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# THE GLOBAL SOLVABILITY TO THE EQUATIONS OF MOTION OF VISCOUS GAS WITH AN ARTIFICIAL VISCOSITY

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It is known that weak solutions to the Navier-Stokes equations for incompressible liquid exist globally in time (due to the work of E. Hopf from the year 1951). In spite of a lot of attempts, it remains an open question wheather the same may be said about the Navier-Stokes equations for compressible fluid. Especially the variability of density is a source of troubles: eventual "very small" as well as "very large" values of density together with insufficient informations about its regularity make impossible to follow the approach of E. Hopf. Failures lead often as far as to doubts about the system of the Navier-Stokes equations as about a convenient model of motion of a viscous compressible fluid.

We shall deal with the following system of equations:

(1) $(\varrho u_i)_{,t} + (\varrho u_j u_i)_{,j} = -p(\varrho)_{,i} + \mu[u_{i,jj} + \frac{1}{3} u_{j,ji}] + [f(\varrho)_{,j} u_i]_{,j}$
$(i = 1, 2, 3)$

(2) $\varrho_{,t} + (\varrho u_j)_{,j} = f(\varrho)_{,jj}$

in $Q_T = \Omega \times (0,T)$. We consider boundary conditions

(3) $u_i|_{\partial\Omega\times(0,T)} \equiv 0 \quad (i = 1,2,3), \qquad \frac{\partial}{\partial \nu} f(\varrho)|_{\partial\Omega\times(0,T)} \equiv 0$

(where $\nu$ is an outer normal vector) and initial conditions

(4) $\varrho|_{t=0} = \varrho_0, \qquad (\varrho u_i)|_{t=0} = \varrho_0 u_{0i} \quad (i = 1,2,3).$

The term on the right hand side of (2) represents a so called *artificial viscosity*. We assume that there exist $\delta > 0$, $\varrho_1 > 0$, $\varrho_2 > \varrho_1$, $\varrho_3 > \varrho_2$, $C_1 > 0, \ldots$ $\ldots, C_6 > 0$ so that $p$ and $f$ are twice continuously differentiable and non-decreasing functions on $\langle 0, +\infty)$ such that $p(\varrho) \le C_1 \varrho$ for $\varrho \in \langle 0, \varrho_1 \rangle$, $f \equiv 0$ on $\langle 0, \varrho_2 \rangle$ and $\varrho\, p''(\varrho) \le C_2\, p'(\varrho)$, $p(\varrho) \le C_3\, f(\varrho)^{1-\delta}$, $f(\varrho) \ge C_4 \varrho^2$, $\varrho\, f'(\varrho) \le$ $\le C_5\, f(\varrho)$, $f(\varrho) \le C_6\, \varrho^5$ for $\varrho \ge \varrho_3$. These conditions are satisfied for example if $p(\varrho) = const.\, \varrho^\alpha$ (where $1 \le \alpha < 3$) for $\varrho \ge 0$ and $f(\varrho) = const.\,(\varrho - \varrho_2)^3$ for $\varrho \ge \varrho_2$. $\Omega$ is supposed to be a bounded domain in $\mathbf{R}^3$ with a sufficiently smooth boundary. $T$ is a given positive number. For each $t \in (0,T)$, the artificial viscosity is acting only on a set $\Omega'_t = \{x \in \Omega;\ \varrho(x,t) > \varrho_2\}$. Since the ammount of mass in $\Omega$ is constant ($=M$), we can derive: $meas(\Omega_t') \le \le M/\varrho_2$. Thus the artificial viscosity can act at most on a set as small as we want if we choose $\varrho_2$ large enough. The system (1),(2) coincides with the system of the Navier-Stokes equations for barotropic fluid and the continuity equation in the range of "reasonable values" of density (i.e. the values $\le \varrho_2$).

The problem (1)-(4) enables to derive various apriori estimates. The most important one (which may be called an *energy inequality* in accordance with the "incompressible case") has the form

$$(5)\quad \int_\Omega \tfrac{1}{2}\varrho u_i u_i\big|_\tau \,dx + \int_\Omega Q(\varrho)\big|_\tau \,dx + \int_0^\tau\int_\Omega \mu[u_{i,j}u_{i,j} + \tfrac{1}{3}(u_{j,j})^2]\,dx\,dt + \int_0^\tau\int_\Omega g(\varrho)_{,j}\,g(\varrho)_{,j}\,dx\,dt \le \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i}u_{0i}\,dx + \int_\Omega Q(\varrho_0)\,dx + const.$$

