EQUADIFF 5

A. F. Izé

# On a fixed point index method for the analysis of the asymptotic behavior and boundary value problems of process and semidynamical systems

In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 145--154.

Persistent URL: http://dml.cz/dmlcz/702279

**Terms of use:**

© BSB B.G. Teubner Verlagsgesellschaft, 1982

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

# ON A FIXED POINT INDEX METHOD FOR THE ANALYSIS OF THE ASYMPTO - TIC BEHAVIOR AND BOUNDARY VALUE PROBLEMS OF PROCESS AND SEMIDY- NAMICAL SYSTEMS

A. F. Izé
Universidade de Sao Paulo , B r a z i l

## 1. INTRODUCTION

Važewski principle [19] plays an important role in the study of ordinary differential equations. Its applicability is largely due to the fact that in a finite dimensional euclidean space, the unity sphere is not a retract of the closed unit ball. Since this is no longer true in infinite-dimensional Banach space the direct extension of Važewski's principle to processes or semidynamical systems on infinite dimensional Banach spaces has a very limited applicability.

Since in finite dimensional spaces the fact that the unity sphere is not a retract of the closed unit ball is equivalent to the fact that every continuous mapping of the unity closed convex ball has a fixed point, the main idea of this work is to develop a method based on fixed point index properties instead of retraction properties.

Our fixed point formulation, Theorem 2, is essentially equivalent, in finite dimension, to Važewski Theorem. Although in finite dimension, Važewski Theorem is no longer applicable, Theorem 2 and also Theorem 3 are applicable and give deeper results since fixed point methods have proved to be very useful in the solution of differential equations either in finite or infinite dimensional spaces. After that we go further and generalize theorem 2 and 3 using Leray-Schauder degree theory or the fixed point index theory for compact or condensing maps. These generalizations, theorems 4, 5 and 6 are stronger even in finite dimension than Važewski Theorem

After the appearence of Ważewski paper several papers arised applying Ważewski principle to the asymptotic behavior of ordinary differential equations, C. Olech [13] , V. Pliss [16] , Mikolajska [11] , N. Onuchic [14] , A. F. Izé [9] and others. Kaplan, Lasota and Yorke [10] applied Ważewski method to boundary value problem and C. Conley [3] also applied Ważewski method to a boundary value problem for a difusion equations in biology. Since our aproach uses Ważewski basic ideas in conection with degree theory it should give, even in finite dimensions much better results and can be applied also to boundary problems in Hilbert spaces.

## 2. PROCESSES

Definition 1. [2] Suppose $X$ is a Banach space $R^+ = [0, \infty)$, $u: R \times X \times R^+ \to X$ is a given mapping and define $U(\sigma, t): X \to X$ for $\sigma \in R$, $t \in R^+$ by

$$U(\sigma,t)x = u(\sigma,x,t).$$

A process on $X$ is a mapping $u: R \times X \times R^+ \to X$ satisfying the following properties

(i) $u$ is continuous ,

(ii) $U(\sigma,0) = I$ (identity),

(iii) $U(\sigma+s,t)U(\sigma,s) = U(\sigma,s+t)$.

A process is said to be an autonomous process or a semidynamical system if $U(\sigma,t)$ is independent of $\sigma$, that is , $T(t) = U(0,t)$ , $t \geq 0$ . Then $T(t)x$ is continuous for $(t,x) \in R^+ \times X$ .

In a process $u(\sigma,x,t)$ can be considered as the state of a system at time $\sigma + t$ if initially the state at time $\sigma$ was $x$.

Processes arise from ordinary differential equations evolution equations, retarded and neutral functional differential equations and partial differential equations.

Definition 2. Suppose $u$ is a process on $X$. The trajectory $\tau^+(\sigma,x)$ through $(\sigma,x)\in R \times X$ is the set in $R \times X$ defined by

$$\tau^+(\sigma,x) = \{(\sigma+t,\ U(\sigma,t)x \mid t\in R^+\}.$$

The orbit $\gamma^+(\sigma,x)$ through $(\sigma,x)$ is the set in $X$ defined by

$$\gamma^+(\sigma,x) = \{U(\sigma,x)x \quad , \quad t\in R^+\}$$

Definition 3. If $u$ is a process on $X$ then an integral of the process on $R$ is a continuous function $y:R\rightarrow X$ such that for any $\sigma \in R$,

$\tau^+(\sigma,y(\sigma)) = \{(\sigma+t,\ y(\sigma+t)) \mid t \geq 0\}$. An integral $y$ is an integral through $(\sigma,x)\in R \times X$ if $y(\sigma) = x$.

We assume in the following that the integral through each $(\sigma,x)\in R \times X$ is unique.

We define

$\tau^{-1}(x) = (\sigma,y)\in R \times X \mid \exists t > 0$ such that

$U(\sigma,t)y = x$.

