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STABILITY AND AVERAGING PROPERTIES OF STOCHASTIC EVOLUTION EQUATIONS

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The theory of averaging for differential equations with quickly oscillating coefficients has been a subject of interest for many authors since early fifties, see e.g. [1] for ODE's, [2], [7] for stochastic differential equations. Recently the results from [7] on averaging in the quadratic mean have been extended to stochastic differential equations in a Hilbert space with unbounded drift terms and applied to stochastic PDE's ([5], [6]). In [5] some stability results are also included. They make it possible to find effective conditions guaranteeing the required averaging properties on an infinite time interval; however, they also may be of some independent interest.

In the present contribution the main results from [5], [6] are summarized. They are restated in a slightly less general, but more transparent form. Consider a parameter-dependent system of semilinear SDE's

$$(1)_\alpha \qquad dx_\alpha(t) = (Ax_\alpha(t)+f_\alpha(t,x_\alpha(t)))dt + \Phi_\alpha(t,x_\alpha(t))dw_t\ , \quad t\geq t_0\ ,$$
$$x_\alpha(t_0) = \varphi_\alpha\ , \quad \alpha\geq 0\ ,$$

in a real separable Hilbert space H, where $A : H\to H$ is an infinitesimal generator of a strongly continuous semigroup S_t, w_t is a K-valued Wiener process on $(\Omega, \mathcal{A}, P)$ with a nuclear covariance W (K - a real separable Hilbert space), $f_\alpha : \mathbb{R}_+\times H\to H$, $\Phi_\alpha : \mathbb{R}_+\times H \to \mathcal{L}(K,H)$ are measurable and satisfy

$$(2) \qquad \| f_\alpha(t,x)-f_\alpha(t,y) \| + \| \Phi_\alpha(t,x)- \Phi_\alpha(t,y) \| \leq \hat{K}\|x-y\|\ ,$$
$$\| f_\alpha(t,0) \| + \| \Phi_\alpha(t,0) \| \leq \hat{K}\ , \quad t\in\mathbb{R}_+, \quad x,y\in H\ ,$$

for some $\hat{K}>0$ independent of α. It is well known (see e.g. [3]) that under the above assumptions there exists a unique mild solution x_α to $(1)_\alpha$.

Theorem 1 ([6]). Assume

$$(3) \qquad \lim_{\alpha\to 0+}\int_{t_1}^{t_2} S_{t_2-s}(f_\alpha(s+t_0,x)-f_0(s+t_0,x))ds = 0\ ,$$

$$(4) \qquad \lim_{\alpha\to 0+}\int_{t_1}^{t_2} \mathrm{Tr}\{(\Phi_\alpha(s+t_0,x)- \Phi_0(s+t_0,x))W(\Phi_\alpha(s+t_0,x)- -\Phi_0(s+t_0,x))^*\}ds = 0$$

for all $x\in H$, $0\leq t_1\leq t_2$, and $\varphi_\alpha\to\varphi_0$.

Then for any $0<T<\infty$ we have

$$(5)\qquad \lim_{\alpha\to 0+}\ \sup_{t\in\langle t_0,T\rangle} E\|x_\alpha(t)-x_0(t)\|^2 = 0 .$$

In the finite-dimensional case it can be seen ([7]) that a similar statement is valid even for $T=+\infty$ provided the limit solution x_0 is asymptotically stable. The proof from [7] fails for $\dim H=\infty$, however, in [5] we prove the assertion imposing some restrictions on S_t.

Definition. A solution x_0 of the equation $(1)_0$ is said to be asymptotically stable in the mean square if

(i) for every $\varepsilon>0$ there exists $\delta>0$ such that for all $t_0\geq 0$ and all solutions $\tilde{x}$ of $(1)_0$ satisfying $E\|\tilde{x}(t_0)-x_0(t_0)\|^2<\delta$ we have $E\|x(t)-x_0(t)\|^2<\varepsilon$, $t\geq t_0$,

(ii) there exists $A>0$ such that for all $\varepsilon>0$, $\delta\in(0,A)$ there exists $T=T(\varepsilon,\delta)>0$ such that for all $t_0\geq 0$, $\tilde{x}$ satisfying $E\|\tilde{x}(t_0)-x_0(t_0)\|^2<\delta$ we have $E\|\tilde{x}(t)-x_0(t)\|^2<\varepsilon$, $t\geq t_0+T$.

Theorem 2 ([5]). Let (3), (4) be fulfilled uniformly w.r.t. $t_0\in\mathbb{R}_+$ and $x\in H$ and assume $S_{(.)}\in \mathbb{C}((0,+\infty),\mathcal{L}(H))$, $\varphi_\alpha\to\varphi_0$. Then (5) is valid with $T=+\infty$ provided x_0 is asymptotically stable in the mean square and $E\|x_0(t)\|^2$ is bounded for $t\geq t_0$.

