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SOLVABILITY AND BIFURCATIONS OF SOME STRONGLY NONLINEAR EQUATIONS

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1. Solvability of strongly nonlinear equations

Let us consider the equation

$$\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) + \lambda_1 |u(x)|^{p-2}u(x) + f(x,u(x)) = g(x), \quad x \in \Omega, \tag{1.1}$$

with the boundary condition

$$|\nabla u|^{p-2}\nabla u \,.\, \vec{n} = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

or

$$u = 0 \quad \text{on } \partial\Omega \;, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\vec{n}$ is the outer normal, ∇u = grad u, $p > 1$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and λ_1 is the smallest eigenvalue of the problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0 \tag{1.4}$$

with the boundary conditions (1.2) or (1.3), respectively. Denoting by φ the eigenfunction corresponding to λ_1, we can suppose $\varphi > 0$ in Ω (see e. g. [1]).

Put

$$f_{-\infty}(x) = \liminf_{s \to -\infty} f(x,s) \qquad \text{and} \qquad f^{+\infty}(x) = \limsup_{s \to +\infty} f(x,s).$$

Let us denote $p^* = pn(n-p)^{-1}$ for $p < n$, and $p^* = +\infty$ for $p = n$. For $p \leqq n$, we shall assume that

$$|f(x,s)| \leqq m(x) + c|s|^{\alpha-1} \tag{1.5}$$

with an arbitrary $\alpha < p^*$, $c > 0$, $m \in L_{\alpha'}(\Omega)$, $\alpha' = \alpha(\alpha-1)^{-1}$. Set $\alpha' = 1$ in the case $p > n$. Moreover, let there exist $r > 0$ and functions $h_{-\infty}$, $h^{+\infty} \in L_{\alpha'}(\Omega)$ such that

$$\begin{aligned} f(x,s) &\geqq h_{-\infty}(x) \qquad \text{for } s < -r\;, \quad \text{a.a. } x \in \Omega,\\ f(x,s) &\leqq h^{+\infty}(x) \qquad \text{for } s > r\;, \quad \text{a.a. } x \in \Omega. \end{aligned} \tag{1.6}$$

THEOREM 1.1 (Nonlinearities of a 'decreasing type'). Suppose (1.5), (1.6). Then the problem (1.1), (1.2) and (1.1), (1.3) has at least one weak solution for any $g \in L_{\alpha'}(\Omega)$ satisfying the condition

$$\int_\Omega f^{+\infty}(x)\varphi(x)\,dx < \int_\Omega g(x)\varphi(x)\,dx < \int_\Omega f_{-\infty}(x)\varphi(x)\,dx \,, \qquad (1.7)$$

where φ is the positive eigenfunction corresponding to the smallest eigenvalue of the problem (1.4), (1.2) and (1.4), (1.3), respectively.

EXAMPLE 1.1 Consider the BVP (1.1), (1.3), where $f(x,u) = -|u|^{q-2}u$, where $1 < q \leq p$. Then $f^{+\infty}(x) \equiv -\infty$, $f_{-\infty}(x) \equiv +\infty$. Hence BVP (1.1), (1.3) has at least one weak solution $u \in \overset{\circ}{W}{}^1_p(\Omega)$ for any $g \in L_{\alpha'}(\Omega)$. If the nonlinearity in (1.1) has the form

$$f(x,u) = \begin{cases} -|u|^{q-1}u & \text{for } x \in \Omega,\ u \geq 0 \,, \\ 0 & \text{for } x \in \Omega,\ u < 0 \,. \end{cases}$$

$1 < q \leq p$, then (1.1), (1.3) has at least one weak solution $u \in \overset{\circ}{W}{}^1_p(\Omega)$ for any $g \in L_{\alpha'}(\Omega)$ satisfying

$$\int_\Omega g(x)\varphi(x)\,dx < 0 \,.$$

Further, denote

$$f^{-\infty}(x) = \limsup_{s\to-\infty} f(x,s) \quad \text{and} \quad f_{+\infty}(x) = \liminf_{s\to+\infty} f(x,s)$$

and suppose that there exist $r > 0$ and functions $h^{-\infty}$, $h_{+\infty} \in L_{\alpha'}(\Omega)$ such that

$$\begin{aligned} f(x,s) &\leq h^{-\infty}(x) && \text{for } s < -r \,, \text{ a.a. } x \in \Omega, \\ f(x,s) &\geq h_{+\infty}(x) && \text{for } s > r \,, \text{ a.a. } x \in \Omega. \end{aligned} \qquad (1.8)$$

Moreover, assume that

$$\lim_{|s|\to+\infty} |s|^{1-p} f(x,s) = 0 \quad \text{for a.a. } x \in \Omega. \qquad (1.9)$$

THEOREM 1.2 (<u>**Nonlinearities of an "increasing type"**</u>). Let us suppose (1.5), (1.8), (1.9) and $p > n$. Then the problem (1.1), (1.2) has at least one weak solution for any $g \in L_{\alpha'}(\Omega)$ satisfying the condition

$$\int_\Omega f^{-\infty}(x)\,dx < \int_\Omega g(x)\,dx < \int_\Omega f_{+\infty}(x)\,dx \,. \qquad (1.7')$$

Suppose, now, that $1 < p \leq n$ and there exists $h \in L_{\alpha'}(\Omega)$ such that

$$|f(x,s)| \leq h(x) \quad \text{for all } s \in \mathbb{R}, \text{ a.a. } x \in \Omega \,. \qquad (1.10)$$

THEOREM 1.3 (<u>**Nonlinearities of an "increasing type"**</u>). Let us suppose (1.5) and (1.10). Then the problem (1.1), (1.2) has at least one weak solution for any $g \in L_{\alpha'}(\Omega)$ satisfying (1.7').

