be remarked that the two conditions $E \cap Y_0$ $\sigma(Y, Z)$ dense in E and E $\sigma(Y, Z_0)$ closed are in some sense opposed because $\sigma(Y, Z)$ is stronger than $\sigma(Y, Z_0)$. Starting with a subspace D of Y_0, if one wishes to obtain for E_0 the norm closure $\overline{D}$ of D, then the above lemma can be applied if and only if

(1) $$\sigma(Y, Z_0)\ c\ell\ D = \sigma(Y, Z)\ c\ell\ D\ ;$$

simply take for E the common value in (1) and use the fact that in a Banach space the weak and the norm closure of a convex set coincide in order to conclude that $E \cap Y_0 = \overline{D}$.

PROOF OF THE LEMMA. To prove the sufficient part, we have to see that the dual of F_0 can be identified to E and that F_0 is complete.

Consider $A : E \to F_0^*: e \mapsto Ae$ where Ae is defined by

$$Ae : F_0 \to \mathbb{R}: z + E_0 \mapsto \langle e, z + E_0^{\perp}\rangle_{E,F} = \langle e, z\rangle_{Y,Z}$$

for $z \in Z_0$. A is a well defined linear continuous injective mapping from E to F_0^*. If we show that A is onto, then, by the closed graph theorem, A will be a linear homeomorphism between E and F_0^*, which gives us the required identification mapping between E and F_0^*. Take $L \in F_0^*$ and consider the mapping

$$Z_0/(E_0^{\perp} \cap Z_0) \mapsto \mathbb{R}: z + (E_0^{\perp} \cap Z_0) \mapsto L(z + E_0^{\perp})$$

for $z \in Z_0$. It is a well defined linear continuous form on $Z_0/(E_0^{\perp} \cap Z_0)$. But the dual of this space can be identified to the orthogonal of $(E_0^{\perp} \cap Z_0)$ in $Z_0^* \equiv Y$, and so, by the bipolar theorem, to $\sigma(Y, Z_0)$ cℓ E_0, which, by assumption, is equal to E. Consequently there exists $e \in E$ such that

$$L(z + E_0^{\perp}) = \langle e, z + (E_0^{\perp} \cap Z_0)\rangle = \langle e, z\rangle_{Y,Z}$$

for $z \in Z_0$, and thus $L = Ae$.

To verify that $F_0 \subset F$ is complete, we will show that F_0 is isomorphic to $Z_0/(Z_0 \cap E_0^{\perp})$. There is an obvious "inclusion": the mapping

$$Z_0/(Z_0 \cap E_0^{\perp}) \to F_0 : z + (Z_0 + E_0^{\perp}) \mapsto z + E_0^{\perp}$$

is well defined linear, continuous, injective and surjective. To see that its inverse is continuous, it clearly suffices to show that $Z_0/(Z_0 \cap E_0^{\perp})$ and F_0 have the same dual space, with equivalent norms. The mapping $L \in F_0^* \mapsto e \in E \equiv (Z_0/Z_0 \cap E_0^{\perp})^*$ constructed above provides such an identification.

To prove the necessary part of the lemma, it suffices, by the Krein-Smulian theorem [6; p. 429], to prove that the limit $y \in Y$ of a bounded $\sigma(Y, Z_0)$ convergent net $y_i \in E$ lies in E. But the bounded sets in E are $\sigma(E, F_0)$ relatively compact since $E \equiv F_0^*$. Since the restriction to E of $\sigma(Y, Z_0)$ is simply $\sigma(E, Z_0) \equiv$ $\equiv \sigma(E, F_0)$ the conclusion follows. Q. E. D.

From the above proof, we will use later the fact that F_0 is isomorphic to $Z_0/(E_0^{\perp} \cap Z_0)$. It is also worth to notice that $\sigma(E, F)$ and $\sigma(E, F_0)$ are the topologies induced on E by $\sigma(Y, Z)$ and $\sigma(Y, Z_0)$ respectively.

2.2. Duality in Orlicz-Sobolev spaces

Orlicz-Sobolev spaces are defined by means of Orlicz spaces in the same way as standard Sobolev spaces are defined by means of L^p spaces:

$$W^m L_M(\Omega) = \{u \in L_M(\Omega);\ D^\alpha u \in L_M(\Omega) \quad \text{for} \quad |\alpha| \leq m\},$$

$$W^m E_M(\Omega) = \{u \in E_M(\Omega);\ D^\alpha u \in E_M(\Omega) \quad \text{for} \quad |\alpha| \leq m\}.$$

They are Banach spaces for the norm

$$||u|| = \left(\sum_{|\alpha| \leq m} ||D^\alpha u||^2_{L_M(\Omega)} \right)^{\frac{1}{2}} .$$

It will be convenient to identify $W^m L_M$ with a subspace of the product $\Pi_{|\alpha| \leq m} L_M(\Omega) \equiv \Pi L_M$.

We wish to build a complementary system involving those spaces. Since $(\Pi L_M, \Pi E_M; \Pi L_{\overline{M}}, \Pi E_{\overline{M}})$ is, in the obvious way, a complementary system and since $W^m L_M$ is a closed subspace of ΠL_M, we are in the situation of the above lemma. To get a complementary system $(W^m L_M, W^m E_M; *, *)$ we must verify that $W^m L_M$ is $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closed, which is clear, and moreover that functions in $W^m L_M$ can be approximated in the $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ sense by functions in $W^m E_M$. This is possible under the mild assumption that Ω enjoys the so-called *segment property*: there exist a locally finite open covering $\{O_i\}$ of $\partial\Omega$ and corresponding vectors $\{y_i\}$ such that for $x \in \overline{\Omega} \cap O_i$ and $0 <$ $< t < 1$, one has $x + ty_i \in \Omega$.

PROPOSITION. *If* Ω *has the segment property, then* $C^\infty(\overline{\Omega})$ *is* $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ *dense in* $W^m L_M(\Omega)$.

The proof follows the lines of the standard proof that $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$, see e. g. [1; p. 11-14]. This latter proof involves essentially three steps: first using a partition of unity associated with $\{O_i\}$, then making translations near the boundary by means of the vectors y_i, finally regularizing. The first step carries over immediately to our situation, and we have see in Section 1.2 that translations and regularizations behave well with respect to the $\sigma(L_M, L_{\overline{M}})$ topology. For more details, see [8; p. 168-169].

We would like now to define spaces analogous to the $W_0^{m,p}$ spaces. Starting with $\mathcal{D}(\Omega)$, and closing it for the norm ΠL_M topology, we stay inside $W^m E_M$, and the resulting space is thus naturally denoted by $W_0^m E_M(\Omega)$. To define $W_0^m L_M(\Omega)$ we may use either the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ topology or the $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ topology. In any case we stay inside

$W^mL_M(\Omega)$. It is a consequence of the following proposition that the two resulting spaces coincide in general.

PROPOSITION. *If* Ω *has the segment property, then*

$$\sigma(\Pi L_M, \Pi E_{\overline{M}})\ c\ell\ \mathcal{D}(\Omega) = \sigma(\Pi L_M, \Pi L_{\overline{M}})\ c\ell\ \mathcal{D}(\Omega) \quad (1)$$

The space (1) *is denoted* $W_0^mL_M(\Omega)$.

PROOF. The inclusion $\supset$ is obvious. So let us take $u \in \sigma(\Pi L_M, \Pi E_{\overline{M}})\ c\ell\ \mathcal{D}(\Omega)$. Denote by K the support of u . By using the covering $\{0_i\}$ of $\partial\Omega$ and an associated partition of unity, we are reduced to considering two cases: either $K \subset \Omega$ or $K \subset 0_i$ for some i . If $K \subset \Omega$, a simple regularization shows that u can be approximate in the $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ sense by functions in $\mathcal{D}(\Omega)$. We now consider the case where $K \subset 0_i$. First note that the function $\tilde{u}$ obtained by extending u by zero outside Ω belongs to $W^mL_M(\mathbb{R}^n)$. Define $u_t(x) = \tilde{u}(x - ty_i)$ for $0 < t < 1$. Then $u_t \in W^mL_M(\mathbb{R}^n)$ and the support of u_t is contained in Ω by the segment property. Moreover, by the results of Section 1.2, $u_t \to u$ in $W^mL_M(\Omega)$ for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$, so that it suffices to approximate each u_t by functions in $\mathcal{D}(\Omega)$. But this can be done by regularization since $\operatorname{supp} u_t \subset \Omega$. Q. E. D.

As remarked in section 2.1, the intersection of $W_0^mL_M(\Omega)$ with ΠE_M will be the norm closure of $\mathcal{D}(\Omega)$, i. e. $W_0^mE_M(\Omega)$.

The preceding proposition allows us to construct a complementary system $(W_0^mL_M, W_0^mE_M; *, *)$. We are now going to describe the spaces one gets here on the right, i. e. with the notations of the lemma of Section 2.1, F and F_0.

The space F is the dual of $W_0^mE_M$. It is the quotient of $\Pi L_{\overline{M}}$ by $\{(f_\alpha) \in \Pi L_{\overline{M}};\ (f_\alpha) \perp W_0^mE_M\}$, and, looking at F as a space of distributions, one has

$$F = \{g \in \mathcal{D}'(\Omega);\ g = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha g_\alpha \text{ with } g_\alpha \in L_{\overline{M}}\} ,$$

with the quotient norm. This space will be denoted $W^{-m}L_{\overline{M}}(\Omega)$. The subspace F_0 of F is isomorphic, as we have seen, to the quotient of $\Pi E_{\overline{M}}$ by $\{(f_\alpha) \in \Pi E_{\overline{M}}; (f_\alpha) \perp W_0^m E_M\}$. Looking at F_0 as a space of distributions, one has

$$F_0 = \{g \in \mathcal{D}'(\Omega);\ g = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha g_\alpha \quad \text{with} \quad g_\alpha \in E_{\overline{M}}\}$$

with the quotient norm. This space will be denoted $W^{-m}E_{\overline{M}}(\Omega)$.

