GENERIC BIFURCATIONS OF VECTOR FIELDS WITH A SINGULARITY OF CODIMENSION 3

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Consider the vector field

$$\dot{x} = Ax + G(x) , \qquad (1)$$

where $x=(x_1,x_2)$, the matrix A is equivalent to the nilpotent Jordan block S with 1 above the diagonal and zeros elsewhere, $G=(G_1,G_2)$, $G(0)= 0$, $G_i(x)=(P_i x,x) + h_i(x)$, P_i are symmetric matrices, $h_i(x) = o(\|x\|^2)$, $i=1,2$, $(.,.)$ is the scalar product on R^2.

There is a smooth regular mapping transforming the vector field (1) into the form

$$\dot{x}_1= x_2, \quad \dot{x}_2= (Tx, x) + t_{30}x_1^3 + T_3(x) + h(x) , \qquad (2)$$

where $T=(t_{ij})$ is a symmetric matrix, $T_3(x)$ is a homogeneous polynomial of degree 3 in x_1,x_2, which does not contain the power x_1^3 and $h(x)= o(\|x\|^3)$. The property $t_{11}= 0$ is invariant with respect to regular transformations of coordinates keeping the origin fixed. If $t_{11}= 0$ then the number $q= t_{30}t_{12}^{-1}$ is also invariant with respect to these transformations.

Let Γ^∞ be the set of all C^∞-vector fields in R^2 of the form (1) and J^k be the set of k-jets of the vector fields from Γ^∞. The set of 2-jets of the vector fields from Γ^∞ for which the matrix of the linear part at 0 is equivalent to the Jordan block S and $t_{11}=0$ is a smooth submanifold Σ of J^2 of codimension 3.

A critical point of the vector field $v \in \Gamma^\infty$ is called nondegenerate if $t_{12}t_{30} \neq 0$ and degenerate otherwise. The condition of degeneracy defines a subset of J^3, which is an algebraic submanifold of J^3 of codimension 4.

Consider the following family of vector fields

$$\dot{x} = f(x, \varepsilon)\ , \qquad (3)$$

where $x=(x_1,x_2)$, $\varepsilon=(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $f_\varepsilon(x)=f(x,\varepsilon)\in C^\infty$, f_0 is the vector field (1). The set of all such families we denote by H^∞. Let G^∞ be the set of all germs at the origin of the vector fields from Γ^∞. We denote by $\tilde{g}\in G^\infty$ the germ, represented by $g\in\Gamma^\infty$. Given any $f\in H^\infty$ we define the mapping $\rho(f)$: $(x,\varepsilon)\to \pi_2\tilde{f}_\varepsilon(x)$, where $\pi_2: G^\infty \to J^2$ is the natural projection.

The family (3) is called nondegenerate if the critical point $x=0$ of the vector field f_0 is nondegenerate and the mapping $\rho(f)$ is transversal to the manifold Σ at the point $(x,\varepsilon)=(0,0)$.

Theorem 1. There exists an open, dense subset H_1^∞ of H^∞ such that if $f\in H_1^\infty$ then f is nondegenerate and there is a smooth change of coordinates $y=y(x,\varepsilon)$, $\mu=\varphi(\varepsilon)$ such that in these coordinates the family f has the form

$$v_\mu^\sigma:\quad \begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \gamma_1^\sigma(\mu) + \gamma_2^\sigma(\mu)y_1 + \mu_3 y_1^2 + \sigma y_1^3 + b_{11}y_1y_2 + b_{02}y_2^2 + \\ &\quad + b_{21}y_1^2y_2 + y_2^2\,\phi(y,\mu), \end{aligned}$$

where $\phi\in C^\infty$, $\sigma = \operatorname{sign} q$, $\gamma_1^\sigma(\mu)= 2\sigma\mu_1 + \mu_2\mu_3 + \frac{1}{27}\mu_3^3$, $\gamma_2^\sigma(\mu)= \sigma(3\mu_2 + \frac{1}{3}\mu_3^2)$, $b_{11}>0$. The numbers b_{11}, sign N, where $N= b_{11}b_{02} + b_{21}$, are invariants of the germ $\tilde{f}$, represented by the family f.

The critical points of v_μ^σ have the form $(z,0)$, where z is a real root of the algebraic equation

$$\sigma y^3 + \mu_3 y^2 + \gamma_2^\sigma(\mu)y + \gamma_1^\sigma(\mu) = 0\ . \qquad (4)$$

The discriminant of the equation (4) has the form $D=D(\mu)=\mu_1^2 + \mu_2^3$. Denote $\mathcal{D}= \{\mu \mid D(\mu)=0\}$, $\mathcal{D}^+= \{\mu \mid D(\mu)>0\}$, $\mathcal{D}^- = \{\mu \mid D(\mu)<0\}$, $H^\pm=\{\mu \mid \mu_1= \pm h(\mu_2)\}$, $h(\mu_2)=(-\mu_2)^{\frac{3}{2}}$, $\mu_2\leqq 0$, i. e. $\mathcal{D}= H^+\cup H^-$. Let $S_1= \mathcal{D}^+\cup\{0\}$, $S_2= \mathcal{D}\setminus\{0\}$, $S_3= \mathcal{D}^-\setminus\{0\}$,

