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ON THE SOLVABILITY
OF THE NON-HOMOGENEOUS POTENTIAL PROBLEM
AT A CORNER SINGULARITY

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1. The problem

In several branches of applied mathematics one meets the following mixed boundary value potential problem:

$\mathcal{S}$ is a plane infinite sectorial region in $\mathbb{R}^3$, given by $x \geq 0$, $|y| \leq x \tan \gamma$, $z = 0$. We require a function ψ such that
(i) $\nabla^2\psi = 0$ in $\mathbb{R}^3 \setminus \mathcal{S}$ (ii) ψ takes prescribed values on $\mathcal{S}$, i.e. $\psi(x,y,0) = f(x,y)$ for $(x,y) \in \mathcal{S}$ (iii) $\partial\psi/\partial z\big|_{z=0} = 0$ for $(x,y) \notin \mathcal{S}$ (iv) $\psi \to 0$ as $r = (x^2+y^2+z^2)^{\frac{1}{2}} \to \infty$

(There is an analogous problem with ψ, $\partial\psi/\partial z$ interchanged in (ii) and (iii)).

Such problems occur, for instance in
(a) electrostatics (potential due to a charged plate, [8]),
(b) aerodynamics (air flow over a 'delta wing', [4], [10]),
(c) elasticity (indentation of an elastic medium by a wedge-shaped punch, [5], [9]).

2. Observations

It must first be noted that the problem does not have a unique solution. If $f(x,y) \equiv 0$, the problem is homogeneous, with zero boundary conditions, but nevertheless has non-trivial solutions. (The sector $\mathcal{S}$ acts in a similar manner to the 'spine' described in [7], page 285). Hence, if we denote solutions of the homogeneous and non-homogeneous problems by ψ_h and ψ_{nh}, then $\psi_h + \psi_{nh}$ is also a solution of the non-homogeneous problem.

On both mathematical and physical grounds, one expects a solution to have (locally, at least) a dominant component of the form

$$r^\nu \times \text{(angular function)}, \tag{2.1}$$

where the index ν (which depends on the vertex angle γ of $\mathcal{S}$) has to be determined.

There appears to be some confusion in the literature regarding the acceptable values of ν, which requires some comment. Morrison and Lewis [8] argued correctly that for the homogeneous problem ν is necessarily real, but their reasoning does not carry over to the non-homogeneous problem. Moreover, it is found that for a solution ψ to be real, it is only necessary that the quantity $\nu(\nu+1)$ be real. Besides the obvious possibility that ν is real (we take $\nu \geq -\frac{1}{2}$ without loss of generality), we can also take ν in the range

$$\nu = -\tfrac{1}{2} + ip \quad (p \geq 0) \tag{2.2}$$

and the main purpose of this paper is to show that such values of ν suffice to solve the non-homogeneous problem. Such a solution is unique in the sense that the function $f(x,y)$, subject only to reasonable smoothness and suitable behaviour at infinity, determines the solution completely. The non-uniqueness of the solution arises only from the possible occurrence of solutions of the homogeneous equation with ν real.

When considered in a physical context, there may be grounds on which a solution with complex ν must be excluded. For example, in the wedge punch problem, this would theoretically imply loss of contact between the punch and the elastic material near the vertex. Such an argument on physical grounds is, however, not wholly convincing because in the real world a completely sharp vertex cannot be achieved or would, at least, have effects such that the assumption of linearity, implied in the formulation of the problem, ceases to hold. (For example, in the wedge punch problem, the material would cease to behave elastically and would become plastic.) It seems reasonable, therefore, to regard a solution with ν complex as quite possibly valid in a region away from the vertex, where the solution is in any case indeterminate both physically and mathematically. This is the viewpoint adopted here.

The analysis was done jointly by the author and Dr.A.Darai of the University of Western Illinois.

3. Solution of the problem

The method is to embed the sector $\mathcal{S}$ in a system of elliptic conal coordinates, which allows separation of variables. Morrison and Lewis [8] use a trigonometric form, but here we use the more compact version which employs Jacobian elliptic functions.

Setting

$$x = \frac{r}{k'}\,\mathrm{dn}\alpha\,\mathrm{dn}\beta\ ,\ y = kr\ \mathrm{sn}\alpha\ \mathrm{sn}\beta\ ,\ z = \frac{ik}{k'}\mathrm{cn}\alpha\ \mathrm{cn}\beta \tag{3.1}$$

with ranges

$$r \in [0,\infty),\quad \alpha \in (-2K,2K],\quad \beta \in [K,K+2iK'], \tag{3.2}$$

the sector $\mathcal{S}$ is given by $\beta = K+2iK'$, and the remainder of the plane $z=0$ is made up of the regions $\beta = K$, $\alpha = \pm K$. The vertex angle of $\mathcal{S}$ is $2\sin^{-1} k$, where k ($\in (0,1)$) is the modulus of the elliptic functions. In these coordinates, $\nabla^2\psi = 0$ becomes

$$k^2(\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\psi}{\partial r}\right) = \frac{\partial^2\psi}{\partial\alpha^2} - \frac{\partial^2\psi}{\partial\beta^2} \qquad (3.3)$$

([1], p.24,[2], sect. 12).