REMARK. In the L^p variational theory of boundary value problems, one starts with an *arbitrary* closed subspace V lying between $W_0^{m,p}(\Omega)$ and $W^{m,p}(\Omega)$. Here, in our situation, we are limited to spaces V satisfying

$$W_0^m L_M(\Omega) \subset V \subset W^m L_M(\Omega) \ ,$$

$$V \ \sigma(\Pi L_M, \Pi E_{\overline{M}}) \ \text{closed in} \ W^m L_M(\Omega) \ ,$$

$$V \cap \Pi E_M \ \ \sigma(\Pi L_M, \Pi L_{\overline{M}}) \ \text{dense in} \ V \ .$$

The last two conditions are in some sense opposed, as already remarked, and it should be of interest to give some interpretation of those conditions, probably in terms of boundary conditions. It should also be of interest to find classes of examples where those conditions are satisfied; this is so in the extreme cases $V = W_0^m L_M$) Dirichlet boundary conditions) and $V = W^m L_M$ (Neumann boundary conditions), as seen above; see [10] for the treatment of the third problem. Of course when $\overline{M}$ has the Δ_2 property, things are much easier because then $E_{\overline{M}} = L_{\overline{M}}$ and thus one can take for V the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure in $W^m L_M$ of any space containing $\mathcal{D}(\Omega)$.

2.3. <u>Boundary values of functions in $W^m L_M(\Omega)$</u>

In this section we assume that the boundary Γ of our open bounded set Ω is sufficiently good so that questions in Ω, near Γ, can be transformed, by using a partition of unit and local charts, into similar questions in $\mathbb{R}_+^n$, near $\mathbb{R}^{n-1}$. This will be certainly so, for our purpose below, if we assume Γ to be C^1. We will also limit ourselves here to $m = 1$, i. e. to the study of $W^1 L_M(\Omega)$.

Consider the "restriction to Γ" mapping:

$$\tilde{\gamma} : C^\infty(\overline{\Omega}) \to C(\Gamma) : u \mapsto u|_\Gamma \ .$$

We will show that it is continuous for the following topologies on $C^\infty(\overline{\Omega})$ and $C(\Gamma)$ respectively:

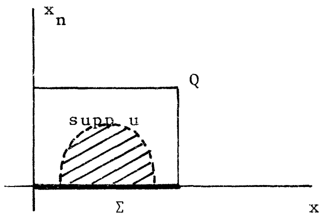
(1) $$\| \ \|_{W^1L_M(\Omega)} \to \| \ \|_{L_M(\Omega)} ,$$

(2) $$\sigma(\Pi L_M, \Pi E_{\overline{M}}) \to \sigma(L_M, E_{\overline{M}}) ,$$

(3) $$\sigma(\Pi L_M, \Pi L_{\overline{M}}) \to \sigma(L_M, L_{\overline{M}}) .$$

It follows from (2) and the results of the previous section that $\tilde{\gamma}$ can be extended into a continuous mapping, denoted by γ, from $W^1L_M(\Omega)$, $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ to $L_M(\Gamma)$, $\sigma(L_M, E_{\overline{M}})$. Condition (3) implies that γ is also continuous from $W^1L_M(\Omega)$, $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ to $L_M(\Gamma)$, $\sigma(L_M, L_{\overline{M}})$, and condition (1) implies that γ is continuous from $W^1E_M(\Omega)$, $\| \ \|$ to $E_M(\Omega)$, $\| \ \|$.

PROOF OF (1), (2), (3). By using a partition of unity and local charts, we are reduced to the following situation: $u \in C^1(\overline{Q})$, with support intersecting only the part Σ of ∂Q, where Q is, say, a cube in $\mathbb{R}^n_+$ and $\Sigma = \partial Q \cap \mathbb{R}^{n-1}$:



We have

$$u(x', 0) = -\int_0 \frac{\partial u}{\partial x_n}(x', x_n)\,dx_n$$

and so, for $v(x') \in E_{\overline{M}}(\Sigma)$,

(4) $$\int_\Sigma u(x', 0)v(x')\,dx' = -\int_Q \frac{\partial u}{\partial x_n}(x', x_n)v(x')\,dx'dx_n .$$

If we note that $v(x') \in E_{\overline{M}}(\Sigma)$ implies $\tilde{v}(x', x_n) \in E_{\overline{M}}(Q)$, where $\tilde{v}(x', x_n) = v(x')$, we immediately deduce (2) from (4). By going to the supremmum when $v(x')$ varies in a bounded set of $E_{\overline{M}}(\Sigma)$ and after noting that the mapping $v(x') \in E_{\overline{M}}(\Sigma) \mapsto \tilde{v}(x', x_n) \in E_{\overline{M}}(Q)$ is bounded, we derive (1) from (4). Finally, by taking $v(x')$ in $L_{\overline{M}}(\Sigma)$, we obtain (3). Q. E. D.

<u>Green's formula</u> holds: if $u \in W^1L_M(\Omega)$ and $v \in W^1L_{\overline{M}}(\Omega)$, then

$$(5) \qquad \int_\Omega u \frac{\partial v}{\partial x_i}\, dx + \int_\Omega v \frac{\partial u}{\partial x_i}\, dx = \int_\Gamma u\, v\, \nu_i\, d\Gamma ,$$

where ν_i denotes the i^{th} component of the exterior normal to Γ.

Indeed first note that each of three terms in (5) has a meaning. Now (5) is true if u and v belongs to $C^\infty(\overline{\Omega})$. Since $C^\infty(\overline{\Omega})$ is $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ dense in $W^1L_M(\Omega)$ and since (2) holds, we derive (5) for $u \in W^1L_M(\Omega)$ and $v \in C^\infty(\overline{\Omega})$. Since $C^\infty(\overline{\Omega})$ is $\sigma(\Pi L_{\overline{M}}, \Pi L_M)$ dense in $W^1L_{\overline{M}}(\Omega)$ and since (3) holds (with M and $\overline{M}$ interchanged) we derive (5) for $u \in W^1L_M(\Omega)$ and $v \in W^1L_{\overline{M}}(\Omega)$.

We will now show that, as in the usual L^p case, $W_0^1L_M(\Omega)$ $(W_0^1E_M(\Omega))$ can be interpreted as the space of functions in $W^1L_M(\Omega)$ $(W^1E_M(\Omega))$ which are zero on the boundary Γ.

PROPOSITION. *The kernel of the trace mapping* $\gamma: W^1L_M(\Omega) \to L_M(\Gamma)$ *is* $W_0^1L_M(\Omega)$. *The kernel of the trace mapping* $\gamma: W^1E_M(\Omega) \to E_M(\Gamma)$ *is* $W_0^1E_M(\Omega)$.

PROOF. Since $W_0^1E_M = W_0^1L_M \cap W^1E_M$, the first assertion implies the second. And to prove the first assertion, it suffices to show that $\ker\gamma \subset W_0^1L_M$ since the other inclusion follows from the continuity properties of γ. So let us take $u \in W^1L_M(\Omega)$ with $\gamma u = 0$. We will show that $\tilde{u}$ defined by $\tilde{u} = u$ in Ω, $\tilde{u} = 0$ outside Ω,

$$\tilde{u} = \begin{cases} u & \text{in } \Omega , \\ 0 & \text{outside } \Omega \end{cases}$$

belongs to $W^1L_M(R^n)$. Once this is done, the result follows by using standard arguments (partition of unity, translations near the boundary, regularization), exactly as in the proof of the second proposition of section 2.2.

It is clear that $\tilde{u} \in L_M(\mathbb{R}^n)$. Write $v_i = \frac{\partial u}{\partial x_i}$ in Ω, $v_i = 0$

outside Ω . Of course $v_i \in L_M(\mathbb{R}^n)$, and we have to show that $\partial\tilde{u}/\partial x_i = v_i$ in the distribution sense on all $\mathbb{R}^n$. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$. We have, using Green's formula and the fact that $\gamma u = 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} v_i \phi dx &= \int_\Omega \frac{\partial u}{\partial x_i} \phi \, dx \\ &= - \int_\Omega u \frac{\partial \phi}{\partial x_i} dx + \int_\Gamma u\phi\nu_i \, d\Gamma \\ &= -\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = - \int_{\mathbb{R}^n} \tilde{u} \frac{\partial \phi}{\partial x_i} dx ,\end{aligned}$$

which concludes the proof. Q. E. D.

2.4. Poincaré's inequality

PROPOSITION. *Suppose* Ω *bounded. Then there exists a constant* c *such that*

$$||u||_{L_M(\Omega)} \leq c \sum_{i=1}^{n} ||\frac{\partial u}{\partial x_i}||_{L_M(\Omega)}$$

for all u *in* $W_0^1 L_M(\Omega)$.

We will need in the proof some properties of the functionals

$$\mathcal{M}(u) = \int_\Omega M(u(x))dx \quad \text{for} \quad u \in L_M(\Omega) ,$$

$$\overline{\mathcal{M}}(v) = \int_\Omega \overline{M}(v(x))dx \quad \text{for} \quad v \in L_{\overline{M}}(\Omega) .$$

LEMMA 1. (i) $\mathcal{M}$ *is convex,* $\sigma(L_M, E_{\overline{M}})$ *lower semicontinuous,* $\not\equiv +\infty$ *on* $L_M(\Omega)$*; its domain is the class* $\mathcal{L}_M(\Omega)$*; similarly for* $\overline{\mathcal{M}}$.
(ii) $\mathcal{M}$ *and* $\overline{\mathcal{M}}$ *are conjugate one of the other in the duality* $(L_M, L_{\overline{M}})$*;* $\mathcal{M}$ *on* L_M *is also the conjugate of* $\overline{\mathcal{M}}$ *restricted to* $E_{\overline{M}}$ *and similarly,* $\overline{\mathcal{M}}$ *on* $L_{\overline{M}}$ *is the conjugate of* $\mathcal{M}$ *restricted to* E_M.
(iii) $\mathcal{M}$ *is (norm) continuous on the interior of* $\mathcal{L}_M(\Omega)$*; similarly for* $\overline{\mathcal{M}}$.

