Karel Kozel

Some remarks to numerical solution of the Euler and Navier-Stokes equation

In: Jaroslav Kurzweil (ed.): Equadiff 7, Proceedings of the 7th Czechoslovak Conference on Differential Equations and Their Applications held in Prague, 1989. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1990. Teubner-Texte zur Mathematik, Bd. 118. pp. 271--274.

Persistent URL: http://dml.cz/dmlcz/702333

SOME REMARKS TO NUMERICAL SOLUTION OF THE EULER AND NAVIER-STOKES EQUATION

KOZEL K., PRAGUE, Czechoslovakia

The work deals with numerical solution of 2D system of the Euler equations used for computation of steady transonic flows through a cascade and with numerical solution of 2D system of Navier-Stokes equations used for computation of steady laminar compressible viscous flows past a flat plate. The problems are solved by finite volume time dependent method using three explicit difference schemes.

Numerical results of 2D steady transonic flows through a compressor cascade ($M_\infty > 1$) and through a turbine cascade ($M_2 > 1$) are compared to experimental results. Numerical solution of 2D Navier-Stokes equations used for simulation of steady laminar viscous flows over a flat plate is compared to Blasius solution with Reynolds number $Re \in \langle 10^2, 10^4 \rangle$.

I. Governing equations

Consider 2D system of Navier-Stokes equations for compressible flow fields

$$U_t + F_x + G_y = R_x + S_y ,$$
$$U = \mathrm{col}\,\|\varrho, \varrho u, \varrho v, e\| = \mathrm{col}\,\|\varrho, \varrho u, \varrho v, \varrho E\|, \qquad (1)$$

where ϱ is density, (u,v) is velocity vector in cartesian coordinates, e is total energy per unit volume, E is specific total internal energy. Vectors $F=F(U)$, $G=G(U)$, $R=R(U,U_x,U_y)$, $S=S(U,U_x,U_y)$ are defined f.e. in [1]. The system (1) with $R=S=0$ is called 2D system of Euler equations. System of Euler equations is nonlinear system of first order and hyperbolic type, eigenvalues $\lambda(A)$, $\lambda(B)$ are real; $A = \partial F/\partial U$, $B = \partial G/\partial U$ are Jacobian matrices.

II. Numerical solution and used difference schemes

A finite volume approach is applied to discretize the system equations (1). In our case the computational domain Ω is divided into quadrilateral cells Ω_{ij} fixed in time. For numerical solution of the Euler equations and a case of transonic cascade flows a non-regular H-grid was used (see [2]); in our case of numerical solution of Navier-Stokes equations a non-regular orthogonal grid was used. For each computational cell the governing equations are considered in following integral form

$$U_t = \frac{1}{\mu_{ij}} \left\{ - \oint_{\partial\Omega_{ij}} (F\,dy - G\,dx) + \oint_{\partial\Omega_{ij}} (R\,dy - S\,dx) \right\}, \quad \mu_{ij} = \iint_{\Omega_{ij}} dx\,dy . \qquad (2)$$

For computation of the steady state solution a time dependent method is used and following relation is fulfilled

$$- \oint_{\partial\Omega_{ij}} (F\,dy - G\,dx) + \oint_{\partial\Omega_{ij}} (R\,dy - S\,dx) = 0 \tag{3}$$

In the case of numerical solution of the Euler equations $R=S=0$. Mac Cormack explicit predictor-corrector finite volume difference scheme was. used for numerical solution of Euler and Navier-Stokes equations:

$$U_{ij}^{n+1/2} = U_{ij}^{n} - \frac{\Delta t}{\mu_{ij}} \left\{ \sum_{k=1}^{4} (F_k^n \Delta y_k - G_k^n \Delta x_k) - (R_k^n \Delta y_k - S_k^n \Delta x_k) \right\} \tag{4a}$$

$$\overline{U_{ij}^{n+1}} = \frac{1}{2} (U_{ij}^{n} + U_{ij}^{n+1/2}) - \frac{\Delta t}{\mu_{ij}} \left\{ \sum_{k=1}^{4} (F_k^{n+1/2} \Delta y_k - G_k^{n+1/2} \Delta x_k) - (R_k^{n+1/2} \Delta y_k - S_k^{n+1/2} \Delta x_k) \right\} \tag{4b}$$

$$U_{ij}^{n+1} = \overline{U_{ij}^{n+1}} + DU_{ij}^{n}, \tag{4c}$$

where U_{ij}^n is mean value of U in the cell Ω_{ij}, $F_k^n, F_k^{n+1/2}, G_k^n, G_k^{n+1/2}$ are defined in [2] and R_k, S_k are approximated by central differencing of second order; DU_{ij}^n is artifitial damping term described in [2]. This difference scheme is possible to write in rezidual form