In order to obtain effective results on infinite time intervals we still need verifiable criteria for mean-square asymptotic stability. The standard application of Liapunov method leads to some difficulties as the mild solutions of $(1)_0$ need not possess a stochastic differential. This can be overcome by approximating mild solutions by strong solutions similarly as in [4]. For $v\in\mathbb{C}_{1,2}(\mathbb{R}_+\times H)$ set

$$\mathcal{L}v(t,x,y)=\frac{\partial}{\partial t}v+\langle v_x(t,x-y),Ax-Ay+f_0(t,x)-f_0(t,y)\rangle+\frac{1}{2}\mathrm{Tr}(\Phi_0(t,x)-$$
$$-\Phi_0(t,y))^* v_{xx}(t,x-y)(\Phi_0(t,x)-\Phi_0(t,y))W\ ,\quad (t,x,y)\in\mathbb{R}_+\times\mathcal{D}(A)\times\mathcal{D}(A)\ .$$

Proposition 3. Assume $\mathcal{L}v(t,x,y)\leq\varphi(t,v(t,x-y))$, $t\in\mathbb{R}_+$, $x,y\in\mathcal{D}(A)$, where $v\in\mathbb{C}_{1,2}(\mathbb{R}_+\times H)$ is such that

$$d_1\|x\|^2\leq v(t,x)\leq d_2\|x\|^2\ ,\quad \|v_x\|+\|v_{xx}\|\leq d_3(1+\|x\|^p),\quad x\in H\ ,$$

for some $d_1', d_2, d_3, p>0$ and $\varphi:\mathbb{R}_+^2\to\mathbb{R}$ is measurable, $\varphi(t,.)$ is Lipschitzian and concave, $\varphi(t,0)=0$ for all $t\geq 0$. Then all solutions x_0 of $(1)_0$ are asymptotically stable in the mean square provided the trivial solution $x\equiv 0$ of the equation $\dot{x}=\varphi(t,x)$ is asymptotically stable.

Example. The stochastic parabolic problem described by

(6) $$\frac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon + \frac{r_1(t/\varepsilon)u_\varepsilon}{1+|u_\varepsilon|} + \frac{r_2(t/\varepsilon)u_\varepsilon}{1+|u_\varepsilon|}\dot{\bar{W}}(t,x) \ , \quad t \geqq t_0 \ , \quad x \in D$$

(D - a bounded region in $\mathbb{R}_n$ with $\mathbb{C}_2$ boundary), $u_\varepsilon(0,x) = u_0(x)$, $u_\varepsilon|_{\partial D} = 0$ can be formally rewritten in the form

(7) $$dx_\varepsilon(t) = (Ax_\varepsilon(t)+f(t/\varepsilon,x_\varepsilon(t)))dt + \Phi(t/\varepsilon,x_\varepsilon(t))dw_t \ , \quad x_\varepsilon(t_0) = \varphi_\varepsilon \ ,$$

in the space $H = L_2(D)$, with $K = H^k(D)$ - valued Wiener process w_t $(k > 2n)$, where $A = \Delta|_{H^2(D)\cap H^1_0(D)}$, $f(t,x)(\theta) = r_1(t)x(\theta).(1+|x(\theta)|)^{-1}$, $\Phi(t,x)h(\theta) = r_2(t)x(\theta)h(\theta)(1+|x(\theta)|)^{-1}$, $\theta \in D$, $h \in K$. Assume

$$\frac{1}{T}\int_{\beta T}^{\beta T+T} r_1(t)dt \to r_1 \ , \quad \frac{1}{T}\int_{\beta T}^{\beta T+T} (r_2(t)-r_2)^2 dt \to 0 \ , \quad T \to \infty \ ,$$

uniformly in $\beta \geqq 0$ for some $r_1, r_2 \in \mathbb{R}$, and $-\lambda_0 + \max(0,r_1) + 1/2\, r_2^2 k^2 \mathrm{Tr} W < 0$, where $\lambda_0 > 0$ is the first eigenvalue of $-A$ and $k > 0$ is such that $\|\ \|_{\mathbb{C}(D)} \leqq k\|\ \|_K$. Then it can be checked that Theorem 2 and Proposition 3 yield

$$\sup_{t \geqq t_0} E\|x_\varepsilon(t)-\bar{x}(t)\|^2 \to 0 \quad \text{as} \quad \varepsilon \to 0 \ , \quad \varphi_\varepsilon \to \bar{\varphi} \ ,$$

where $\bar{x}$ is the solution of the limit equation $d\bar{x} = (A\bar{x} + r_1\bar{x}/1+|\bar{x}|)dt + (r_2\bar{x}/1+|\bar{x}|)dw_t$, $\bar{x}(t_0) = \bar{\varphi}$ (see [5] for a similar example).

<u>Remark.</u> Some extensions of the above results (e.g. averaging in $L_p(\Omega)$ for $p \geqq 2$, averaging in probability, statements analogous to Theorems 1, 2 for a cylindrical Wiener process, etc.) can be found in [5],[6].

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