PROOF. It is clear that $\mathcal{M}$ is convex, $\not\equiv +\infty$, with domain (the set where it is $< \infty$) $\mathcal{L}_M(\Omega)$. We will show that for any $v \in L_{\overline{M}}(\Omega)$,

(1) $$\overline{\mathcal{M}}(v) \geq \sup \{\textstyle\int_\Omega uv \, dx - \mathcal{M}(u);\ u \in L_M\} ,$$

(2) $$\overline{\mathcal{M}}(v) \leq \sup \{\textstyle\int_\Omega uv \, dx - \mathcal{M}(u);\ u \in E_M\} ,$$

which implies (i) and (ii). Let $u \in L_M$ and $v \in L_{\overline{M}}$. By Young's inequality, $\overline{M}(v) \geq uv - M(u)$ pointwise, and (1) follows by integration. Now take $v \in L_{\overline{M}}$ and write $u_n = \psi(v_n)$, where ψ is the odd function equal on $\mathbb{R}^+$ to the right derivative of $\overline{M}$ and v_n is the troncated function of v (i. e. $v_n(x) = v(x)$ if $|v(x)| \leq n$, $v_n(x) = 0$ if $|v(x)| > n$). Clearly $u_n \in L \subset E_M$, and one has Young's equality,

$$\overline{M}(v_n) = v_n\, \psi(v_n) - M(\psi(v_n))$$

pointwise. By integration we get

$$\int_\Omega \overline{M}(v_n) = \int_\Omega v_n . u_n \, dx - \mathcal{M}(u_n)$$
$$= \int_\Omega v \, . u_n \, dx - \mathcal{M}(u_n) .$$

Since, by the monotone convergence theorem, the left-hand side converges to $\overline{\mathcal{M}}(v) \leq +\infty$, we conclude that (2) holds. Finally, to verify (iii), we note that $\mathcal{M}(u)$ is ≤ 1 on the closed unit ball of $L_M(\Omega)$ (with respect to the Luxemburg norm); this follows directly from the definition of $\|\ \|_{(M)}$. But a convex functional which is bounded from above on an open set (of a locally convex space) is automatically continuous on the interior of the set of points where it is finite. Thus (iii) follows. Q. E. D.

LEMMA 2. *Suppose* Ω *bounded, of diameter* d . *Then*

(3) $$\int_\Omega M(u(x))dx \leq \int_\Omega M(2d \frac{\partial u}{\partial x_1})dx$$

for all $u \in W_0^1 L_M(\Omega)$.

Poincaré's inequality follows easily from (3). Indeed, if $\|\partial u/\partial x_1\|_{(M)}$ remains bounded, then there exists k such that

$$\int_\Omega M(\frac{1}{k} \frac{\partial u}{\partial x_1})dx \leq 1 ,$$

and it follows from (3) that

$$\int_\Omega M(\frac{u}{2dk}) \leq 1 ,$$

which shows that $\|u\|_{(M)}$ remains $\leq 2dk$.

PROOF OF LEMMA 2. First assume $u \in \mathcal{D}(\Omega)$. Then

$$M(u(x_1, x_2, \dots, x_n)) = M\left(\int_{\infty}^{x_1} \frac{\partial u}{\partial x_1}(\xi, x_2, \dots, x_n)d\xi\right)$$

$$\leq \frac{1}{d}\int_{-\infty}^{+\infty} M(d \frac{\partial u}{\partial x_1}(\xi, x_2, \dots, x_n))d\xi$$

by Jensen's inequality. By integrating both sides over Ω , we deduce

$$\int_\Omega M(u(x))dx \leq \int_\Omega M(d \frac{\partial u}{\partial x_1})\, dx . \tag{4}$$

Assume now that $u \in W_0^1 L_M(\Omega)$ with compact support $\subset \Omega$. Then $u_\varepsilon \in \mathcal{D}(\Omega)$ and we have

$$\int_\Omega M(u_\varepsilon)dx \leq \int_\Omega M(d \frac{\partial u_\varepsilon}{\partial x_1})\, dx . \tag{5}$$

Since $u_\varepsilon \to u$ for, say, $\sigma(L_M, E_{\overline{M}})$, lemma 1 implies

$$\int_\Omega M(u(x))dx \leq \liminf \int_\Omega M(u_\varepsilon(x))\, dx .$$

For the right-hand side of (5), we have, by Jensen's inequality

$$\int_\Omega M((\frac{\partial du}{\partial x_1})_\varepsilon)dx \leq \int_\Omega \left[M(\frac{\partial du}{\partial x_1})\right]_\varepsilon dx . \tag{6}$$

Two cases are now distinguished either $\frac{\partial du}{\partial x_1} \notin \mathcal{L}_M$ or $\frac{\partial du}{\partial x_1} \in \mathcal{L}_M$. In the first case, (4) obviously holds for u . In the second case, $\left[M(\frac{\partial du}{\partial x_1})\right]_\varepsilon$ converges in the L^1 sense to $M(\frac{\partial du}{\partial x_1})$, which allows us to pass to the limit in the right-hand side of (6). Consequently (4) holds for u . Finally if $u \in W_0^1 L_M(\Omega)$, then consider an open set Ω^1 containing $\overline{\Omega}$, of diameter $2d$, and extend u into $\tilde{u}$ by putting $\tilde{u} = 0$ on $\Omega^1 \setminus \Omega$. We know that $\tilde{u} \in W_0^1 L_M(\Omega^1)$, with compact support in Ω^1, and so we can write

$$\int_{\Omega^1} M(\tilde{u})dx \leq \int_{\Omega^1} M(2d \frac{\partial \tilde{u}}{\partial x_1})dx ,$$

which reduces immediately to (3). Q. E. D.

Chapter 3. Strongly nonlinear elliptic problems

3.1. Introduction

It is our purpose now to study the following Dirichlet problem: find a function $u(x)$ on Ω satisfying

(1) $$\begin{cases} -\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left[\phi\left(\frac{\partial u}{\partial x_j}\right)\right] = f \quad \text{in } \Omega , \\ u = 0 \quad \text{on } \Gamma , \end{cases}$$

where f is given. The assumptions made on ϕ are the following: $\phi : \mathbb{R} \to \mathbb{R}$ is continuous, odd, strictly increasing, and $\phi(+\infty) = +\infty$. *No restriction* is imposed on the nature of the *growth* of ϕ at infinity.

With a function ϕ as above we associate the N-function:

$$M(t) = \int_0^t \phi(\tau)d\tau .$$

The typical examples for ϕ are the following:

(i) $\phi(t) = |t|^{p-2}t$ where $1 < p < \infty$. This is a case of polynomial growth, M and $\overline{M}$ satisfy the Δ_2 condition. We are in a reflexive situation, the classical theory of monotone operators in reflexive Banach can be applied.

(ii) $\phi(t) = \text{sgn}\, t.(e^{|t|} - 1)$. This is a case of rapid growth, M does not satisfy the Δ_2 condition but $\overline{M}$ does.

(iii) $\phi(t) = \text{sgn}\, t.\log(1 + |t|)$. This is a case of slow growth, M satisfies the Δ_2 condition but $\overline{M}$ does not.

(iv) There are functions ϕ for which neither M nor $\overline{M}$ have the Δ_2 property, see [13; p. 28].

3.2. Nemyckii operator

Let us consider the Nemyckii operator:

(1) $$u(x) \mapsto \phi(u(x)) .$$

Our purpose in this section is to show how properties of this nonlinear operator such as everywhere definess, boundedness, coercivity, surjectivity, are influenced by the nature of the growth of ϕ. We will look at (1) as a mapping T from $L_M(\Omega)$ into $L_{\overline{M}}(\Omega)$, with domain

$$D(T) = \{u \in L_M(\Omega);\ \phi(u(x)) \in L_{\overline{M}}(\Omega)\} .$$

T is obviously a monotone mapping.

LEMMA 1. (i) $E_M \subset D(T) \subset \mathscr{L}_M$ *and* $E_{\overline{M}} \subset R(T) \subset \mathscr{L}_{\overline{M}}$.

(ii) T *transforms a bounded set into a bounded set if and only if* M *has the* Δ_2 *property.*

(iii) T *is coercive (i. e.* $\langle u, Tu\rangle/\|u\| \to +\infty$ *as* $\|u\| \to +\infty$ *) if and only if* $\overline{M}$ *has the* Δ_2 *property.*

Of course other properties may be of interest, such as continuity, or maximal monotonicity. We will discuss those two later in sections 3.4 and 3.5.

PROOF OF LEMMA 1. If $u \in D(T)$, then $u \in L_M$ and $\phi(u) \in L_{\overline{M}}$, so that $u.\phi(u) \in L^1$; but

$$u\phi(u) = M(u) + \overline{M}(\phi(u)) \geq M(u) ,$$

which implies $M(u) \in L^1$, i. e. $u \in \mathscr{L}_M$. On the other hand one has

$$\overline{M}(\phi(t)) \leq M(2t) \quad \text{for} \quad t \in \mathbb{R} \tag{2}$$

because, for $t \geq 0$,

$$M(2t) = \int_0^{2t} \phi(\tau)d\tau \geq \int_t^{2t} \phi(\tau)d\tau \geq t\phi(t)$$

$$= M(t) + \overline{M}(\phi(t)) \geq \overline{M}(\phi(t)) .$$

So if $u \in E_M$, then $2u \in E_M \subset \mathscr{L}_M$, and we see from (2) that $\phi(u) \in \mathscr{L}_{\overline{M}} \subset L_{\overline{M}}$. We have thus proved that $E_M \subset D(T) \subset \mathscr{L}_M$. (Remark that a slight modification of the preceding argument shows that if $u \in \operatorname{int} \mathscr{L}_M$, i. e. if $(1 + \varepsilon)u \in \mathscr{L}_M$ for some $\varepsilon > 0$, then $u \in D(T)$). Since T^{-1} is the Nemyckii operator from $L_{\overline{M}}$ into L_M associated with the function ψ reciprocal to ϕ , we immediately deduce corresponding informations on the range of T , which completes the proof of part (i).

