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In: Jaroslav Kurzweil (ed.): Equadiff 7, Proceedings of the 7th Czechoslovak Conference on Differential Equations and Their Applications held in Prague, 1989. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1990. Teubner-Texte zur Mathematik, Bd. 118. pp. 88--91.

Persistent URL: http://dml.cz/dmlcz/702386

A NEW VIEW OF CENTER MANIFOLDS

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1 Introduction

We consider ode.'s which are written as a pair of coupled de.'s (= differential equations)

$$\dot{x} = g(t,x,y), \dot{y} = h(t,x,y), \dot{} = d/dt, \tag{1.1}$$

x, y being vectors in general. Let M be a set in the (t,x)-space which is bounded and open relative to this space and $(t,x) \to s(t,x)$ a mapping into the y-space. We are interested in a smooth manifold with these properties

(i) it has a representation $y = S(t,x), (t,x) \in \mathrm{M}$,

(ii) it is (locally) invariant with respect to the de. (1.1), (1.2)

(iii) $S(t,x) = s(t,x)$ if $(t,x) \in \partial\mathrm{M}$.

For shortness we call a manifold satisfying these three conditions a center manifold (abbr. *cmf.*). The novelty of our definition lies in the explicit occurence of an additional side condition namely the boundary condition (iii). It is obvious that one cannot resort anymore to our basic knowledge about *cmf's*. What one would expect is - roughly speaking - aggravated and relaxed conditions respectively for existence and uniqueness respectively. We are concerned in this paper with existence only and wish to formulate two fundamental hypotheses - concerning the de. (1.1), the set M and the boundary data - which are sufficient in order that one can actually construct a mapping S which has all properties (1.2). The second hypothesis is discussed in detail in Sec. 2, the first one runs as follows

$$\mathrm{M} = \{(t,x) : W(t,x) < 0\}, \dot{W}(t,x,s(t,x)) > 0 \quad \text{if } (t,x) \in \partial\mathrm{M}. \tag{1.3}$$

$\dot{W}(t,x,y)$ denotes the derivative of W with respect to (1.1), i.e.

$$\dot{W}(t,x,y) := W_t(t,x) + W_x(t,x)g(t,x,y). \tag{1.4}$$

(1.3) admits a simple geometric interpretation. Note that the set $\{t,x,y : (t,x) \in \partial\mathrm{M}, y = s(t,x)\}$ is the boundary of any manifold having the properties (i), (iii), cf. (1.2). (1.3) means that this boundary is crossed by trajectories of (1.1) in such a way that (t,x) changes from the interior of M to the exterior of t increases. Therefore if the manifold is locally invariant - property (ii) - then it is "outflowing". As time decreases one can never reach the boundary by following a trajectory starting in the interior. This kind of internal stability (as $t \to -\infty$) is the main motivation for adding (iii) to the standard definition of a *cmf.* One of its implications will be discussed in Sec.4.

2 Dichotomy along solutions.

Dichotomy in this spaper is seen in connection with the natural boundary value problem (abbr.: bvp.) for the de. (1.1): Given $(t_1, x_1), (t_2, y_2)$ find a solution $x(t), y(t)$ satisfying $x(t_1) = x_1, y(t_2) = y_2$. There is a local version of this problem where x_1, y_2 are restricted to neighborhoods of the respective boundary values $\bar{x}(t_1), \bar{y}(t_2)$ of some reference solution $\bar{x}(t), \bar{y}(t)$. A possible approach to the modified bvp. follows then the standard pattern and this means first of all: Linearization. So let us consider the variational equation (abbr.: ve.) along the reference solution. As it is suggested by the representation (1.1) of the given de. we will write it as a coupled pair of linear de.'s:

$$\dot{w} = A(t)w + A_1(t)z, \quad \dot{z} = B_1(t)w + B(t)z, \tag{2.1}$$

where $A(t) = g_x$, $B(t) = h_y$, $A_1(t) = g_y$, $B_1(t) = h_x$ and the argument behind g_x etc. is $t, \bar{x}(t), \bar{y}(t)$. The corresponding bvp. runs as follows: Find a solution $w(t), z(t)$ of (2.1) satisfying $w(t_1) = w_1$ and $z(t_2) = z_2$. It is solvable for arbitrary w_1, z_2 iff this statement holds true for any solution $(w(t), z(t))$ of (2.1):

$$w(t_1) = 0 \text{ and } z(t_2) = 0 \Rightarrow w(t) = 0 \text{ and } z(t) = 0 \text{ for all } t. \tag{2.2}$$

This property is of particular interest for us since it prepares the stage for applying the implicit function theorem. Hence if (2.2) holds true then the localized bvp. for the original de. admits a solution which depends smoothly upon the data x_1, y_2. A possible way to ensure correctness of (2.2) is by means of estimates for $\| w(t) \|, \| z(t) \|$, the bounds being linear functions of $\| w(t_1) \|$ and $\| z(t_2) \|$. Such an estimate can easily be obtained from two basic inequalities which are stated in the next lemma.

