K. Doppel

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ON THE FLOATING BODY PROBLEM

DOPPEL K., BERLIN, Berlin West

1. Introduction and statement of the problem

1.1. The floating body problem is a boundary-value problem concerned with determining the surface waves produced by a vertically oscillating body floating in an inviscid incompressible fluid of finite depth with a free surface extending to infinity. This problem was introduced by F.John in 1949 (cf.[11]). In his classic work, John showed how this boundary problem could be reduced to an integral equation over the wetted portion of the body by employing a Green's function, which satisfies both the boundary condition at the bottom of the fluid and the linearized free surface condition on the entire fluid-air boundary. In spite of the very complicated form of this Green's function, John demonstrated the existence of irregular frequencies, i.e. frequencies for which the integral equation was not uniquely solvable. Of course, the complicated form of the Green's function and the non-unique solvability of the resulting integral equations are both undesirable features, and hence there has been a bulk of literature dealing with the floating body problem in order to improve the situation. However, almost all of them are based on potential theoretical approaches (see e.g. Angell/Hsiao/Kleinman [2]).

In the present paper we shall treat a simpler version of the classical 2-dimensional floating body problem namely the floating beam problem by using the Hilbert space approach (cf. Euvrard/Jami/Morice/Dusset [10], Doppel [3], Doppel/Hsiao [5], Doppel/Schomburg [6],[7]).

1.2. We denote by $p = (x, y)$ the elements in the two-dimensional Euclidean space $\mathbf{R}^2$. Define the fluid domain $\Omega := \mathbf{R} \times]-h, 0[$ with the bottom surface $S_B := \mathbf{R} \times \{-h\}$. Fix an open intervall $S_I :=]p_1, p_2[\subset \mathbf{R} \times \{0\}$ and call it the *floating beam*. A basic problem in linear hydrodynamics is now formulated as follows (cf. John [11]):

<u>**Problem A**</u>. *(Classical formulation of the floating beam problem.) Find all $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that*

$$\Delta u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \begin{cases} 0 & \text{on } S_B \\ f & \text{on } S_I \\ \lambda u & \text{on } S_F \end{cases}, \tag{1.2}$$

where $\partial/\partial n$ denotes the outer normal derivative to the domain Ω, f is a given function on the ship hull S_I and $\lambda \in \mathbf{C}$ is the wave number with $\mathrm{Im}\lambda \geq 0$.

1.3. In his paper [11], p.50, F.John expressed the idea that the discovery of a variational formulation of Problem A could facilitate the existence proof for more general surfaces S_I and also permit the construction of approximate expressions for the solutions. Therefore in Doppel [3], Doppel/Hsiao [5] the following weak formulation of Problem A was given. Consider the sesquilinear form

$$(u, v)_E = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dp + \int_{S_F} u\overline{v}\, ds$$

on the Sobolev space $H^1(\Omega)$ where ds denotes the line element on S_F. It can be shown that $(\cdot,\cdot)_E$ is a well-defined inner product on $H^1(\Omega)$ which is equivalent to the usual one $(\cdot,\cdot)_1$ (cf. Doppel/Hsiao [5]). Then $a_\lambda(\cdot,\cdot)$ with

$$a_\lambda(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dp - \lambda \int_{S_F} u\overline{v}\, ds. \tag{1.3}$$

is a continuous sesquilinear form on $H^1(\Omega)$. Furthermore for $f \in L^2(S_I)$ define the bounded anti-linear form l_f on $H^1(\Omega)$ by

$$l_f(v) = \int_{S_I} f\overline{v} ds\,.$$

Now we are able to formulate

Problem B. *(Hilbert space formulation of the floating beam problem.) For each $f \in L^2(S_I)$ and $\lambda \in \mathbf{C}$ find all $u \in H^1(\Omega)$ such that*

$$a_\lambda(u,v) = l_f(v)$$

holds for all $v \in H^1(\Omega)$.

Remark. The relation $u \in H^1(\Omega)$ can be interpreted as a generalization of a radiation condition of the type

$$u(x,y) = o(1)\,,\quad \frac{\partial u}{\partial |x|}(x,y) = O(|x|^{-(1+\epsilon)})\,,\quad -h \le y \le 0\,,\ |x| \to \infty.$$

2. Existence of weak solutions

2.1. Denote by $H^1(\Omega)^*$ the set of all anti-linear continuous functionals on $H^1(\Omega)$. Define the linear and continuous operator $A_\lambda : H^1(\Omega) \to H^1(\Omega)^*$ as usual by

$$(A_\lambda u)(v) \;=\; a_\lambda(u,v)\quad,\quad u,v \in H^1(\Omega),\ \lambda \in \mathbf{C},$$

where $a_\lambda(\cdot,\cdot)$ is defined by (1.3). For A_λ we have a Gårding's inequality (cf.Doppel/Hsiao [5]):

To each $\lambda \in \mathbf{C}$ there exist real constants $\gamma > 0$ and $\mu \ge 0$ such that

$$|A_\lambda u(u)| = \;|a_\lambda(u,u)| \ge\; \gamma\; \|u\|^2_{1,\Omega} - \mu\|u\|^2_{0,\Omega}$$

holds for all $u \in H^1(\Omega)$.