EXAMPLE 1.2 Consider the BVP (1.1), (1.2), where $f(x,u) = \operatorname{arctg} u$. Then $f^{-\infty}(x) \equiv -\frac{\pi}{2}$, $f_{+\infty}(x) \equiv \frac{\pi}{2}$. Hence BVP (1.1), (1.2) has at least one weak solution $u \in W^1_p(\Omega)$ for any $g \in L_{\alpha'}(\Omega)$ satisfying

$$-\frac{\pi}{2} < [\operatorname{meas}\Omega]^{-1} \int_\Omega g(x)\,dx < \frac{\pi}{2} \,.$$

Consider the equation of the type

$$\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x)) + \lambda_1 |u(x)|^{p-2}u(x) - |u(x)|^{q-2}u(x) + $$
$$+ f(x,u(x)) = g(x) , \quad x \in \Omega \qquad \textbf{(1.11)}$$

with $q > p$ and f having the growth not stronger then the $(p - 1)$ - th power.

THEOREM 1.4 (Equations with a "higher order term"): Let $q > p$. Then the problem (1.11), (1.2) and (1.11), (1.3) has at least one weak solution $u \in W^1_p(\Omega) \cap L_q(\Omega)$ and $u \in \mathring{W}^1_p(\Omega) \cap L_{q_1}(\Omega)$, respectively, for any right hand side $g \in L_{q'}(\Omega)$ $(q' = q(q - 1)^{-1})$.

REMARK 1.1. The proofs of Theorems 1.1 - 1.4 can be found in Boccardo, Drábek and Kučera [3] . In fact, more general results are proved in [3] , where the terms div $(|\nabla u|^{p-2}\nabla u)$ and $|u|^{q-2}u$ in our equations (1.1) and (1.11) may be replaced by more general ones. In Anane and Gossez [2] the assertion similar to our Theorem 1.1 is proved by using a different approach.

2. Bifurcations of strongly nonlinear equations.

Let $h = h(\lambda, x, s)$ be a Carathéodory's function defined on $\mathbb{R} \times \Omega \times \mathbb{R}$ such that $h(\lambda, x, 0) \equiv 0$ and

$$\lim_{s \to 0} |s|^{1-p} h(\lambda, x, s) = 0$$

uniformly for a.a.$x \in \Omega$ and λ from a bounded interval.

Consider the equation

$$\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x)) + \lambda |u(x)|^{p-2}u(x) = h(\lambda, x, u(x)),$$
$$x \in \Omega , \qquad \textbf{(2.1)}$$

with the boundary condition (1.2) or (1.3). Let us denote $X = W^1_p(\Omega)$ or $X = \mathring{W}^1_p(\Omega)$ if (1.2) or (1.3) is considered, respectively. We say that $C = \{ (\lambda, u) \in \mathbb{R} \times X, (\lambda, u)$ solves (2.1), (1.2) or (2.1), (1.3)$\}$ is a **continuum** of nontrivial weak solutions of (2.1), (1.2) or (2.1), (1.3), respectively, if it is connected in $\mathbb{R} \times X$.

THEOREM 2.1 (Global bifurcation). Let us suppose that all the assumptions stated above are fulfilled. Then there exists a continuum C of nontrivial weak solutions of (2.1), (1.2) or (2.1), (1.3) which contains in its closure the point $(\lambda_1, 0) \in \mathbb{R} \times X$ and C is either unbounded in $\mathbb{R} \times X$ or C contains in its closure a point $(\lambda_0, 0) \in \mathbb{R} \times X$, where $\lambda_0 > \lambda_1$ is an eigenvalue of (1.4), (1.2) or (1.4), (1.3), respectively.

REMARK 2.1. The proof can be found in Drábek [5]. It is based on the degree theory and on some ideas from Rabinowitz [7]. Theorem 2.1

generalizes analogous results for semilinear problems. It generalizes also 'local bifurcation results' for homogeneous problem, proofs of which are based on the Ljusternik - Schnirelmann theory (see e. g. Fučík et al. [6]).

REMARK 2.2. It is possible to strengthen the assertion of Theorem 2.1 by using the simplicity of λ_1 and some ideas from Dancer [4]. Essentially, under the same assumptions as in Theorem 2.1 it is possible to prove that there exist two maximal connected subsets C^+, C^- of C containing $(\lambda_1, 0) \in \mathbb{R} \times X$ in their closure, C^+ (C^-) 'bifurcating in the direction of' φ ($-\varphi$) and such that either

(i) both C^+, C^- are unbounded in $\mathbb{R} \times X$, or

(ii) both C^+, C^- contain in their closure a common point different from $(\lambda_1, 0) \in \mathbb{R} \times X$.

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