$$(U_{ij}^{n+1} - U_{j}^{n}) / \Delta t = -\mathrm{Rez}\, U_{ij}^{n} = - \frac{1}{\mu_{ij}} \left\{ \sum_{k=1}^{4} (\tilde{F}_k^n \Delta y_k - \tilde{G}_k^n \Delta x_k) - (\tilde{R}_k^n \Delta y_k - \tilde{S}_k^n \Delta x_k) \right\}, \tag{5}$$

where $\tilde{F}, \tilde{G}, \tilde{R}, \tilde{S}$ are the approximations of F,G,R,S and Rez U_{ij}^n is approximation of

$$\frac{1}{\mu_{ij}} \cdot \oint_{\partial\Omega_{ij}} (F\,dy - G\,dx + R\,dy - S\,dx). \tag{6}$$

We can define $\|\mathrm{Rez}\, U_{ij}\|_{L_2}$, $\|\mathrm{Rez}\, U_{ij}\|_C$, $\|\Delta M\|_C = \max_{ij} |M_{ij}^{n+1} - M_{ij}^{n}|$ (M is the Mach number) and a convergence of time dependent process to the steady state solution is controlled by the values $\log \|\mathrm{Rez}\, U_{ij}^n\|_{L_2}$, $\|\mathrm{Rez}\, U_{ij}^n\|_C$, $\|\Delta M\|_C$.

Runge-Kutta time stepping difference schemes were also used for numerical solution of the Navier-Stokes equation:

$$\begin{aligned} U_{ij}^{n} &= U_{ij}^{(0)}, \\ U_{ij}^{(p)} &= U_{ij}^{(0)} - \Delta t.\, \alpha_p .\, \widetilde{\mathrm{Rez}}\, U_{ij}^{(p-1)}, \quad (p = 1,2,\dots,m), \alpha_p \text{ real}, \\ U_{ij}^{n+1} &= U_{ij}^{m}, \end{aligned} \tag{7}$$

$$\widetilde{\mathrm{Rez}}\, U_{ij}^{(p)} = \frac{1}{\mu_{ij}} \left\{ \sum_{k=1}^{4} (F_k^{(p)} \Delta y_k - G_k^{(p)} \Delta x_k) - (R_k^{(p)} \Delta y_k - G_k^{(p)} \Delta x_k) \right\} - \widetilde{D}U_{ij}^{n}, \text{ where}$$

$F_p^{(k)}, G_p^{(k)}, R_p^{(k)}, S_p^{(k)}$ are defined in [1] as well as artifitial damping term $\widetilde{D}U_{ij}^n$. $\widetilde{\mathrm{Rez}}\, U_{ij}^n$ is also an approximation of (6). For example in regular orthogonal grid we can use ($\mu_{ij} = \Delta x . \Delta y$)

$$\mathrm{Rez}\, U_{ij}^{n} = \frac{F_{i+1,j}^n - F_{i-1,j}^n}{2 .\, \Delta x} + \frac{G_{i,j+1}^n - G_{i,j-1}^n}{2 . \Delta y} - \frac{R_{i+1/2,j}^n - R_{i-1/2,j}^n}{\Delta x} + \frac{S_{i,j+1/2}^n - S_{ij-1/2}^n}{\Delta y} - \widetilde{D}U_{ij}^{n}, \tag{8}$$

where $R^n_{i+1/2} = \frac{1}{2}(R^n_{i+1,j} + R^n_{ij})$, $S^n_{ij+1/2} = \frac{1}{2}(S^n_{ij+1} + S^n_{ij})$, ... and difference approximation of second order for all space derivatives is used. Using regular orthogonal grid Mac Cormack scheme (4) is $O(\Delta t^2, \Delta x^2, \Delta y^2)$, Runge-Kutta scheme (8) with $\alpha_1=1/4$, $\alpha_2=1/3$, $\alpha_3=1/2$, $\alpha_4=1$, (RK4) is $O(\Delta t^4, \Delta x^2, \Delta y^2)$ and with $\alpha_1=1/2$, $\alpha_2=1/2$, $\alpha_3=1$ (RK2) is $O(\Delta t^2, \Delta x^2, \Delta y^2)$ A stability limitation of mentioned difference schemes is given in [2],[5].

III. Numerical solution of steady inviscid transonic flows

In this part some numerical results of steady inviscid transonic flows through a cascade of compressor and turbine type are presented and compared to experimental results. Presented results are achieved by numerical solutin of the Euler equations using Mac Cormack finite volume difference scheme. A detailed mathematical formulation of a weak solution of the problem of transonic flows through a plane cascade and the details of numerical method one can find in [2]. Our numerical solution of transonic flows through a compressor cascade is improved by multi--level grid solution (FAS-algorithm) using three-level grids.

