THE DIMENSION OF A SET OF SINGULARITIES OF WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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The flow of a viscous incompressible fluid in a domain $\Omega \subset R^3$ may be described by the Navier-Stokes equations

$$(1) \qquad \frac{\partial u_i}{\partial t} + u_j\, u_{i,j} = f_i - p_{,i} + \Delta u_i \qquad (i \in \{1,2,3\}) ,$$

(where $u_{i,j} = \partial u_i / \partial x_j$) and the equation of continuity

$$(2) \qquad u_{i,i} = 0 .$$

It is well known that a weak solution u to the problem, given by the equations (1), (2), by the initial condition

$$(3) \qquad u(x,0) = u_0(x) \qquad (x \in \Omega)$$

and the boundary condition

$$(4) \qquad u\big|_{\partial\Omega} = 0$$

exists on any time interval (0,T) (under certain assumptions about the smoothness of $\partial\Omega$ and the functions f, u_0). It was shown by J.Leray [2] that a "bad behaviour" of the weak solution u is concentrated to a set M of time instants ($\subset (0,T)$) so that (0,T) - - M can be expressed as a unification of at most countably many open intervals I_n ($n \in \{1,2,3,\dots\}$) such that u becomes after an eventual redefinition on a set of a measure zero smooth on each of these intervals. Moreover, the Hausdorff dimension of M is at most 1/2.

V.Scheffer [4] and C.Foias, R.Temam [1] showed that there exists a subset $\Omega_0 \subset \Omega$ so that

$$\sup_{t \in (0,T)} \operatorname{ess} |u(x,t)| < +\infty \qquad \text{if } x \in \Omega - \Omega_0$$

and the Hausdorff dimension of Ω_0 is less or equal to 5/2.

In this contribution, we deal with the case $\Omega = R^3$. We suppose that u is a weak solution to the problem (1), (2), (3) on a time interval (0,T) as it is defined for example in [2] and so

$$\int_0^T \int_{R^3} (u_{i,j}\, u_{i,j})\, dx\, dt < +\infty ,$$

$$\sup_{t \in (0,T)} \operatorname{ess} \int_{R^3} u_i\, u_i\, dx < +\infty .$$

We shall use the following representation formula for our solution:

$$(5)\quad u_i(x,t) = \int_{R^3} E_{ij}(x-y,t)\cdot u_j(y,0)\,dy -$$

$$- \int_0^t \int_{R^3} E_{ij}(x-y,t-\tau)\cdot u_k(y,\tau)\cdot u_{j,k}(y,\tau)\,dy\,d\tau +$$

$$+ \int_0^t \int_{R^3} E_{ij}(x-y,t-\tau)\cdot f_j(y,\tau)\,dy\,d\tau \qquad (i\in\{1,2,3\})$$

where E_{ij} are components of a fundamental solution tensor to the linearized problem, corresponding to (1), (2), (3). It is due to Oseen [3] that these components have a form

$$E_{ij}(x,t) = -\Delta\varphi(x,t)\cdot\delta_{ij} + \varphi_{,ij}(x,t),$$

where

$$\varphi(x,t) = \frac{1}{4\pi^{3/2}|x|}\int_0^{|x|}\frac{1}{\sqrt{t}}\exp\left(-\frac{a^2}{4\cdot t}\right)da \qquad (t>0).$$

In fact, weak solutions satisfy the identity (5) for almost all $[x,t]\in R^3\times(0,T)$. Suppose that u is modified in such a way that it satisfies (5) for all $[x,t]\in R^3\times(0,T)$.

The formula (5) is used in a little diferent form also by V.Scheffer in [4].

In the following, we shall estimate for each $t\in(0,T)$ the dimension of a set of such $x\in R^3$, where the solution u may have eventually an infinite value.

It may be shown that

$$\Delta\varphi(x,t) = -\frac{1}{8\pi^{3/2}}\frac{1}{t^{3/2}}\exp\left(-\frac{|x|^2}{4\cdot t}\right),$$

$$(6)\quad |\Delta\varphi(x,t)| \leq \text{const}/(t+|x|^2)^{3/2}.$$

In order to estimate $|u_i(x,t)|$, we shall estimate each term on the right-hand side of (5). The most important of these terms is the second one because we may achieve the first term and the third term to be as "good" as we need choosing u_0 and f sufficiently smooth. The second term may be expressed as

$$\int_0^t \int_{R^3} \Delta\varphi(x-y,t-\tau)\cdot u_k(y,\tau)\cdot u_{i,k}(y,\tau)\,dy\,d\tau -$$

$$- \int_0^t \int_{R^3} \varphi_{,ij}(x-y,t-\tau)\cdot u_k(y,\tau)\cdot u_{j,k}(y,\tau)\,dy\,d\tau .$$