Now assume that T is a bounded mapping and let $u \in L_M$. We have, for the troncated function u_n , $|u_n| \leq |u|$, and so u_n varies in a bounded set in $E_M \subset D(T)$. Consequently $\phi(u_n)$ varies in a bounded set in $L_{\overline{M}}$, and there exists K such that

$$\int_\Omega \phi(u_n)u_n \, dx \leqq K .$$

But the left hand side is $\geqq \int_\Omega M(u_n)dx$, so that, by Fatou's lemma, we get $\int_\Omega M(u)dx < \infty$, i. e. $u \in \mathscr{L}_M$. We have thus shown that L_M is contained in $\mathscr{L}_M$, which implies the Δ_2 property for M. Let us now assume that M has the Δ_2 property, and let u vary in a bounded set in L_M:

$$(3) \qquad ||u||_{(M)} \leqq \frac{k}{2} \text{, i. e. } \int_\Omega M(\frac{2u}{k})dx \leqq 1 .$$

The number k above can always be taken > 1. Since $M(kt) \leq cM(t)$ for $t \geqq t_0$, we derive from (3)

$$1 \geqq \int_{\frac{|2u|}{k} \geqq t_0} M(\frac{2u}{k})dx + \int_{\frac{|2u|}{k} < t_0} M(\frac{2u}{k})dx \geqq \frac{1}{c} \int_{\frac{|2u|}{k} \geqq t_0} M(2u)dx ,$$

from which it follows, since Ω is bounded, that $\int_\Omega M(2u)dx$ remains bounded. Inequality (2) then implies that $\int_\Omega \overline{M}(\phi(u))dx$ remains bounded, i. e. $\phi(u)$ remains bounded "in the mean" in $L_{\overline{M}}$ and so remains bounded in $L_{\overline{M}}$.

Finally, since coercivity of T implies boundedness of T^{-1}, and since T^{-1} enjoys, as already remarked, properties similar to those of T, we see that coercivity of T implies that $\overline{M}$ has the Δ_2 property. The converse is a consequence of lemma 2 below and of the inequality

$$\int_\Omega u\phi(u)dx \geqq \int_\Omega M(u(x))dx .$$

Q. E. D.

LEMMA 2. *One has*

$$(4) \qquad \frac{1}{||u||_{(M)}} \int_\Omega M(u(x))dx \to +\infty \text{ when } ||u||_{(M)} \to +\infty$$

if and only if $\overline{M}$ *satisfies the* Δ_2 *condition.*

PROOF. Since (4) implies that the operator T above is coercive, the necessary condition follows from the parts of Lemma 1 already proved.

To prove the sufficient condition, let us first assume that $\overline{M}$ satisfies the Δ_2 condition for all t, i. e.

(5) $$\overline{M}(2t) \leqq k\overline{M}(t) \quad \text{for all} \quad t \in \mathbb{R}$$

(and not only for large t). Defining a function $f: [1,+\infty[\to [k,+\infty[$ by

$$f(r) = r[(1-\lambda)k^{n+1} + \lambda k^{n+2}]$$

if $r \in [2^n, 2^{n+1}]$ and $r = (1-\lambda)2^n + \lambda 2^{n+2}$, we obtain $\overline{M}(rt) \leqq$ $\leqq f(r)\overline{M}(t)$ for $t \in \mathbb{R}$ and $r \geqq 1$, and so, by passing to the conjugate convex functions, $M(f(r)r^{-1}t) \geqq f(r)M(t)$ for $t \in \mathbb{R}$ and $r \geqq 1$. Since $f(r)r^{-1}$ strictly increases from k to $+\infty$ as r goes from 1 to $+\infty$, its reciprocal function $g(s)$ is well defined and strictly increases from 1 to $+\infty$ as s goes from k to $+\infty$, and we have

(6) $$M(st) \geqq sg(s)M(t)$$

for $t \in \mathbb{R}$ and $s \geqq k$. Now take $u \in L_M(\Omega)$ with $||u||_{(M)} > k$. If $\varepsilon > 0$ satisfies $||u||_{(M)} - \varepsilon > k$, then it follows from (6) that

$$\begin{aligned}\int_\Omega M(u)dx &\geqq (||u||_{(M)} - \varepsilon)g(||u||_{(M)} - \varepsilon)\int_\Omega M(u(||u||_{(M)} - \varepsilon)^{-1})dx \\ &\geqq (||u||_{(M)} - \varepsilon)g(||u||_M - \varepsilon)\end{aligned}$$

by definition of $||\ ||_{(M)}$. Letting $\varepsilon \to 0$, we obtain

$$\int_\Omega M(u)dx \geqq ||u||_{(M)}\ g(||u||_M) ,$$

which proves the lemma under condition (5).

To get rid of (5),we will use the following result about *equivalent N-functions*: two N-functions N and M are said to be equivalent if there exist $\alpha_1, \alpha_2 > 0$ and t^* such that

(7) $$N(\alpha_1 t) \leqq M(t) \leqq N(\alpha_2 t)$$

for $t \geqq t^*$. One can prove (cf. [13; §13]) that two Orlicz spaces $L_M(\Omega)$ and $L_N(\Omega)$ are equal, with equivalent norms, if and only if M and N are equivalent N-functions.Moreover given an N-function

M satisfying the Δ_2 condition (i. e. for $t \geqq$ some t_0), there always exists an equivalent N-function N satisfying the Δ_2 condition for all $t \in \mathbb{R}$.

So let us start with M satisfying the Δ_2 condition for $t \geqq t_0$ and let us consider an equivalent N-function N satisfying the Δ_2 condition for all t . For this N we have

(8) $$\frac{1}{||u||_{(M)}} \int_\Omega N(u(x))dx \to +\infty$$

as $||u||_{(M)} \to \infty$. Now, by (7),

$$\frac{1}{||u||_{(M)}} \int_\Omega M(u(x))dx \geqq \frac{1}{||u||_{(M)}} \int_{|u(x)|\geqq t^*} N(\alpha_1 u)dx$$
$$= \frac{1}{||\alpha_1 u||_{(M)}} \int_\Omega N(\alpha_1 u)dx - \frac{1}{||u||_{(M)}} \int_{|u(x)|\geqq t^*} N(\alpha_1 u)dx \to +\infty$$

as $||u||_{(M)} \to +\infty$, because of (8). Q. E. D.

We remark that the last part of the proof of lemma 2 as given in our paper [8; p. 183] contains a little gap. It should be replaced by the preceding argument involving equivalent N-functions.

3.3. <u>An abstract existence theorem</u>

Let $(Y, Y_0; Z, Z_0)$ be a complementary system and consider a mapping T with domain in Y and values in Z . The following four assumptions will be made:

(i) $D(T) \supset Y_0$ and T is *hemicontinuous* on Y_0, i. e. continuous from the finite dimensional subspaces of Y_0 to Z , $\sigma(Z, Y_0)$,

(ii) T is *monotone*, i. e. $\langle y_1 - y_2, Ty_1 - Ty_2\rangle \geqq 0$ for all y_1, y_2 in D(T) ,

(iii) T is *pseudo-monotone* in the following sense: for any bounded net $y_i \in D(T)$ such that $y_i \to y \in Y$ for $\sigma(Y, Z_0)$, $Ty_i \to z \in Z$ for $\sigma(Z, Y_0)$ and such that $\limsup \langle y_i, Ty_i\rangle \leqq \langle y, z\rangle$, it follows that $y \in D(T)$, $Ty = z$ and $\langle y_i, Ty_i\rangle \to \langle y, z\rangle$,

(iv) T^{-1} is *locally bounded* near any point of Z_0, i. e. for any $z \in Z_0$, there exists a (norm) neighbourhood $\mathcal{N}$ of z in Z such that $T^{-1}\mathcal{N} = \{y \in Y;\ Ty \in \mathcal{N}\}$ is bounded in Y.

THEOREM. *Suppose* $T : D(T) \subset Y \to Z$ *satisfy the above conditions. Suppose also that two additional technical assumptions are satisfied (see below). Then* $R(T) \supset Z_0$.

The first of the two additional assumptions mentioned above requires that $T(Y_0)$ meets Z_0. The second is an assumption of geometric nature on the complementary system. Let us consider a (equivalent) norm $\|\ \|_Y$ on Y and let us denote by $\|\ \|_{Y_0}$ the restriction of $\|\ \|_Y$ to Y_0, by $\|\ \|_Z$ the dual norm on Z and by $\|\ \|_{Z_0}$ the restriction of $\|\ \|_Z$ to Z_0. We call $\|\cdot\|_Y$ *admissible* if $\|\ \|_Y$ is the norm on Y dual to $\|\ \|_{Z_0}$ on Z_0 and if the inequality

$$(1) \qquad \langle y, z\rangle \leqq \|y\|_Y \ \|z\|_Z$$

holds for all $y \in Y$ and $z \in Z$. The second additional assumption simply requires that such a norm exists on Y.

We do not know whether an admissible norm always exists in an arbitrary complementary system. Non admissible norms exist when $Z \neq Z_0$: simply take $\|y\| + |\langle y, z\rangle|$ where $\|\ \|$ is any norm on Y and $z \in Z \setminus Z_0$. In the complementary system $(L_M, E_M; L_{\overline{M}}, E_{\overline{M}})$ both the Luxemburg norm and the Orlicz norm are admissible, as is immediately verified. And it is easy to see that if $\|\ \|_Y$ is admissible in $(Y, Y_0; Z, Z_0)$ and if $(E, E_0; F, F_0)$ is obtained from $(Y, Y_0; Z, Z_0)$ by the procedure of section 2.1, then the restriction of $\|\ \|_Y$ on E is admissible in $(E, E_0; F, F_0)$. Consequently the usual norms on $(W^1L_M, W^1E_M; *, *)$ and $(W_0^1 L_M, W_0^1 E_M; W^{-1}L_{\overline{M}}, W^{-1}E_{\overline{M}})$ are also admissible.

