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PERTURBED DISCRETE TIME DYNAMICAL SYSTEMS

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Introduction. Our purpose is to study randomly perturbed discrete time dynamical systems from the statistical point of view. Thus we consider the behaviour of sequences of distributions corresponding to a given system and we show some sufficient conditions for the existence of a stationary distribution and its asymptotical stability.

These results are applied to integral recurrences. We show the existence, uniqueness and stability of solutions of some Volterra equations with advanced argument and correspnding differential equations. Such equation appear in mathematical models of the cell cycle.

1. General system. We consider a discrete time dynamical system of the form

$$x_{n+1} = S(x_n, \xi_n), \qquad \text{for } n = 0,1,\dots \tag{1}$$

where S is a given transformation defined on a subset $A \subset V$ of $R^d \times R^k$ with values in the set A. We shall always assume that:

(i) For every fixed y the function S(x,y) is continuous in x and for every fixed x is Borel measurable in y. The set $A \subset R^d$ is closed and $V \subset R^k$ is Borel measurable.

(ii) The random vectors $\xi_0, \xi_1, \dots$ are independent with the same distribution

$$G(B) = \text{Prob}(\xi_n \in B) \qquad \text{for } B \subset V, \quad B \text{ Borelian.}$$

(iii) The initial random vector x_0 is independent of the perturbation sequence ξ_n.

The asymptotic behaviour of a solution (x_n) of system (1) is described by the sequence of distributions

$$F_n(B) = \text{Prob}(x_n \in B) \qquad \text{for } B \subset A, \quad B \text{ Borelian.}$$

It is easy to find a recurrence relation between F_{n+1} and F_n. Namely setting

$$F_n(h) = \int_A h(x)\, F_n(dx)$$

we have

(2) $$F_{n+1}(h) = \int_A \left[\int_V h(S(x,y))\, G(dy) \right] F_n(dx) \qquad \text{for } h \in C_0(A),$$

where $C_0(A)$ is the space of real valued continuous functions on A with compact support.

We write relation (2) shortly as $F_{n+1} = PF_n$ and we call P the transition operator for (1). The operator P is defined on the space of distributions (probabilistic Borel measures on A). In some cases P has the property that for any absolutely continuous distribution F its image PF is also absolutely continuous.In this case we shall also write $f_{n+1} = Pf_n$ where $f_n = dF_n/dx$.

2. <u>Asymptotic stability.</u> Recall that a sequence of distributions F_n converges weakly to a distribution F_* if $F_n(h) \to F_*(h)$ for every $h \in C_0(A)$. A sequence F_n converges strongly to F_* if $\|F_n - F_*\| \to 0$ where $\|.\|$ denotes the total variation.

We say that sytem (1) is weakly (strongly) asymptotically stable if there is a unique distribution F_* (called stationary) such that $F_* = PF_*$ and if P^nF converges weakly (strongly) to F_* for every initial distribution F.

The following theorem gives a sufficient condition for the weak asymptotic stability. A condition for strong stability can be found in [4].

<u>Theorem 1.</u> Assume that

(3) $$E(|S(x,\xi_n) - S(z,\xi_n)|) < |x - z| \quad \text{for } x \neq z, \quad \text{and}$$

(4) $$E(|S(x,\xi_n)|^p) \leq a|x|^p + b, \quad a < 1, \quad p > 1,$$

where a, b, p are nonegative constants and E denotes the mathematical expectation. Then the system (1) is weakly asymptotically stable.

The proof given in [6] is based on Chebyshev type inequalities. In this proof the assumption $p > 1$ is essential.It is not known if Theorem 1 is valid for $p = 1$.

3. <u>Additive case.</u> Consider now a special case when $S(x,y) = T(x) + y$ or

(5) $$x_{n+1} = T(x_n) + \xi_n \qquad \text{for } n = 0,1,\ldots$$

We assume that T maps a closed set $A \subset R^d$ into itself, $\xi_n \in V \subset R^d$ with probability one and $A + V \subset A$. In this case we may formulate a sufficient condition for strong asymptotic stability.