Fig.1 shows comparison of our numerical results of transonic flows through a compressor cascade (full line) using distribution of modified pressure coefficients along upper and lower profile surface to experimental results of DFVLR Cologne (see [3]) for slightly supersonic upstream Mach number $M_\infty = 1{,}03$. Computing this case of transonic flows we have to use multigrid procedure.

Fig. 2 shows comparison of our numerical results of transonic flows through a turbine cascade (full line) using distribution of Mach number along upper and lower profile surface to experimental results [4] of Institute of Thermomechanics of Czechoslovak Academy of Sciences ($M_\infty = 0{,}395$; $p_2 = 0{,}45\ p_1$; $M_2 = 1{,}2$).

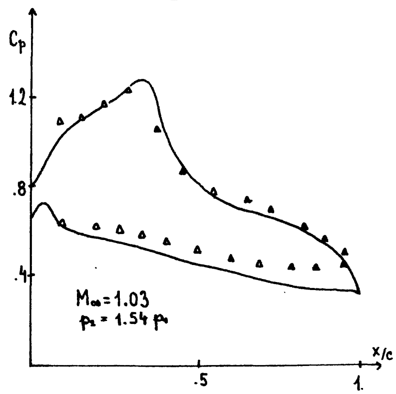

Fig.1

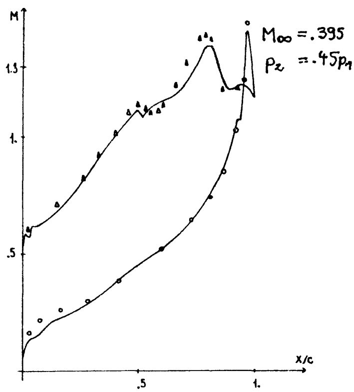

Fig.2

IV. Numerical solution of steady laminar viscous flows past a flat plate

This type of the solution is based on numerical solution of the system of Navier-Stokes equations (1) by three difference schemes: Mac Cormack, RK4, RK2. The problem was solved in domain Ω (oblong) using orthogonal grid regular in x-direction and regular or non-regular in y-direction. A detailed mathematical description of the problem is possible to find in [5]. The problem was solved for $M_1=0,2$ and Reynolds numbers $Re \in (100,10000)$ and one can compare the agreement of our numerical solution to Blasius solution, efficiency of three numerical methods in computation of the steady state solution and convergence of mentioned three methods to the steady state solution using the change $\|u\|_C = \max_{ij} |u_{ij}^{n+1} - u_{ij}^{n}|$. Qualitatively good results was observed for all three difference schemes but the best for RK4 method. Fig.3 shows comparison of our numerical solution by RK4 (full line) to Blasius solution ($\eta = (y/x)/Re^{1/2}$) for Re=1000. Fig. 4a,4b shows the convergence of RK4 and MC using the change of u ($\|u\|_C = \|\Delta u\|_C = \max_{ij} \|u_{ij}^{n+1} - u_{ij}^{n}\|$) during the iteration process for Re=100 (Fig. 4a) and Re=1000 (Fig.4b).

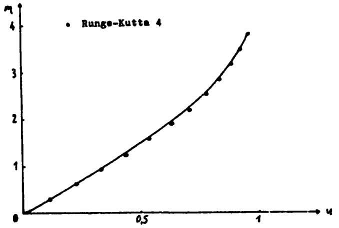

Fig.3

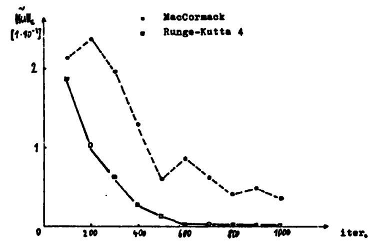

Fig.4a

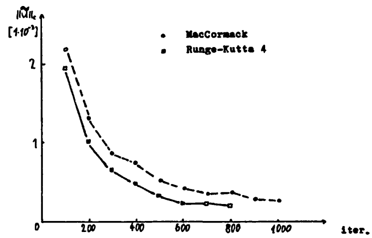

Fig.4b

References

[1] Arnone A., Swanson R.C.: ICASE Report No 88-32.

[2] Kozel K., Nhac N., Vavřincová M.: Lecture Notes in Numerical Fluid Mechanics, Vol.24, Vieweg.

[3] Schreiber H.A., Starken H.: Journal of Eng. for Gas Turbines and Power, 1984, April, Vol 106/289.

[4] Šťastný M., Šafařík P.: Strojnický časopis, 34, 1983, No 1-2 (in Czech).

[5] Kozel K., Lain J.: Zpravodaj VZLÚ, 1989 (in Czech).