(for a.a. $\tau \in (0,T)$), where $Q(\varrho) = max\{P(\varrho);0\}$, $P(\varrho) = \varrho\int_1^\varrho p(\sigma)/\sigma^2\,d\sigma$ (for $\varrho > 0$), $P(0) = 0$, $g(\varrho) = \int_0^\varrho [p'(\sigma)\cdot f'(\sigma)/\sigma]^{1/2}\,d\sigma$. We can obtain the inequality (5) if we multiply the $i$-th equation in (1) by $u_i$, integrate by parts and use the equation (2) and the relation $\varrho P'(\varrho) - P(\varrho) = -p(\varrho)$ in an appropriate way. Moreover, by means of a similar technics, imbedding theorems, Gronwall inequality, etc., it can be shown, that

$$(6)\ \|\varrho\|_{\mathcal{J}_1} \le const., \qquad (7)\ \|\varrho U\|_{\mathcal{J}_2} \le const., \qquad (8)\ \|f(\varrho)\|_{\mathcal{J}_3} \le const.,$$

where $\mathcal{J}_1 = \mathcal{J}(0,T;+\infty,2;L_G(\Omega),H^{-1}(\Omega))$, $\mathcal{J}_2 = \mathcal{J}(0,T;2,1;L^2(\Omega)^3,H^{-3}(\Omega)^3)$, $\mathcal{J}_3 = \mathcal{J}(0,T;2,1;H^1(\Omega),H^{-3}(\Omega))$, $U \equiv (u_1,u_2,u_3)$ and the constants on the right hand sides of (6)-(8) depend only on $T$, $\Omega$, $\varrho_0$ and $U_0 \equiv (u_{01},u_{02},u_{03})$. If $X_0$ and $X_1$ are Banach spaces then $\mathcal{J}(0,T;\alpha_0,\alpha_1;X_0,X_1)$ denotes the space of functions $v \in L_0$ ($\equiv L^{\alpha_0}(0,T;X_0)$) such that $v' \in L_1$ ($\equiv L^{\alpha_1}(0,T;X_1)$), with the norm equal to $\|v\|_{L_0} + \|v'\|_{L_1}$. $L_G(\Omega)$ is the Orlitz space corresponding to the Young function $G(\varrho) = max\{Q(\varrho); \int_0^\varrho f(\sigma)\,d\sigma\}$.

We want to prove the global existence of a weak solution to the problem (1)-(4). It will be a couple of functions $\varrho \in L^\infty(0,T;L_G(\Omega))$, $U \equiv (u_1,u_2,u_3) \in L^2(0,T;H_0^1(\Omega)^3)$ such that $\varrho \ge 0$, $f(\varrho) \in L^2(0,T;H^1(\Omega))$,

$$(9)\quad \int_0^T\int_\Omega [\varrho u_i\varphi_{i,t} + \varrho u_j u_i\varphi_{i,j} + f(\varrho)u_{i,j}\varphi_{i,j} + f(\varrho)u_i\varphi_{i,jj} + p(\varrho)\varphi_{i,i} - \mu u_{i,j}\varphi_{i,j} - \tfrac{1}{3}\mu u_{j,j}\varphi_{i,i}]\,dx\,dt = -\int_\Omega \varrho_0 u_{0i}\cdot(\varphi_i\big|_{t=0})\,dx$$

for all $\varphi \equiv (\varphi_1,\varphi_2,\varphi_3) \in C^\infty(\overline{Q}_T)^3$ such that $\varphi_i\big|_{\partial\Omega\times(0,T)} \equiv 0$, $\varphi_i\big|_{t=0} \equiv 0$ $(i=1,2,3)$ and

$$(10)\quad \int_0^T\int_\Omega [\varrho\psi_{,t} + \varrho u_j\psi_{,j} - f(\varrho)_{,j}\psi_{,j}]\,dx\,dt = -\int_\Omega \varrho_0\cdot(\psi\big|_{t=0})\,dx$$

for all $\psi \in C^\infty(\overline{Q}_T)$ such that $\psi\big|_{t=0} \equiv 0$.

It is necessary to construct a sequence of approximations at first. That is why we formulate a problem which approximates (1)-(4) in a convenient way:

(11) $(\varrho u_i)_{,t} + (\varrho \tilde{u}_j u_i)_{,j} = -\tilde{p}(\varrho)_{,i} + \mu[u_{i,jj} + \frac{1}{3} u_{j,ji}] + [f(\varrho)_{,j} u_i + \frac{1}{n} \varrho_{,j} u_i]$
$(i = 1,2,3;$ in $Q_T)$,

(12) $\varrho_{,t} + (\varrho \tilde{u}_j)_{,j} = f(\varrho)_{,jj} + \frac{1}{n} \varrho_{,jj}$ (in $Q_{T,n}$),

(13) $u_i|_{\partial\Omega \times (0,T)} \equiv 0 \quad (i = 1,2,3), \qquad \frac{\partial}{\partial \nu}[f(\varrho) + \frac{1}{n}\varrho]|_{\partial\Omega_n \times (0,T)} \equiv 0,$