The purpose of this geometric assumption is to insure that the *duality mapping* is monotone and pseudo-monotone. Let $\|\ \|_Y$ be a (equivalent) norm on Y and let $\|\ \|_{Y_0}, \|\ \|_Z, \|\ \|_{Z_0}$ be defined as above. The corresponding duality mapping $J : D(J) \subset Y \to 2^Z$ is defined by

$$Jy = \{z \in Z;\ ||z||_Z = ||y||_Y \text{ and } \langle y, z\rangle = ||y||_Y\, ||z||_Z\},$$

with $D(J) = \{y \in Y;\ Jy \neq \emptyset\}$. Clearly J is coercive, bounded, and hemicontinuous on Y_0, and the restriction of J to $Y_0 \subset D(J)$ is the usual duality mapping from Y_0 to 2^{Y_0}. Note that the meaning of the conditions of hemicontinuity and pseudo-monotonicity here have to be suitably modified since J may be multivalued.

LEMMA. *If* $||\ ||_Y$ *is admissible, then* J *is monotone and pseudo-monotone.*

PROOF. Monotonicity follows easily, as in the standard reflexive situation by using inequality (1). Let now (y_i, z_i) be a net such that $z_i \in Jy_i$, y_i bounded, $y_i \to y \in Y$ for $\sigma(Y, Z_0)$, $z_i \to z \in Z$ for $\sigma(Z, Y_0)$ and $\limsup \langle y_i, z_i\rangle \leqq \langle y, z\rangle$, Since $||\ ||_Y$ is admissible, we have

$$||y||_Y\, ||z||_Z \geqq \langle y, z\rangle \geqq \limsup \langle y_i, z_i\rangle \geqq \liminf \langle y_i, z_i\rangle =$$
$$= \liminf ||y_i||_Y\, ||z_i||_Z \geqq \text{both } ||y||_Y^2 \text{ and } ||z||_Z^2 \ ;$$

we have also used above the facts that $||\ ||_Y$ is $\sigma(Y, Z_0)$ l. s. c. and that $||\ ||_Z$ is $\sigma(Z, Y_0)$ l. s. c. From the above relation, we deduce $||y||_Y = ||z||_Z$ and $\langle y, z\rangle = ||y||_Y\, ||z||_Z$, i. e. $z \in Jy$. Moreover $\langle y_i, z_i\rangle \to \langle y, z\rangle$. Q. E. D.

Before going into the proof of the theorem we mention that the Nemyckii operator $u(x) \mapsto \phi(x)$ from L_M to $L_{\overline{M}}$ considered in the previous section satisfies assumptions (i) - (iv). This will be clear from the results of section 3.4. We note also that in the particular reflexive situation where $Y = Y_0$ and $Z = Z_0$, condition (iii) is implied by (i) and (ii), as is well known, so that our theorem reduces to the statement that an everywhere defined monotone hemicontinuous operator from a reflexive Banach space into its dual is onto if its inverse is locally bounded. This is a particular case of a result by ROCKAFELLAR [20].

The following lemma will be needed in the proof of the Theorem.

Its proof (cf. [8]) is identical to that of the corresponding "reflexive" result in [2] and we omit it here.

LEMMA. *Suppose that* S *and* T *are two mappings with domains in* Y *and values in* Z *(or more generally in* 2^Z*). If both* S *and* T *are pseudo-monotone and if one of them is bounded, then* $S + T$ *is pseudo-monotone.*

PROOF OF THE THEOREM. We first consider the case where T is coercive, which is a more restrictive condition than (iv). Then the result $R(T) \supset Z_0$ follows easily, by standard arguments. Let us sketch them. Let $j_F^* T j_F$ be the Galerkin approximant of T, where F is a finite dimensional subspace of Y_0, $j_F : F \to Y_0$ is the injection mapping and $j_F^* : Z \to F^*$ the dual projection. The mapping $j_F^* T j_F : F \to F^*$ is continuous and coercive, and so, given $z \in Z_0$, there exists $y_F \in F$ solution of $j_F^* T j_F(y_F) = j_F^* z$. Coercivity of T implies that y_F remains bounded in Y, so that, passing to a subnet if necessary, $y_F \to y \in Y$ for $\sigma(Y, Z_0)$. It is then easy, by means of condition (iii), to verify that $Ty = z$.

Now consider the general case. Since $T(Y_0)$ meets Z_0, we can always assume, by making a translation if necessary, that $T(0) \in Z_0$. Let us consider for $\varepsilon > 0$ the mapping

$$T_\varepsilon = \varepsilon J + T : D(T) \cap D(J) \to 2^Z .$$

It satisfies the same assumptions as T ((iii) follows from the lemma) but is in addition coercive. Consequently it follows from the case considered above (suitably modified in order to deal with multivalued mappings) that $R(T_\varepsilon) \supset Z_0$. So, given $z \in Z_0$, there exists $y_\varepsilon \in Y$ such that $z \in (\varepsilon J + T)(y_\varepsilon)$. We will show that y_ε remains bounded in Y as $\varepsilon \to 0$. It will then be easy to deduce, using (iii), that $z \in R(T)$.

To prove that y_ε remains bounded, let us consider the segment $[z, T(0)]$ in Z_0. Since T^{-1} is locally bounded near any point of this segment, it follows by a simple compacity argument that there exists $\delta > 0$ such that T^{-1} is bounded in Y, where

$$\mathcal{N} = \{u \in Z;\ \text{dist}\ (u;\ [z,\ T(0)]) \leq \delta\}.$$

Let K be such that $||T^{-1}\mathcal{N}|| \leq K$. We will show that

$$(5) \qquad ||T_\varepsilon^{-1}u|| \leq 2K$$

for any $u \in [z, T(0)]$ and $0 < \varepsilon < \delta/2K$, which implies that $y_\varepsilon \in T_\varepsilon^{-1}(z)$ remains bounded in Y as $\varepsilon \to 0$. Assume by contradiction that $||T^{-1}u|| > 2K$ for some $u \in [z, T(0)]$ and some ε with $0 < \varepsilon < \delta/2K$. The mapping $v \in R(T_\varepsilon) \mapsto ||T_\varepsilon^{-1}v||$ is singlevalued and (norm) continuous. Indeed, if $y_1 \in T_\varepsilon^{-1}(z_1)$ and $y_2 \in T_\varepsilon^{-1}(z_2)$, then

$$\langle y_1 - y_2, z_1 - z_2\rangle = \langle y_1 - y_2, a_1 - a_2\rangle + \varepsilon\langle y_1 - y_2, b_1 - b_2\rangle,$$

where $z_1 = a_1 + \varepsilon b_1$ with $a_1 \in T(y_1)$ and $b_1 \in J(y_1)$, and similarly for z_2; consequently

$$\langle y_1 - y_2, z_1 - z_2\rangle \geq \varepsilon(||y_1|| - ||y_2||)^2,$$

and we see that if $z_1 = z_2$, then necessarily $||y_1|| = ||y_2||$, and in addition, if $z_1 \to z_2$, then $||y_1|| \to ||y_2||$. Consequently, since $||T_\varepsilon^{-1}(T(0))|| = 0$ and $||T_\varepsilon^{-1}u|| > 2K$, there must exist v in the segment $[u, T(0)]$ such that $||T_\varepsilon^{-1}v|| = 2K$. Let $y \in T_\varepsilon^{-1}v$ and write $v = a + \varepsilon b$ with $a \in T(y)$ and $b \in J(y)$. We have $||y|| = ||T_\varepsilon^{-1}v|| = 2K$, thus $||b|| = 2K$, and consequently $||a - v|| = \varepsilon 2K < \delta$, so that $a \in \mathcal{N}$. But then y, which lies in $T^{-1}a$, must have a norm $\leq K$, which is a contradiction. Q. E. D.

Several variants of the above theorem can be given. We mention briefly a few of them which are of interest in applications. For the precise statements and proofs, see [8] and [9].

The *monotonicity* condition (ii) may be replaced by the condition that T is *odd* for large values of its argument. Condition (ii) may also be replaced by the condition that T is *positive* for large values of its argument: $\langle y, Ty\rangle \geq 0$ for $||y||$ large. Instead of assuming as above that T is monotone, odd or positive, it suffices to assume that T is *homotopic* to a mapping T_1 which is monotone, odd or positive. The precise definition of the homotopy used here involves the pseudo-monotonicity condition (iii). In this situation, the local a priori bound condition (iv) is required to hold uniformly with respect to the homotopy.

When Y_0 and Z_0 are separable, the above theorem as well as its variants admit *sequential* versions: it suffices to assume that condition (iii) holds for ordinary sequences (a fact which is useful in concrete applications where measure theoretical arguments are sometimes used to verify (iii), as in Theorem 2 of section 3.5). However some additional assumption must then be imposed, for instance T should be strongly quasi bounded in the sense that y bounded and $\langle y, Ty \rangle$ bounded from above imply Ty bounded; one could also, instead of this boundedness restriction, use a modified condition (iii) which involves dense subspaces of Y_0.

3.4. Application

THEOREM. *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$, *with the segment property. Let* $\phi: \mathbb{R} \to \mathbb{R}$ *be continuous, strictly increasing, odd, with* $\phi(+\infty) = +\infty$. *Write* $M(t) = \int_0^t \phi(\tau)d\tau$. *Then for any* $f \in W^{-1}E_{\overline{M}}(\Omega)$ *there exists an unique* $u \in W_0^1L_M(\Omega)$ *such that* $\phi(\partial u/\partial x_j) \in L_{\overline{M}}(\Omega)$ *for* $j = 1, \ldots, N$ *and*

$$- \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left[\phi\left(\frac{\partial u}{\partial x_j}\right)\right] = f$$

in the distribution sense in Ω.