Let us denote $J_1(x,t)$ the first of these integrals and $J_2(x,t)$ the second of them. Using (6), we can derive:

$$(7)\quad |J_1(x,t)| \leq$$
$$\leq \text{const} \int_0^t \int_{R^3} \frac{1}{((t-\tau)+|x-y|^2)^{3/2-\alpha}} \frac{|u_k(y,\tau)\cdot u_{i,k}(y,\tau)|}{(t-\tau)^\alpha}\, dy\, d\tau .$$

If $\alpha < 1/2$ then

$$\int_0^t \int_{R^3} \frac{|u_k(y,\tau)\cdot u_{i,k}(y,\tau)|}{(t-\tau)^\alpha}\, dy\, d\tau < +\infty .$$

Let $a \in (2,3)$. There exists $\alpha \in (0,1/2)$ so that $3-2\alpha < a$. Let us denote by $G_a(t)$ the set

$$\left\{ x \in R^3 \,\middle|\, \exists m_x : \int_{B(x,2^{-m})} \int_0^t \frac{|u_k(y,\tau)\cdot u_{i,k}(y,\tau)|}{(t-\tau)^\alpha}\, d\tau\, dy \leq 2^{-am} \text{ for } m \geq m_x \right\}$$

(where $B(x,2^{-m}) = \{ y \mid |x-y| < 2^{-m} \}$). It is a consequence of Lemma 4.2 in [1] that the a-dimensional Hausdorff measure of $(R^3 - G_a(t))$ is equal to zero and so $\dim(R^3 - G_a(t)) \leq a$.

Let $x \in G_a(t)$. We denote $U_0 = R^3 - B(x,2^{-m_x})$, $U_1 = B(x,2^{-m_x}) - B(x,2^{-m_x-1})$, $U_2 = B(x,2^{-m_x-1}) - B(x,2^{-m_x-2})$, ...

Using (7), we get

$$|J_1(x,t)| \leq$$
$$\leq \text{const} \sum_{r=0}^{+\infty} \int_{U_r} \int_0^t \frac{1}{((t-\tau)+|x-y|^2)^{3/2-\alpha}} \frac{|u_k(y,\tau)\cdot u_{i,k}(y,\tau)|}{(t-\tau)^\alpha}\, dy\, d\tau \leq$$
$$\leq \text{const} \sum_{r=0}^{+\infty} \int_{U_r} \int_0^t 2^{(3-2\alpha)(m_x+r)} \frac{|u_k(y,\tau)\cdot u_{i,k}(y,\tau)|}{(t-\tau)^\alpha}\, dy\, d\tau \leq$$
$$\leq \text{const}\ 2^{(3-2\alpha)m_x} \int_{U_0} \int_0^t \frac{|u_k(y,\tau)\cdot u_{i,k}(y,\tau)|}{(t-\tau)^\alpha}\, dy\, d\tau +$$
$$+ \text{const} \sum_{r=1}^{+\infty} 2^{(3-2\alpha)(m_x+r)} \int_{B(x,2^{-m_x-r+1})} \int_0^t \frac{|u_k(y,\tau)u_{i,k}(y,\tau)|}{(t-\tau)^\alpha}\, dy\, d\tau \leq$$
$$\leq C_1 + \text{const} \sum_{r=1}^{+\infty} 2^{(3-2\alpha)(m_x+r)}\, 2^{(-m_x-r+1)a} < +\infty .$$

We proved the following lemma:

Lemma Let $Q(t)$ be a set of such $x \in R^3$ that $|J_1(x,t)| < +\infty$. Then $\dim(R^3 - Q(t)) \leq a$ for each $a \in (2,3)$ and hence $\dim(R^3 - Q(t)) \leq 2$.

It is possible to prove the same lemma if we consider $J_2(x,t)$ instead of $J_1(x,t)$. If $f \in L^{3/2}(R^3 \times (0,T))^3$ then the analogous result could be proved also in the case of the third term on the right-hand side of (5). If $u_0 \in L^2(R^3)^3$ and $\operatorname{div} u_0 = 0$ (in the sence of distributions) then the first term on the right-hand side of (5) is finite for $t > 0$. Thus, the following theorem holds:

Theorem Let $u_0 \in L^2(R^3)^3$ so that $\operatorname{div} u_0 = 0$ in the sence of distributions, $f \in L^{3/2}(R^3 \times (0,T))^3$. Suppose that u is a weak solution of the problem (1), (2), (3) which is modified on a set of a measure zero in such a way that it satisfies (5) for all $[x,t] \in R^3 \times (0,T)$. Let $G(t)$ (for $t \in (0,T)$) be a set of such $x \in R^3$ that $|u(x,t)| < +\infty$. Then $\dim(R^3 - G(t)) \leq 2$.

References

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