Lemma 2.1 Let $\phi_w(t,\tau)$ and $\phi_z(t,\tau)$ respectively be the principal matrix solutions (abbr.: pms.) of the linear homogeneous de. $\dot{w} = A(t)w$ and $\dot{z} = B(t)z$ respectively. Let $\alpha(t), \beta(t), \rho(t)$ be functions on some time interval I which satisfy these conditions for all $t, \tau \in \mathrm{I}$ and with some constants γ_w, γ_z :

$$\|\phi_w(t,\tau)\| \le \gamma_w e^{\int_\tau^t \alpha(s)ds} \text{ if } \quad \tau \le t, \ \|\phi_z(t,\tau\| \le \gamma_z e^{\int_\tau^t \beta(s)ds} \text{ if } \quad t \le \tau,$$
$$\alpha(t) < \rho(t) < \beta(t). \tag{2.3}$$

Furthermore assume that

$$\kappa := \gamma_w \gamma_z d_w d_z < 1 \tag{2.4}$$

where

$$d_w := \sup_{\mathrm{I}} \frac{\|A_1(t)\|}{\rho(t) - \alpha(t)}, d_z := \sup_{\mathrm{I}} \frac{\|B_1(t)\|}{\beta(t) - \rho(t)}. \tag{2.5}$$

Conclusion. If conditions (2.3) - (2.5) are satisfied and if $w(t), z(t)$ is a solution of (2.1) then these statements hold true:

$$w(t_1) = 0 \Rightarrow \|w(t)\| \le (1-\kappa)^{-1} \gamma_z \gamma_w d_w \|z(t)\| \quad \text{if} \quad t \in \mathrm{I}, t \ge t_1, \tag{2.6}$$

$$z(t_2) = 0 \Rightarrow \|z(t)\| \le (1-\kappa)^{-1} \gamma_z \gamma_w d_z \|w(t)\| \quad \text{if} \quad t \in \mathrm{I}, t \le t_2. \tag{2.7}$$

Remarks. 1.) Note that (2.2) follows from (2.6) or from (2.7) in case $t_1 \le t_2$. This proviso cannot be removed. Hence the lemma introduces a time preference into the discussion of bvp's and does not allow to interchange the role of initial and terminal constraint. This lack of symmetry becomes apparent in the hypothesis (2.3). 2.) The proof of the lemma can be carried out in a straightforward way if one borrows the method from [1], Ch. V, Sec. 4 where it is assumed that α, β, ρ are constants. One first considers the case that ρ vanishes identically. Then one simply has to redo the proof of Hilfssatz 4.1 ([1],p.221). Next one turns to the situations treated in [1] on p. 222-224. They all can be reduced to the case discussed in the first lemma by means of the substitution

$$w = \tilde{w}\bar{\rho}(t) + \phi_w(t,t_1)w(t_1), z = \tilde{z}\bar{\rho}(t) + \phi_z(t,t_2)z(t_2) \tag{2.8}$$

where $\bar{\rho}(t) := exp(\int_{t_1}^t \rho(s)ds)$. The pair $(\tilde{w}(t), \tilde{z}(t))$ then satisfies an inhomogeneous linear de. and the boundary conditions $\tilde{w}(t_1) = 0, \tilde{z}(t_2) = 0$.

3 Generating a manifold out of trajectories.

We return to the discussion of the de. (1.1) and use the notation introduced in Sec.1. Our aim is to show that certain sets of "outflowing" trajectories constitute a cmf. in the sense of definition (1.2). To this purpose we introduce the sets

$$\mathrm{M}_\lambda := \{(t,x) \in \mathrm{M} : \lambda < t < T\} \tag{3.1}$$

(cf. (1.3)), where $\lambda \geq -\infty$ and $T > \lambda$ is a fixed real number. The sets

$$\{(t,x) : \lambda < t < T, W(t,x) = 0\} \quad \text{and} \quad \{(T,x) : W(T,x) < 0\} \tag{3.2}$$

belong to $\partial \mathrm{M}_\lambda$. Except for a sufficiently small neighborhood of their intersection their union can be identified with the set

$$\hat{\mathrm{X}}_0 := \{(t,x) : t > \lambda,\ \hat{V}(t,x) = 0\}. \tag{3.3}$$

Here $\hat{V}(t,x)$ is a sufficiently smooth function satisfying

$$\hat{V}_x(T,x) = 0 \quad \text{if} \quad \hat{V}(T,x) = 0; \dot{\hat{V}}(t,x,s(t,x)) > 0 \quad \text{if} \quad \hat{V}(t,x) = 0 \tag{3.4}$$