Unicity is easily verified, using the fact that $\mathcal{D}(\Omega)$ is $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ dense in $W_0^1L_M(\Omega)$.

PROOF OF EXISTENCE. We will apply the abstract theorem of section 3.3 to the complementary system $(W_0^1L_M, W_0^1E_M; W^{-1}L_{\overline{M}}, W^{-1}E_{\overline{M}})$ and the operator $T : D(T) \subset W_0^1L_M \to W^{-1}L_{\overline{M}}$ defined by

$$D(T) = \{u \in W_0^1L_M;\ \phi\left(\frac{\partial u}{\partial x_j}\right) \in L_{\overline{M}} \quad \text{for } j = 1, \ldots, N\},$$

$$Tu = - \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left[\phi\left(\frac{\partial u}{\partial x_j}\right)\right] \quad \text{for } u \in D(T).$$

We note that

$$\langle v, Tu\rangle = \sum_{j=1}^{N} \int_{\Omega} \phi(\frac{\partial u}{\partial x_j}) \frac{\partial v}{\partial x_j}\, dx$$

for $u \in D(T)$ and $v \in W_0^1L_M$.

The two technical assumptions of the abstract theorem are clearly satisfied. Also $D(T) \supset Y_0$ (see the study of the Nemyckii operator in section 3.2) and T is monotone. The hemicontinuity of T will follow from Lemma 1 bellow. So we have just to verify the pseudo-monotonicity condition (iii) and the local a priori bounded condition (iv).

Pseudo-monotonicity condition. Let $u_i \in D(T)$ be a bounded net with $u_i \to u \in W_0^1L_M$ for $\sigma(W_0^1L_M, W^{-1}E_{\overline{M}})$, $Tu_i \to g \in W^{-1}L_{\overline{M}}$ for $\sigma(W^{-1}L_{\overline{M}}, W_0^1E_M)$ and

(2) $$\limsup \langle u_i, Tu_i\rangle \leqq \langle u, g\rangle.$$

We must show that $u \in D(T)$, $Tu = g$ and $\langle u_i, Tu_i\rangle \to \langle u, g\rangle$. It clearly suffices to prove the last convergence for a subnet.

First $\phi(\partial u_i/\partial x_j)$ remains bounded in $L_{\overline{M}}$ for each $j = 1,\dots,N$. Indeed, from (2) we have

$$\text{const.} \geqq \langle u_i, Tu_i\rangle = \sum_{j=1}^{N} \int_{\Omega} \phi(\frac{\partial u_i}{\partial x_j}) \frac{\partial u_i}{\partial x_j}\, dx$$
$$\geqq \sum_{j=1}^{N} \int_{\Omega} \overline{M}(\phi(\frac{\partial u_i}{\partial x_j}))\, dx ,$$

which implies that each $\phi(\partial u_i/\partial x_j)$ remains bounded "in the mean" in $L_{\overline{M}}$, and so remains bounded in $L_{\overline{M}}$. Consequently, passing to a subnet if necessary, we can assume that $\phi(\partial u_i/\partial x_j) \to h_j \in L_{\overline{M}}$ for $\sigma(L_{\overline{M}}, E_M)$. It follows from

$$\langle v, Tu_i\rangle = \sum_{j=1}^{N} \int_{\Omega} \phi(\frac{\partial u_i}{\partial x_j}) \frac{\partial v}{\partial x_j}\, dx$$

for $v \in \mathcal{D}(\Omega)$ that

(3) $$\langle v, g\rangle = \sum_{j=1}^{N} \int_{\Omega} h_j \frac{\partial v}{\partial x_j}\, dx$$

for $v \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ dense in $W_0^1L_M$, (3) also holds for $v \in W_0^1L_M$.

Now, by monotonicity, we have

(4) $$\sum_{j=1}^{N} \int_{\Omega} (\phi(\frac{\partial u_i}{\partial x_j}) - \phi(w_j))(\frac{\partial u_i}{\partial x_j} - w_j)dx \geqq 0$$

for all (w_j) in, say, ΠL^{∞}. Going to the limit and using (2) and (3), we obtain

(5) $$\sum_{j=1}^{N} \int_{\Omega} (h_j - \phi(w_j))(\frac{\partial u}{\partial x_j} - w_j)\ dx \geqq 0$$

for all $(w_j) \in \Pi L^{\infty}$. One would like now to use Minty's classical argument, i. e. putting w_j equal to $(\partial u/\partial x_j) + tw_j$ in (5) and letting $t \to 0$. But this is not allowed because $\partial u/\partial x_j \notin L^{\infty}$ in general. So let us introduce

$$\Omega_k = \{x \in \Omega;\ |\frac{\partial u}{\partial x_j}(x)| \leqq k \quad \text{for} \quad j = 1, \ldots, N\}$$

and let χ_k denote the characteristic function of Ω_k. We replace w_j in (5) by $w_j\chi_k - (\partial u/\partial x_j)\chi_k + (\partial u/\partial x_j)\chi_\ell$, where $\ell \geqq k$:

$$\begin{aligned}
0 \leqq & \sum_{j=1}^{N} \int_{\Omega} (h_j - (w_j\chi_k - \frac{\partial u}{\partial x_j}\chi_k + \frac{\partial u}{\partial x_j}\chi_\ell))(\frac{\partial u}{\partial x_j} - w_j\chi_k + \frac{\partial u}{\partial x_j}\chi_k - \frac{\partial u}{\partial x_j}\chi_\ell)dx \\
= & \sum_{j=1}^{N} \int_{\Omega} h_j(\frac{\partial u}{\partial x_j} - \frac{\partial u}{\partial x_j}\chi_\ell)\ dx \\
& - \sum_{j=1}^{N} \int_{\Omega} \phi(w_j\chi_k - \frac{\partial u}{\partial x_j}\chi_k + \frac{\partial u}{\partial x_j}\chi_\ell)(\frac{\partial u}{\partial x_j} - \frac{\partial u}{\partial x_j}\chi_\ell)\ dx \\
& + \sum_{j=1}^{N} \int_{\Omega} (h_j - \phi(w_j\chi_k - \frac{\partial u}{\partial x_j}\chi_k + \frac{\partial u}{\partial x_j}\chi_\ell))(\frac{u}{x_j}\chi_k - w_j\chi_k)\ dx\ .
\end{aligned}$$

The first integral goes to zero as $\ell \to \infty$, the second integral is zero because $\phi(w_j\chi_k - \frac{\partial u}{\partial x_j}\chi_k + \frac{\partial u}{\partial x_j}\chi_\ell)$ is zero outside Ω_ℓ since $\ell \geqq k$, and the last integral is equal to

$$\sum_{j=1}^{N} \int_{\Omega} (h_j - \phi(w_j\chi_k - \frac{\partial u}{\partial x_j}\chi_k + \frac{\partial u}{\partial x_j}\chi_k))(\frac{\partial u}{\partial x_j}\chi_k - w_j\chi_k)\ dx\ .$$

Hence, fixing k and letting $\ell \to +\infty$, we obtain

(6) $$\sum_{j=1}^{N} \int_{\Omega_k} (h_j - \phi(w_j))(\frac{\partial u}{\partial x_j} - w_j)\ dx \geqq 0$$

for any $(w_j) \in \Pi L^{\infty}$. Here, on Ω_k, we can apply Minty's argument and derive from (6) that $h_j = \phi(\partial u/\partial x_j)$ on Ω_k. Since k is arbitrary, we obtain $h_j = \phi(\partial u/\partial x_j)$ on Ω, for $j = 1, \ldots, N$. Consequently, $u \in D(T)$ and (3) implies that $Tu = g$.

It remains to see that $\langle u_i, Tu_i\rangle \to \langle u, g\rangle$. We first deduce from (4) that

$$L \equiv \liminf \sum_{j=1}^{N} \int_\Omega \phi(\frac{\partial u_i}{\partial x_j}) \frac{\partial u_i}{\partial x_j}\, dx$$
$$\geqq \sum_{j=1}^{N} \int_\Omega \phi(w_j)(\frac{\partial u}{\partial x_j} - w_j)dx + \sum_{j=1}^{N} \int_\Omega \phi(\frac{\partial u}{\partial x_j})w_j\, dx$$

for all $(w_j) \in \Pi L^\infty$. Let Ω_k and χ_k be as above. Then, putting w_j equal to $\frac{\partial u}{\partial x_j}\chi_k$, we get

$$L \geqq \sum_{j=1}^{N} \int_\Omega \phi(\frac{\partial u}{\partial x_j}\chi_k)(\frac{\partial u}{\partial x_j} - \frac{\partial u}{\partial x_j}\chi_k)dx + \sum_{j=1}^{N} \int_\Omega \phi(\frac{\partial u}{\partial x_j})\frac{\partial u}{\partial x_j}\chi_k\, dx ,$$

where the first integral is zero. Letting $k \to \infty$, we obtain

$$L \geqq \sum_{j=1}^{N} \int_\Omega \phi(\frac{\partial u}{\partial x_j}) \frac{\partial u}{\partial x_j}\, dx .$$

This inequality, combined with (2), implies $\langle u_i, Tu_i\rangle \to \langle u, g\rangle$.

Local a priori bound condition. Let

$$g = g_0 - \sum_{j=1}^{N} \frac{\partial g_j}{\partial x_j} \in W^{-1}E_{\overline{M}}(\Omega)$$

be given. We must prove that there exists a (norm) neighbourhood of g in $W^{-1}L_{\overline{M}}(\Omega)$ such that $\{u \in W_0^1L_M(\Omega);\ Tu \in \mathcal{N}\}$ is bounded in $W_0^1L_M(\Omega)$. We will make use of Poincaré's inequality (cf. section 2.4):

$$\int_\Omega M(v)dx \leqq a \int_\Omega \sum_{j=1}^{N} M(b\, \frac{\partial v}{\partial x_j})\, dx$$

for $v \in W_0^1L_M$.