$G_i = \{\mu \mid \gamma_i^-(\mu) = 0\}$, $G_i^+ = \{\mu \mid \gamma_i^-(\mu) > 0\}$, $G_i^- = \{\mu \mid \gamma_i^-(\mu) < 0\}$, $M_k = \{\mu \mid \gamma_k^+(\mu) = 0\}$, $M_k^+ = \{\mu \mid \gamma_k^+(\mu) > 0\}$, $M_k^- = \{\mu \mid \gamma_k^+(\mu) < 0\}$, $i,k = 1,2$, $\alpha^- = G_1 \cap G_2$, $\alpha^+ = M_1 \cap M_2$. The sets G_1, G_2, M_1, M_2 are smooth surfaces in R^3.

Theorem 2. If $f \in H_1^\infty$ then there exists a neighbourhood U of the origin in the parameter space and a neighbourhood V of the origin in the phase space such that for $\mu \in U \cap S_k$ ($k = 1,2,3$) the vector field v_μ^σ has exactly k critical points in V.

Zero eigenvalues. If $\mu \in U \setminus \mathfrak{D}$, where U is a sufficiently small neighbourhood of the origin, then for any critical point K the matrix $L(K)$ of the linear part of v_μ^σ computed at K has no zero eigenvalue. If $\mu \in \mathfrak{D}$ there is a critical point K_1, for which the matrix $L(K_1)$ has a zero eigenvalue (it has multiplicity 2 only if $\mu \in \alpha^\sigma$) and for the second critical point K_2 the matrix $L(K_2)$ has no zero eigenvalue.

Pure imaginary eigenvalues. Let K be a critical point of v_μ^+ (v_μ^-). The matrix $L(K)$ has pure imaginary eigenvalues if and only if $K=(0,0)$, $\mu \in M_1 \cap M_2^-$ ($G_1 \cap G_2^-$).

Bifurcations for v_μ^+. By [1, Theorem 6.2.1], for $\mu \in S_1$ the only critical point is a saddle. Let P_0 be the plane through the point $\mu_0 \in \mathfrak{D}^-$ parallel to the (μ_1,μ_3)-plane. Let $w_\mu^+ = v_\mu^+$ for $\mu \in P_0$ and let $Q_1 \in H^+$, $Q_2 \in H^-$ be the end-points of the curve $h = P_0 \cap M_1 \cap M_2^- \cap (\mathfrak{D}^- \cup \mathfrak{D})$. Each of the vector fields $w_{Q_1}^+$ and $w_{Q_2}^+$ has two critical points: a saddle K_1 and a saddle node K_2, for which the matrix $L(K_2)$ has zero eigenvalue of the multiplicity 2. There exist neighbourhoods U_1, U_2, V of Q_1, Q_2 and K_2, respectively, such that the bifurcation diagram for $w_\mu^+|_V$ in U_1 and U_2 corresponds to the bifurcation diagram of Bogdanov's normal form with positive and negative signature, respectively (see [3, Theorem 1]). For $\mu \in h \cap U_1$ ($h \cap U_2$) two critical points are saddles and there is one critical point K, for which the matrix L(K) has pure imaginary eigenvalues and the first Ljapunov focus number L_1 [2] is positive (negative). It is possible to show that there is exactly one point Q on h, where L_1 changes its sign and sign L_2= sign N, where L_2 is the second Ljapunov focus number (for $\mu = Q$). The number N is generically nonzero. The bifurcation diagram in a neighbourhood of the point Q looks like the one described in [2, p.p. 208, p.p. 243].

Bifurcations for v_μ^-. For $\mu \in G_1 \cap \mathcal{D}^+$ the critical point is a focus. There are curves $\gamma_1, \gamma_2 \subset G_1 \cap \mathcal{D}^+ \cap \{\mu | \mu_2 < 0\}$, $\gamma_3 \subset \subset G_1 \cap \mathcal{D}^+ \cap \{\mu | \mu_2 > 0\}$, $\overline{\gamma}_i \setminus \gamma_i = \{0\}$, i= 1,2,3, such that for any $Q_0 \in \gamma_1 \cup \gamma_2 \cup \gamma_3$ there is $L_1 = 0$ and sign L_2= sign N. The bifurcations near this point can be described using the results from [2]. For $\mu \in [G_1 \cap \mathcal{D}^+ \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)] \cup [G_1 \cap \mathcal{D}^- \cap G_2^-]$ there is $L_1 \neq 0$. Let $\tilde{P}_0$ be the plane through $\mu_0 \in \mathcal{D}^-$ parallel to the (μ_1, μ_3)-plane. The set $\tilde{P}_0 \cap G_1 \cap G_2^- \cap (\mathcal{D}^- \cup \mathcal{D})$ consists of two components with endpoints $\tilde{Q}_1 \in H^-$, $R_1 \in H^+$ and $\tilde{Q}_2 \in H^+$, $R_2 \in H^-$, respectively. The bifurcation diagram in a neighbourhood of $\tilde{Q}_1$ and $\tilde{Q}_2$ corresponds to the bifurcation diagram of Bogdanov's normal form with positive and negative signature, respectively.

References

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