(14) $\varrho|_{t=0} = \varrho_0$ (in $\Omega_n$), $\qquad (\varrho u_i)|_{t=0} = \varrho_0 u_{0i}$ $(i = 1,2,3;$ in $\Omega)$,

where $\Omega_n = \{x \in \mathbf{R}^3;\ dist(x,\Omega) < \frac{1}{n},\ Q_{T,n} = \Omega_n \times (0,T)$ and $\sim$ denotes the regularization defined in a following way:

$$\tilde{\zeta}(x,t) = \int_{\Omega_n} \omega(n(x-y))\ \zeta(y,t)\ dy$$

(for $\zeta$ defined a.e. in $Q_{T,n}$ and such that $\zeta(.,t) \in L^1(\Omega_n)$ for a.a. $t \in (0,T)$; $\omega$ is a fixed function from $C^\infty(\mathbf{R}^3)$ such that $supp\ \omega = \{x \in \mathbf{R}^3;\ |x| \le 1\}$, tne integral of $\omega$ over $\mathbf{R}^3$ is equal to 1 and $\omega(x) \ge 0$ for all $x \in \mathbf{R}^3$.) It may be shown that the problem (11)-(14) has a weak solution (which will be denoted by $U^n \equiv (u_1^n, u_2^n, u_3^n)$, $\varrho^n$ in the following). It is rather complicated from the technical point of view, the used apparatus involves the method of a discretization in time, theory of nonlinear elliptic and parabolic equations, fixed point theorems, etc. The details may be found in [2]. The same estimates as (5)-(8) may be derived for the aoproximations $U^n$, $\varrho^n$. Thus, we shall refer to (5)-(8) as to estimates related to these approximations.

It follows from (5)-(8) and imbeddings $\mathcal{J}_1 \subset\subset L^2(0,T;H^{-1}(\Omega))$, $\mathcal{J}_2 \subset\subset L^2(0,T;H^{-1}(\Omega)^3)$, $\mathcal{J}_3 \subset\subset L^2(Q_T)$ that there exist subsequnces of $\{U^n\}$, $\{\varrho^n\}$ (denoted by $\{U^n\}$, $\{\varrho^n\}$ again) and functions $U, \varrho, V, \hat{f}$ so that $U^n \rightharpoonup U$ in $L^2(0,T;H_0^1(\Omega)^3)$, $\varrho^n \overset{*}{\rightharpoonup} \varrho$ in $\mathcal{J}_1$, $\varrho^n \to \varrho$ in $L^2(0,T;H^{-1}(\Omega))$, $\varrho^n U^n \to V$ in $L^2(0,T;H^{-1}(\Omega)^3)$, $f(\varrho^n) \to \hat{f}$ in $L^2(Q_T)$. It is possible to show that $V = \varrho U$, $\varrho \ge 0$ and $\hat{f} = f(\varrho)$. Moreover, we have: $\tilde{U}^n \rightharpoonup U$ in $L^2(0,T;H_0^1(\Omega)^3)$.

The functions $U^n$, $\varrho^n$ satisfy the integral relations analogous to (9), (10). The convergences mentioned above enable to pass to limits (containing $U, \varrho$ instead of $U^n, \varrho^n$, integrals over $\Omega$ instead of over $\Omega_n$ and having the regularization $\sim$ vanished) in all terms in these integral relations except the term

(15) $$\int_0^T \int_{\Omega_n} \tilde{p}(\varrho^n)\ \varphi_{i,i}\ dx\ dt\,.$$

It is possible to pass to the limit also in this term if the condition

(16) $p(\varrho) \doteq C_7 \cdot \varrho$ (for an **appropriate** constant $C_7 > 0$ and $\varrho \in \langle 0, \varrho_2 \rangle$)

is satisfied. Thus, we can see that $U$, $\varrho$ represent the weak solution to the problem (1)-(4) and we can state:

**Theorem.** *Let the condition* (16) *be fulfilled and let* $\varrho_2 \in L_Q(\Omega)$, $\varrho_2 \geq 0$ *a.e. in* $\Omega$, $U_0 \equiv (u_{01}, u_{02}, u_{03}) \in L^2(\Omega)^3$. *Then there exists a weak solution* $U \equiv (u_1, u_2, u_3)$, $\varrho$ *to the problem* (1)-(4).

It is possible to show that the weak solution $U$, $\varrho$ from the last theorem satisfies the energy inequality (5).

Without the condition (16), we are able to prove the solvability of the problem (1)-(4) in the following sense: There exist $U \equiv (u_1, u_2, u_3) \in L^2(0,T; H_0^1(\Omega)^3)$ and a measurable function $[x,t] \to \nu_{[x,t]}$ which assigns to a.e. $[x,t] \in Q_T$ a nonnegative Radon measure $\nu_{[x,t]}$ on $\langle 0, +\infty)$ so that this measure has a so called "unit mass" and if we put

$$\varrho(x,t) = \int_0^{+\infty} \lambda \, \nu_{[x,t]}(d\lambda), \qquad \Pi(x,t) = \int_0^{+\infty} p(\lambda) \, \nu_{[x,t]}(d\lambda)$$

then $\varrho \in L^\infty(0,T; L_G(\Omega))$, $f(\varrho) \in L^2(0,T; H^1(\Omega))$ and the relations (9), (10) are satisfied (with $\Pi$ instead of $p(\varrho)$ in (9)). We can obtain this result if we use the ideas of L. Tartar [3] and Di Perna [1].

The mentioned results may be generalized for the case of a nonzero body force and nonhomogeneous boundary conditions.

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