Take $r > \max\{b, 1+ab\}$ and choose c such that

$$\int_\Omega \overline{M}(rg_j)dx \leq c \quad \text{for } j = 0, 1, \ldots, N .$$

This is possible since $g_j \in E_{\overline{M}}$ so that $rg_j \in E_{\overline{M}} \subset L_{\overline{M}}$. Now consider

$$\mathcal{N} = \{h = h_0 - \sum_{j=1}^{N} \frac{\partial h_j}{\partial x_j} \in W^{-1}L_{\overline{M}};\ \int_\Omega \overline{M}(rh_j)dx \leq c + 1 \quad \text{for } j = 0, 1, \ldots, N\} .$$

Clearly $g \in \mathcal{N}$, and since the functional $\int_\Omega \overline{M}(w(x))dx$ is norm continuous on the interior of $\mathcal{L}_{\overline{M}}$ (cf. section 2.4), $\mathcal{N}$ is a (norm) neighbourhood of g in $W^{-1}L_{\overline{M}}$. If $u \in W_0^1L_M$ verifies $Tu = h \in \mathcal{N}$, then

$$(7) \qquad \sum_{j=1}^{N} \int_\Omega \phi\left(\frac{\partial u}{\partial x_j}\right) \frac{\partial u}{\partial x_j}\, dx = \sum_{j=1}^{N} \int_\Omega h_j \frac{\partial u}{\partial x_j}\, dx + \int_\Omega h_0 u\, dx .$$

The left-hand side of (7) is greater than

$$\sum_{j=1}^{N} \int_\Omega M\left(\frac{\partial u}{\partial x_j}\right) dx$$

by Young's equality. The right-hand side is less than

$$\sum_{j=1}^{N} \int_\Omega \overline{M}(rh_j)dx + \sum_{j=1}^{N} \int_\Omega M\left(\frac{1}{r}\frac{\partial u}{\partial x_j}\right)dx + \int_\Omega \overline{M}(rh_0)dx + \int_\Omega M\left(\frac{u}{r}\right)dx$$

$$\leq N(c+1) + \frac{1}{r}\sum_{j=1}^{N} \int_\Omega M\left(\frac{\partial u}{\partial x_j}\right)dx + (c+1) + \frac{ab}{r}\sum_{j=1}^{N} \int_\Omega M\left(\frac{\partial u}{\partial x_j}\right)dx .$$

From the choice of r, we deduce that $\sum_{j=1}^{N} \int_\Omega M\left(\frac{\partial u}{\partial x_j}\right)dx$ remains bounded and so, by Poincaré's inequality again, u remains bounded in $W_0^1L_M$. Q. E. D.

LEMMA 1. *Let* M *and* N *be* N*-functions and let* $f(x, t)$ *satisfy the Caratheodory conditions on* $\Omega \times \mathbb{R}$. *Assume that there exist* $a(x) \in L_N$, *constants* b *and* c *such that*

$$(8) \qquad |f(x, t)| \leq a(x) + bN^{-1}M(ct)$$

for $x \in \Omega$ *and* $t \in \mathbb{R}$. *Then* $f(x, u(x)) \in L_N$ *for* u *in some strip* B *around* E_M *:*

$$B = \{u \in L_M;\ \text{dist}\ (u, E_M) < \frac{1}{2c}\},$$

and the mapping $u(x) \in B \rightarrow f(x, u(x)) \in L_N$ *is continuous on the finite dimensional simplexes of* B *with values in* L_N, $\sigma(L_N, E_{\overline{N}})$. *(The distance in the definition of* B *is measured by means of the Luxemburg norm.)*

We remark that this lemma does not follow from the standard continuity results on Nemyckii operators in Orlicz spaces as given in [13; §17].

PROOF OF LEMMA 1. It is well known (cf. [13]; p. 82) that $\mathscr{L}_M$ contains the strip

$$A = \{u \in L_M;\ \text{dist}\ (u, E_M) < \frac{1}{2}\}.$$

Let $u \in B$. Then $cu \in A \subset \mathscr{L}_M$, so that $M(cu) \in L^1$ and $N^{-1}M(cu) \in$ $\in \mathscr{L}_N$. Consequently (8) implies that $|f(x, u(x))|$ is majorized by a function in L_N , and so belongs to L_N .

Now we show that if u runs over a simplex $S \subset B$, then $f(x, u(x))$ remains bounded in L_N . Let us write

$$S = \{ \sum_{i=1}^{R} \lambda_i u_i ; \lambda_i \geqq 0 , \sum_{i=1}^{R} \lambda_i = 1\}$$

where $u_i \in B$. If $u = \sum_{i=1}^{R} \lambda_i u_i$, then

$$N^{-1}M(cu) = N^{-1}M(\sum_{i=1}^{R} \lambda_i cu_i) \leqq N^{-1} \sum_{i=1}^{R} \lambda_i M(cu_i) ,$$

so that

$$N(N^{-1}M(cu)) \leqq \sum_{i=1}^{R} \lambda_i M(cu_i) \leqq \sum_{i=1}^{R} M(cu_i) .$$

Since the right-hand side is a fixed (i. e. independent of u) element of L^1, $N^{-1}M(cu)$ remains bounded in the mean in L_N , and so remains bounded in L_N . The conclusion that $f(x, u(x))$ also remains bounded in L_N follows then immediately from (8).

Let now u_i, $u \in B$, $u_i \to u$, u_i in a finite dimensional simplex of B . Thus $f(x, u_i(x))$ remains bounded in L_N . Taking a subsequence if necessary, we can assume $u_i \to u$ a. e., so that $f(x, u_i(x)) \to$ $\to f(x, u(x))$ a. e. The conclusion then follows from lemma 2 below. Q. E. D.

LEMMA 2. *Let* u_i *be a bounded sequence in* L_N *and assume that* $u_i \to u$ *a. e. Then* $u \in L_N$ *and* $u_i \to u$ *for* $\sigma(L_N, E_{\overline{N}})$.

PROOF. There exists λ such that $||u_i||_{(N)} \leqq \lambda$, i. e. $\int_\Omega N(u_i/\lambda)dx \leqq 1$. This implies, by Fatou's lemma, that $\int_\Omega N(u/\lambda)dx \leqq$ $\leqq 1$, i. e. $u \in L_N(\Omega)$. Now, since u_i is bounded in $L_N = (E_{\overline{N}})^*$, to prove that $u_i \to u$ for $\sigma(L_N, E_{\overline{N}})$, it suffices to prove that $\langle u_i, v\rangle \to \langle u, v\rangle$ for v in some (norm) dense subset of $E_{\overline{N}}$. Thus, here, it suffices to prove that for any A measurable $\subset \Omega$,

$$\int_A u_i(x)dx \to \int_A u(x)dx . \tag{9}$$

We know that $\int_\Omega N(u_i/\lambda)dx \leq 1$. This implies, by De La Vallée - Poussin's theorem (cf. [18; p. 159]) that the functions u_i are equiabsolutely integrable on Ω . Since they converge almost everywhere, it follows from Vitali's theorem (cf. [6; p. 122]) that they converge in $L^1(\Omega)$, which implies (9). Q. E. D.

Lemma 1 yields the desire hemicontinuity result for T because it follows from inequality (2) of section 3.2, that

$$|\phi(t)| \leq \overline{M}^{-1}M(2t) \quad \text{for} \quad t \in \mathbb{R} ,$$

which is a condition of type (8).

3.5. <u>Comments</u>

a. A more general result can be proved by using essentially the same arguments.

THEOREM 1. *Let* Ω *be an open bounded set in* $\mathbb{R}^N$, *with the segment property. Let* $A_\alpha(x, \xi)$, $|\alpha| \leq m$, *be functions defined for* $x \in \Omega$ *and* $\xi = (\xi_\beta)_{|\beta|\leq m}$, $\xi_\beta \in \mathbb{R}$, *with the usual Carathéodory conditions. Suppose:*

(i) there exist an N-*function* M, $a(x) \in E_{\overline{M}}(\Omega)$, *constants* b, c *such that*

$$|A_\alpha(x, \xi)| \leq a(x) + b \sum_{|\beta|\leq m} \overline{M}^{-1}M(c\xi_\beta)$$

for all $|\alpha| \leq m$, x *and* ξ ,

(ii) for all x *and* ξ, ξ', *one has*

$$\sum_{|\alpha|\leq m}(A_\alpha(x, \xi) - A_\alpha(x, \xi'))(\xi_\alpha - \xi'_\alpha) \geq 0 ,$$

(iii) there exist $a_\alpha(x)$ *in* $E_{\overline{M}}(\Omega)$ *for* $|\alpha| = m$, *in* $L_{\overline{M}}(\Omega)$ *for* $|\alpha| < m$, $b(x) \in L^1(\Omega)$, *constants* d, $e > 0$ *such that*

$$\sum_{|\alpha|\leq m}(A_\alpha(x, \xi) - a_\alpha(x))\xi_\alpha \geq d \sum_{|\alpha|=m} M(e\xi_\alpha) - b(x)$$

for all x *and* ξ .

Then, for any given f *in* $W^{-m}E_{\overline{M}}(\Omega)$, *there exists* $u \in W_0^mL_M(\Omega)$ *such that* $A_\alpha(x, \xi(u)) \in L_{\overline{M}}(\Omega)$ *for all* $|\alpha| \leq m$ *and*

$$\sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha A_\alpha(x, \xi(u)) = f$$

in the distribution sense in Ω . *Here* $\xi(u)$ *stands for* $(D^{\beta}u)_{|\beta|\leq m}$.

Other existence theorems can be given along the lines of LERAY-LIONS [16], where the monotonicity assumptions involve only the top-order terms. Their proofs depend on more complicated abstract theorems than the one of section 3.3. Here is an example of such an existence theorem (for details see [9]).