Note that A_λ for $\lambda \in \mathbf{C}\setminus[0,\infty[$ is strongly coercive, and therefore Problem B is uniquely solvable (cf. Doppel/Hsiao [5]). In the case $\lambda = 0$ one has to restrict the functions f to be in a certain smaller space $L^2_{\mathrm{ad}}(\Omega)$ and to replace $H^1(\Omega)$ by $F^1(\Omega)$ which is obtained by the completion of the Schwartz space $\mathcal{S}(\Omega) := \mathcal{S}(\mathbf{R}^2) \cap C^\infty(\Omega)$ with respect to the norm

$$|\phi|_{1,\Omega} = (\int_\Omega |\nabla\phi|^2 dp)^{1/2} \tag{2.1}$$

(for details cf. Doppel/Hsiao [5]).

2.2. Since the space $H^1(\Omega)$ is not compactly imbedded in $L^2(\Omega)$ we cannot get a Fredholm alternative by Gårding's inequality for the case $\lambda > 0$. But we can try to prove the existence in this case via the limiting absorption principle. For this it seems to be necessary to prove a Green representation formula for (weak) solutions of Problem B as in the classical case (cf. John [11]). And for this again we have to study the regularity of the weak solutions.

3. Regularity away from the corners $\mathbf{p_1}, \mathbf{p_2}$

It was shown in Doppel/Hsiao [5] that Weyl's Lemma implies the following interior regularity :

Each weak solution of Problem B lies in $C^\infty(\Omega)$.

This can be strengthened to (cf. Doppel/Schomburg [6]):

Each weak solution of Problem B lies in $C^\infty(\Omega \cup S_F \cup S_B)$.

Using interpolation theory one can show (cf. Doppel/Schomburg [6]):

The solutions u of Problem B lie in $H^{3/2}(\Omega \cap K)$ for each compact $K \subset \mathbf{R}^2$. Especially the restrictions $u|_{\partial\Omega}$ lie in $H^1_{\text{loc}}(\partial\Omega)$.

4. Weighted regularity in the corners $\mathbf{p_1}, \mathbf{p_2}$

In this section we will study the (weighted) H^2 regularity of solutions of Problem B in the corners p_1, p_2. To be more precise, define the weighted Sobolev space $H^2(\Omega;\rho)$ as follows. Let α be a multi-index, i.e. $\alpha = (\alpha_1, \alpha_2), \alpha_i \in \mathbf{N}_0$, $|\alpha| = \alpha_1 + \alpha_2$. For a nonnegative (positive almost everywhere) measurable function ρ on Ω let us denote by $H^2(\Omega;\rho)$ the space of all functions $u \in H^1(\Omega)$ such that

$$\sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u(p)|^2 \rho(p) dp < \infty.$$

Now take the special weight function $\rho_\epsilon, \epsilon > 0$ defined by

$$\rho_\epsilon(p) := (\text{dist}(p, \{p_1, p_2\}))^{2+\epsilon} |\phi(p)|^2,$$

where $\phi \in C_0^\infty(\mathbf{R}^2)$ is an arbitrary but fixed test function such that $\{p_1, p_2\} \subset \text{supp}\ \phi$, and obtain the following regularity result:

If $f \in H^{3/2}(S_I)$ then each solution of Problem B lies in $H^2(\Omega;\rho_\epsilon)$ for all $\epsilon > 0$.

We remark that in this field the pioneering work was done by V.A.Kondrat'ev [12] in the sixtees. As far as we know inhomogeneous mixed boundary value problems, especially of the Robin-Neumann type as Problem B, have not been investigated in the literature (cf. Kufner/Sändig [13]).

5. Singularity functions of weak solutions

In order to investigate the singularity behaviour of solutions of Problem B in the corner points we assume without loss of generality $p_1 = 0$ and cut off the domain Ω in a suitable neighbourhood of $p_1 = 0$. Then the resulting system is transformed into a parameter-dependent boundary value problem for an ordinary differential operator. A careful spectral analysis of this boundary value problem leads to the following theorem:

Fix $\lambda_0 > 0$. Then there exists a bounded open neighbourhood $V_0 (\subset \mathbf{C})$ of λ_0, such that for all λ in V_0 with $\text{Im}\lambda > 0$ the equation

$$z \tanh(\pi z) = \lambda$$

has exactly two complex roots $z_j = x_j + iy_j$, $j = 1, 2$ in the strip

$$S = \{z \in \mathbf{C} \mid -1 < \text{Im} z < 0\}.$$

Furthermore, for all λ in V_0 with $\mathrm{Im}\lambda > 0$ each solution u of Problem B has in a suitable neighbourhood of $p_1 = 0$ the representation

$$u(r,\omega) = c_1 \exp(ix_1 \ln r)\cosh(z_1\omega)r^{-y_1} + c_1 \exp(ix_2 \ln r)\cosh(z_2\omega)r^{-y_2} + v,$$

where (r,ω) denote the polar coordinates in $\mathbb{R}^2$, $v \in H^2$ and the coefficients c_1, c_2 are depending on u and f.

For the proof cf. Doppel/Schomburg [9]. The proof is based on techniques developed by V.A.Kondrat'ev [12] and M.S.Agranovich/M.I.Vishik [1].

6. Numerical results

In order to use finite-element-methods for Problem B one has to cut off the fluid domain Ω. In K.Doppel/R.Hochmuth/B.Schomburg [4] the existence of solutions as well as the convergence of a finite-element-method for the cut-off-problem is proved. We remark that contrary to D.Euvrard/ A.Jami/C.Morice/Y.Dusset [10] more general (esp. locally refined) triangulations are admitted. Furthermore, one can obtain improved convergence rates for the finite-element-method with respect to weaker norms without using regularity results.

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