(for the definition of $\dot{\hat{V}}$ etc. cf. (1.4)). How to construct such a function (given $W(t,x)$) is demonstrated in [2], Sec.3. For any $(t_0,x_0) \in \hat{\mathrm{X}}_0$ let $x_0(t), y_0(t)$ be the solution of (1.1) satisfying $x_0(t_0) = x_0, y_0(t_0) = s(t_0,x_0), \mathrm{J}_0$ the maximal interval $(\ldots,t_0)$ such that $(t,x_0(t)) \in \mathrm{M}_\lambda$ for $t \in \mathrm{J}_0$ and $\mathrm{k}(t_0,x_0)$ the trace of the curve $\{(t,x_0(t),y_0(t)), t \in \mathrm{J}_0$. Furthermore if the hypotheses of Lemma 2.1 - with $\mathrm{I} = \mathrm{J}_0$ and the ve. along $(x_0(t),y_0(t))$ as underlying linear de. (2.1) - hold we put $\Gamma_\lambda(t_0,x_0) := (1-\kappa)^{-1}\gamma_z\gamma_w d_w$ (cf.(2.6)), otherwise we put $\Gamma_\lambda(t_0,x_0) = \infty$. Finally we denote by S_λ the union of the $\mathrm{k}(t_0,x_0)$:

$$\mathrm{S}_\lambda := \{t,x,y : \exists (t_0,x_0) \in \hat{\mathrm{X}}_0 \quad \text{such that} \quad (t,x,y) \in \mathrm{k}(t_0,x_0)\}. \tag{3.5}$$

S_λ is locally invariant and outflowing.

Theorem 3.1 Hypotheses. (i) (2.4) holds, (ii) we have for $(t,x) \in \hat{\mathrm{X}}_0$

$$\Gamma_\lambda(t,x) \cdot \|\frac{1}{\dot{\hat{V}}(t,x,s(t,x))}[h(t,x,s(t,x)) - \dot{s}(t,x,s(t,x))]\hat{V}_x(t,x) + s_x(t,x)\| < 1.$$

Claim. There exists a mapping $(t,x) \to S(t,x)$ into the y-space which is as smooth as the right hand side of the de. (1.1) such that $\mathrm{S}_\lambda = \{t,x,y : (t,x) \in \mathrm{M}_\lambda, y = S(t,x)\}$.

Corollary. Under the hypotheses of the theorem the following statement holds true for solutions $(x(t),y(t))$ of (1.1) :

$$W(T,x(T)) < 0, y(T) = s(T,x(T)) \Rightarrow W(t,x(t)) < 0 \text{ for } \lambda < t \leq T.$$

The statement of the theorem is somewhat more general than that of Theorem 4.1 in [2], the proof is pracitally the same. Explicit dependence upon t (of $g, h, W, s, \alpha, \beta, \rho$) presents no extra problems. Note that the quantity χG appearing in [2] is an upper bound for $\Gamma_\lambda(t,x), (t,x) \in \partial\mathrm{M}$.

4 Application: Singular Perturbations.

If the right hand sides of (1.1) have the form

$$g(t,x,y) = \epsilon p(t,x,y,\epsilon),\ h(t,x,y) = B(t,x,\epsilon)y + \epsilon q(t,x,y,\epsilon), \epsilon > 0, \tag{4.1}$$

one can verify both hypotheses of Theorem 3.1 quite easily if: (i) The eigenvalues of the matrix B have real parts which are positive and bounded away from zero. (ii) $\epsilon \leq \epsilon_0$, where ϵ_0 can be determined explicitly from conditions stated in Lemma 2.1. (iii) One chooses $s(t,x) = 0$. The conclusion of the theorem and its implications (see [2], Sec.5) comprise then our knowledge concerning the role of imf.-techniques in singular perturbation problems (cf. e.g. [3]). It contains also various new informations (concrete estimates for ϵ_0 and extensions to situations where the "bounded away from zero" - requirement is weakened).

5 Application: Stability, Zero Dynamics .

If $y = 0$ is a "basic" imf. for (1.1), i.e. if $h(t,x,0) = 0$ identically in (t,x) and if one chooses $s(t,x)$ such that $W(t,x) = 0 \Rightarrow s(t,x) = 0$ the hypothesis (ii) of Theorem 3.1 need to be verified only for $t = T$, i.e. under the proviso $t = T, \hat{V}_x = 0$ (cf. (3.4)). Note that the cmf. $y = S(t,x)$ then bifurcates from the basic imf along ∂M. We give an example of stability criteria exploiting this fact and assuming $t \to \infty$: x, y scalar, $h(t,x,y) = b(t,x,y)y$ with $b(t,x,y) \leq 0, x = 0$ is a uniformly asymptotically stable equilibrium of the "zero-dynamics"- equation $\dot{x} = g(t,x,0).(x,y) = (0,0)$ is then a stable equilibrium of (1.1) if h, g and their partial derivatives with respect to x, y are bounded uniformly in t and if $b(t,x,0) - g_x(t,x,0) < 0$ on every set $\{t,x : t_1 \leq t \leq t_2,\ \xi_0 \leq x \leq \xi_1\}$, the bounds depending on $\xi_0 > 0, \xi_1, t_2 - t_1$ only.

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