THEOREM 2. *Let* Ω *be an open bounded set in* $\mathbb{R}^N$*, with the segment property. Let* $A_{\alpha}(x, \xi)$, $|\alpha| \leq m$*, be functions defined for* $x \in \Omega$ *and* $\xi = (\xi_{\beta})_{|\beta|\leq m}$, $\xi_{\beta} \in \mathbb{R}$ *, with the usual Carathéodory conditions. We will split* $\xi = (\xi_{\beta})_{|\beta|\leq m}$ *into its top order part* $\zeta = (\xi_{\beta})_{|\beta|=m}$ *and its lower order part* $\eta = (\xi_{\beta})_{|\beta|<m}$. *Suppose:*

(i) there exist two N-functions M *and* P , P *growing essentially slower than* M *(i. e.* $P(t)/M(\varepsilon t) \to 0$ *as* $t \to \infty$*, for any* $\varepsilon > 0$*), functions* $a_{\alpha}(x)$ *in* $E_{\overline{M}}$ *for* $|\alpha| = m$, *in* $L_{\overline{M}}$ *for* $|\alpha| < m$, *a constant* c *such that for all* x *and* ξ ,

if $|\alpha| = m$: $|A_{\alpha}(x,\xi)| \leq a_{\alpha}(x) + c \sum_{|\beta|=m} \overline{M}^{-1}M(c\xi_{\beta}) + c_{|\beta|<m}\overline{P}M(c\xi_{\beta})$,

if $|\alpha| < m$: $|A_{\alpha}(x,\xi)| \leq a_{\alpha}(x) + c_{|\beta|=m} \overline{M}^{-1}P(c\xi_{\beta}) + c_{|\beta|<m}\overline{M}^{-1}M(c\xi_{\beta})$,

(ii) for each x, η , $\zeta \neq \zeta'$,

$$\sum_{|\alpha|=m} (A_{\alpha}(x, \zeta, \eta) - A_{\alpha}(x, \zeta', \eta))(\zeta_{\alpha} - \zeta'_{\alpha}) > 0 ,$$

(iii) for each x, ζ''

$$\sum_{|\alpha|=m} (A_{\alpha}(x, \zeta, \eta) - \zeta'_{\alpha})(\zeta_{\alpha} - \zeta''_{\alpha}) \to +\infty$$

as $|\zeta| \to +\infty$*, uniformly for bounded* ζ' *and* η ,

(iv) there exist $b_{\alpha}(x) \in E_{\overline{M}}(\Omega)$ *for* $|\alpha| = m$, *in* $L_{\overline{M}}(\Omega)$ *for* $|\alpha| < m$, $b(x) \in L^1(\Omega)$, *constants* d, $e > 0$ *such that*

$$\sum_{|\alpha|\leq m} (A_{\alpha}(x, \xi) - b_{\alpha}(x))\xi_{\alpha} \geq d \sum_{|\alpha|=m} M(e\xi_{\alpha}) - b(x)$$

for all x *and* ξ .

Then, for any given f *in* $W^{-m}E_{\overline{M}}(\Omega)$, *there exists* $u \in W_0^m L_M(\Omega)$

such that $A_\alpha(x, \xi(u)) \in L_{\overline{M}}(\Omega)$ *for all* $|\alpha| \leq m$ *and*

$$\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi(u)) = f$$

in the distribution sense in Ω .

b. The arguments given in section 3.4 to verify the pseudo-monotonicity condition can be used to prove that the Nemyckii operator $u(x) \in L_M \mapsto \phi(u(x)) \in L_{\overline{M}}$ considered in section 3.2 is maximal monotone from L_M to $L_{\overline{M}}$. The question whether the monotone operator $T : D(T) \subset W_0^1 L_M \to W^{-1} L_{\overline{M}}$ of section 3.4 is also maximal monotone is of some interest. Here is a partial answer.

PROPOSITION. *Assume that* $\overline{M}$ *satisfies the* Δ_2 *condition. Then the above operator is maximal monotone.*

PROOF. Since $\overline{M}$ satisfies the Δ_2 condition, $E_{\overline{M}} = L_{\overline{M}}$, and so our complementary system takes a simpler form: $(W_0^1 L_M, W_0^1 E_M; W^{-1} L_{\overline{M}}, W^{-1} E_{\overline{M}}) = (X^{**}, X; X^*, X^*)$ with $X = W_0^1 E_M$. And $T : D(T) \subset X^{**} \to X^*$. Denote by T_1 the restriction of T to X . So

$$T_1 : X \to X^* : u \mapsto - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \phi(\frac{\partial u}{\partial x_j}) .$$

It is monotone, hemicontinuous, everywhere defined, and so maximal monotone from X to X^* . And T appears as a monotone extension of T_1 to the bidual X^{**} (for some results about such extensions, see [11]).

The operator T_1 is the (sub) gradient of the functional

$$\Phi(u) = \sum_{j=1}^{N} \int_\Omega M(\frac{\partial u}{\partial x_j}) \, dx$$

on X . Indeed, Φ is a convex continuous functional on X (cf. Lemma 1 of section 2.4), and so $\partial\Phi$ is a maximal monotone mapping. But, using the equality $M' = \phi$, one easily verifies that $T_1 \subset \partial\Phi$, and consequently, $T_1 = \partial\Phi$.

Now we apply a result from convex analysis (cf. ROCKAFELLAR [19]) which says that if Φ is a lower semicontinuous, convex, $\not\equiv +\infty$ function on a Banach space X , then $\partial\Phi$ has an unique maximal monotone extension T_2 to the bidual X^{**}; moreover the graph of $\partial\Phi$ is dense in the graph of T_2 in the sense that given any $(x^{**}, x^*) \in \in \text{gr}\, T_2$, there exists a net $(x_i, x_i^*) \in \text{gr}\, \partial\Phi$ such that x_i is bounded in X, $x_i \to x^{**}$ for $\sigma(X^{**}, X^*)$ and $x_i^* \to x^*$ in norm. In our situation, we have seen that T is pseudo-monotone; pseudo-monotonicity is a closedness condition which clearly implies that the graph of T is closed for the above convergence. Consequently T contains T_2, and since T is monotone and T_2 maximal monotone, $T = T_2$, which shows that T is maximal monotone. Q. E. D.

We remark that the proof of the above proposition, based on results from convex analysis, does not extend to the case of the operator

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi(u))$$

considered in theorem 1.

c. Some partial results for the Dirichlet problem for the equation

$$(1) \qquad - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} |\phi(\frac{\partial u}{\partial x_j})| = f$$

in the case where ϕ now is no longer odd at infinity have been obtained in [12].

References

[1] S. AGMON: Lectures on elliptic boundary value problems. Van Nostrand Math. Studies, no. 2, Van Nostrand, Princeton, J.J., 1965.

[2] F. E. BROWDER and P. HESS: Nonlinear mappings of monotone type in Banach spaces. J. Functional Analysis 11 (1972), 251-294.

[3] T. DONALDSON: Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces. J. Diff. Equat. 10 (1971), 507-528.

[4] T. DONALDSON: Inhomogenuous Orlicz-Sobolev spaces and nonlinear parabolic initial value problems. J. Diff. Eq. 16 (1974), 201-256.

[5] T. DONALDSON and S. TRUDINGER: Orlicz-Sobolev spaces and imbedding theorems. J. Functional Analysis 8 (1971), 52-75.

[6] N. DUNFORD and J. T. SCHWARTZ: Linear operators. I: General theory. Pure and Appl. Math., vol. 7, Interscience, New York, 1958.

[7] A. FOUGERES: a) Théorèmes de trace et de prolongement dans les espaces de Sobolev et de Sobolev-Orlicz. C. R. Ac. Sc. Paris 274 (1972), 181-184.

b) Operateurs elliptiques du calcul des variations à coefficients très fortement non linéaires. C. R. Ac. Sc. Paris 274 (1972), 763--766,

[8] J. P. GOSSEZ: Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. Trans. Amer. Math. Soc. 190 (1974), 163-205.

[9] J. P. GOSSEZ: Surjectivity results for pseudo-monotone mappings in complementary systems. J. Math. Anal. Appl. 53 (1976), 484--494.

[10] J. P. GOSSEZ: A remark on strongly nonlinear elliptic boundary value problems. Bol. Soc. Brasil. Mat., to appear.

[11] J. P. GOSSEZ: On the extensions to the bidual of a maximal monotone operator. Proc. Am. Math. Soc. 62 (1977), 67-71.

[12] J. P. GOSSEZ and P. HESS: Sur certains problèmes aux limites non linéaires. C. R. Acad. Sc. Paris 278 (1974), 343-345.

[13] M. KRASNOSEL'SKII and J. V. RUTICKII: Convex functions and Orlicz spaces. GITTL, Moscow 1958; English transl., Internat. Monographs on Advanced Math. Phys., Hindustan, Delhi; Noordhoff, Groningen, 1961.

[14] A. KUFNER, O. JOHN, S. FUČÍK: Function Spaces. Academia, Praha, 1977.

[15] T. LACROIX: Espaces de traces des espaces de Sobolev-Orlicz. J. Math. Pures Appl. 53 (1974), 439-458.

[16] J. LERAY and J. L. LIONS: Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. Bull. Soc. Math. France 93 (1965), 97-107.

[17] V. MUSTONEN: An elliptic boundary value problem for strongly nonlinear equations in unbounded domains. Proc. Royal Soc. Edinburgh 76 (1977), 217-230.

[18] I. NATANSQN: Theory of functions of a real variable. Ungar, New York, 1955.

[19] R. T. ROCKAFELLAR: On the maximal monotonicity of subdifferential mappings. Pacific J. Math. 33 (1970), 209-216.

[20] R. T. ROCKAFELLAR: Local boundedness of nonlinear monotone operators. Michigan Math. J. 16 (1969), 397-407.