<u>Theorem 2.</u> Assume that

(6) $$|T(x) - T(z)| < |x - z| \qquad \text{for } x \neq z, \qquad \text{and}$$

(7) $$T(x) \overline{<} a|x| + b, \quad a < 1, \quad E(|\xi_n|) < \infty,$$

where a, b are nonegative constants. Assume, moreover, that the common distribution G of ξ_n is absolutely continuous. Then (5) is strongly asymptotically stable and the stationary distribution F_* is absolutely continuous.

The proof will be given in [7]. It is based on the Komorník decomposition theorem for Markov operators [3]. The assumption that G is absolutely continuous may be released. It is sufficient to assume that in the Lebesgue decomposition of G the absolutely continuous part is not trivial and that T is not singular, i.e. mes A = 0 implies $\mathrm{mes}(T^{-1}(A)) = 0$.

4. Multiplicative case. Consider another special case of system (1) when $S(x,y) = y\,T(x)$ or

(8) $$x_{n+1} = \xi_n T(x_n) \qquad \text{for } n = 0,1,\ldots$$

Assume that $A = V = R_+$ and that $T: R_+ \to R_+$ is continuous.

Using the M. Podhorodyński version [8] of the lower bound function theorem K. Horbacz proved the following result.

Theorem 3. If $\mathrm{Prob}(\xi_n = 0) > 0$ (or $G(\{0\}) > 0$) then system (8) is strongly asymptotically stable.

The assumption $G(\{0\}) > 0$ is quite restrictive. It may be released. Using the Komorník decomposition K. Horbacz proved also the following theorem.

Theorem 4. Assume that $T(x) > 0$ for $x \in R_+$ and that G is absolutely continuous Assume moreover that

(9) $$dG(x)/dx > 0 \qquad \text{for } x \text{ sufficiently large, and}$$

(10) $$T(x) \leq ax + b, \qquad aE(\xi_n) < 1,$$

where a, b are nonegative constants. Then system (8) is strongly asymptotically stable.

5. Applications. Mathematical models of the cell cycle lead to integral Volterra equations with advanced argument. In particular, in [5] the recurrence $f_{n+1} = Pf_n$ was considered with P given by the formula

(11) $$Pf(x) = 4x\,e^{-x^2} \int_0^{2x} e^{y^2/2} f(y)\,dy \qquad \text{for } x \geq 0.$$

In this recurrence f_n denotes the density of the cell size distribution in the n-th generation of cells. It is easy to verify that (11) is the transition operator for the system

(12) $$x_{n+1} = \frac{1}{2}\sqrt{x_n^2 + 2\xi_n}, \qquad dG/dx = e^{-x} \quad \text{for } x \geqslant 0$$

which satisfies the assumptions of Theorem 1. Moreover for $y_n = x_n^2$ the assumptions of Theorem 2 are satisfied. This shows that (12) is strongly asymptotically stable and the stationary distribution F_* is absolutely continuous. The density $f_* = dF_*/dx$ is, therefore, the unique nonnegative and normalized (in L^1) solution of the integral equation $f_* = Pf_*$ and the corresponding differential equation. This result was proved in [6] by a different method.

J.J.Tyson and K.B.Hannsgen [10] cosidered the recurrence $f_{n+1} = Pf_n$ with the operator

(13) $$Pf(x) = \int_0^{x/c} K(x,z)\, f(z)\, dz \quad \text{where} \quad K(x,z) = \begin{cases} (a/c)(x/c)^{-1-a}, & 0 \leqslant z \leqslant 1,\ x \geqslant c, \\ (a/c)(x/c)^{-1-a} z^a, & 1 < z \leqslant x/c, \\ 0, & x < c. \end{cases}$$

J.Tyrcha observed [9] that (13) is the transition operator for the system

(14) $$x_{n+1} = \max(x_n, 1)\,\xi_n, \qquad dG/dx = (a/c)(x/c)^{-1-a}.$$

From Theorem 4 it follows immediately that (14) is strongly asymptotically stable if the inequality $1/a < 1-c$ $(a > 0,\ c > 0)$ is satisfied. This fact was first observed in [10].

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