Separation in the form $\psi = R(r)\,A(\alpha)\,B(\beta)$ (3.4)

gives the three equations

$$\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) - \nu(\nu+1)R = 0, \qquad (3.5a)$$

$$A'' + (\lambda-\nu(\nu+1)k^2\mathrm{sn}^2\alpha)A = 0,\quad B''+ (\lambda-\nu(\nu+1)k^2\mathrm{sn}^2\beta)B=0 \qquad (3.5b,c)$$

in which λ and $\nu(\nu+1)$ are separation constants. Equations (3.5b,c) are Lamé equations.

The equation (3.5a) provides different forms of solution $R(r)$ according to the value of the (real) quantity $\nu(\nu+1)$. According as

$$\text{(a)}\ \nu(\nu+1) \geq -\tfrac{1}{4},\quad \text{(b)}\ \nu(\nu+1) = -\tfrac{1}{4}\quad \text{(c)}\ \nu(\nu+1) \leq -\tfrac{1}{4} \qquad (3.6a,b,c)$$

we have

$$\text{(a)}\ R(r,\nu) = Ar^{\nu}+ Br^{-\nu-1}\quad (\nu\geq -\tfrac{1}{2}),\quad \text{(b)}\ R(r,\nu)=r^{-\frac{1}{2}}(A + B\ln r),$$

$$\text{(c)}\ R(r,\nu) = r^{-\frac{1}{2}}(A\cos(p\ln r)+ B\sin(p\ln r))\ (p\geq 0) \qquad (3.7a,b,c)$$

For simplicity, we now assume that the boundary-value function $f(x,y)$, and hence also the solution ψ, is symmetric about the centre line of $\mathscr{S}$. Then consideration of the coordinate system [5] shows that $A(\alpha)$ must be even in α with period 2K, so we adjoin to (3.5b) the conditions

$$A'(0) = A'(K) = 0. \qquad (3.8)$$

We have thus a regular Sturm-Liouville problem, with the usual infinity of eigenvalues $\lambda^{(m)}$ and corresponding eigenfunctions. Such solutions were investigated by Ince [6], denoted by $Ec_{\nu}^{2m}(\alpha)$, and given either as power series in $\mathrm{sn}^2\alpha$ or as Fourier-Jacobi series. They were partly tabulated by Ince, but may readily be computed by the technique given in [3].

Further analysis shows that $B(\beta)$ must be the same function of β that $A(\alpha)$ is of α. We thus obtain a separated solution, satisfying (i),(iii) of the original problem, for arbitrary ν and arbitrary $m = 0,1,2,\ldots$ A more general solution is obtained by summing over m and integrating with respect to ν, in the form

$$\psi = \psi(r,\alpha,\beta) = \int d\nu \sum_{m=0}^{\infty} C_m(\nu)\ R(r,\nu)\ Ec_{\nu}^{2m}(\alpha)\ Ec_{\nu}^{2m}(\beta) \qquad (3.9)$$

The range of integration with respect to ν is at present unspecified. It remains to satisfy the boundary condition on $\mathscr{S}$, which becomes

$$\psi(r,\alpha,K+2iK') = F(r,\alpha) \quad \text{where } f(x,y)= F(r,\alpha) \qquad (3.10)$$

The integral relationship consisting of (3.9),(3.10) must now be inverted to give $C_m(\nu)$ in terms of $F(r,\alpha)$. In [5] it is shown that this can be done uniquely if the range of integration for ν is taken to be $\nu = -\frac{1}{2}+ip$, $p \in [0,\infty)$, subject only to moderate conditions on the smoothness of F and its behaviour as $r \to \infty$ The mathematics involves only the orthogonality of the Ec_ν^{2m} with respect to m, and a Fourier transform with respect to $s = \ln r$. It is obvious that, since $R \sim r^{-\frac{1}{2}}$ as $r \to \infty$, the condition (iv) is also satisfied so the solution is completed.

R E F E R E N C E S

[1] F.M.Arscott, Periodic Differential Equations, Pergamon Press, 1964.

[2] F.M.Arscott, A.Darai, Curvilinear coordinates in which the Helmholtz equation separates, IMA Jour.Appl.Math 27(1981),33 33-70.

[3] F,M.Arscott, R.Lacroix,W.T.Shymanski, A three-term recursion and the computation of Mathieu functions, Proc.8th. Manitoba conference on numerical mathematics (1978) 107-115.

[4] S.N.Brown, K.Stewartson, Flow near the apex of a plane delta wing, Jour.I.M.A., 5 (1969) 206-216.

[5] A.Darai, Ph.D.Thesis, University of Manitoba, 1985.

[6] E.L.Ince, The periodic Lamé functions, Proc.Roy.Soc.Edinburgh, 60 (1939) 47-63 and Further investigations into the periodic Lamé functions, ibid. 83-94.

[7] O.D.Kellogg, Foundations of Potential Theory, Ungar, 1929.

[8] J.A.Morrison, J.A.Lewis, Charge singularity at the corner of a flat plate, SIAM J.Appl.Math. 31 (1976) 233-250.

[9] V.L.Rvachev, The pressure on an elastic half-space of a stamp with a wedge-shaped planform, PMM (Jour.of Appl.Math. and Mech.) 23 (1959) 229-233.

[10] R.S.Taylor, A new approach to the delta wing problem, J.Inst. Math. Applics 7 (1971) 337-347.