If $P_0 = (\sigma,x) \in R \times X$ and $z \in \gamma^+(\sigma,x)$, we define

$$t_z = \inf\{t \geq 0 \mid U(\sigma,t)x = z\}$$

$$Q_z = (\sigma+t_z\ ,\ U(\sigma,t_z)x)$$

$$[P_0\ ,\ Q_z] = \{(\sigma+t,\ \ U(\sigma,t)x \mid 0 \leq t < t_z\}$$

$$[P_0\ ,\ Q_z] = \{(\sigma+t,\ \ U(\sigma,t)x \mid 0 \leq t < t_z\}$$

$$(P_0\ ,\ Q_z] = \{(\sigma+t,\ \ U(\sigma,t)x \mid 0 < t \leq t_z\}$$

$$(P_0\ ,\ Q_z) = \{(\sigma+t,\ \ U(\sigma,t)x \mid 0 < t < t_z\}$$

Let $\Omega$ be an open set of $R \times X$, $\omega$ an open set of $\Omega$, $\omega\subset\Omega$, $\omega \neq \phi$ and $\partial\omega = \overline{\omega} \cap (\overline{\Omega - \omega})$ the boundary of $\omega$ with respect to $\Omega$. We put:

$S^o = \{P_o = (\sigma, x) \in \partial\omega \mid \exists\ t > 0$ e $z \in \gamma^+(\sigma, x)$ , with $(P_o , Q_z) \neq \emptyset$ and $(P_o , Q_z) \cap \overline{\omega} = \emptyset$

$S = \{Q \in \partial\omega \mid \exists\ P_o = (\sigma, x) \in \omega$ , com $Q \in \tau^+(\sigma, x)$ e $[P, Q) \subset \omega\}$

$S^* = S^o \cap S$ .

The points of $S$ are called egress points, the points of $S^*$ are called strict egress points.

Given a point $P_o = (\sigma, x) \in \omega$ , if the trajectory $\tau^+(\sigma, x)$ of the process is contained in $\omega$ for every $t > 0$ , we say that the trajectory is asymptotic with respect to $\omega$ , if the trajectory is not asymptotic with respect to $\omega$ then there is a $t > 0$ such that $(\sigma + t, U(\sigma, t)x) \in \partial\omega$ . Taking

$t_{P_o} = \{\min t > 0 \mid (\sigma + t, U(\sigma, t)x) \in \partial\omega\}$

$Q = (\sigma + t_{P_o} , U(\sigma, t_{P_o})x = C(P_o . \omega)$

we have

$[P_o , Q) \quad \omega$ .

The point $C(P_o , \omega)$ is called the consequent of $P_o$ .

Define $G$ to be the set of all $P_o = (\sigma, x) \in \omega$ such that there is $C(P_o , \omega)$ and $C(P_o , \omega) \in S^*$ .

Consider the mapping

$k: G \cup S^* \to S^*$ .

$K(P_o) = C(P_o , \omega)$ if $P_o \in \omega$ and $K(P_o) = P_o$ if $P_o \in S^*$ .

The proof of the following is standard, see for example [2] , [13] .

<u>Lemma 1</u>:The mapping $K: G \cup S^* \to S^*$ is continuous.

<u>Theorem 1</u>:(First form) Assume that there exist sets $S \subset \partial\omega$ and $Z \subset \omega \cup S$ , $Z \neq \emptyset$ satisfying the conditions:

(i) $S = S^*$ (that is, the points of $S$ are egress points).

(ii) $Z$ is compact and convex.

(iii) $Z \cap S$ is a retract of $S$.

(iv) There exists a continuous mapping $\phi : Z \cap S \rightarrow Z \cap S$ such that $\phi(P) \neq P$ for every $P \in Z \cap S$.

Then there is at least one point $P_0 = (\sigma, x) \in Z \cap \omega$ such that the trajectory $\tau^+(\sigma, x)$ is contained in $\omega$, that is, the trajectory $\tau^+(\sigma, x)$ is asymptotic with respect to $\omega$.

Proof: Assume the Theorem is not true. Hence for every $P_0 \in Z \cap \omega$ the trajectory through $P_0$ is not asymptotic with respect to $\omega$, that is, there exists $C(P_0, \omega)$. Since $S = S^*$, $C(P_0, \omega) \in S^*$. Then

$Z \cap \omega \subset G$.

From Lemma 1 the mapping $K$ is continuous.

From condition (iii) there is a retraction.

$r : S \rightarrow Z \cap S$

The mapping

$R = r \cdot K : Z \cap \omega \rightarrow Z \cap S$

is continuous and takes $P_0 = (\sigma, x)$ into $C(P_0, \omega) = (\sigma + t_Q, U(\sigma, t_Q)x) \in Z \cap S$.

From condition (iv) the mapping $\phi$ takes $C(P_0, \omega)$ into $S(C(P_0, \omega)) = C'(P_0, \omega) \neq C(P_0, \omega)$ and then the composite mapping

$\phi \cdot rK : Z \rightarrow Z$

is continuous and never has a fixed point. This is in contradiction with Schauder fixed point theorem. Then the trajectory $\tau^+(P_0)$ is asymptotic with respect to $\omega$ and the theorem is proved.

Remark 1: One simple situation in which condition (iv) is satisfied is when $Z \cap S$ is symmetric, that is, $-Z \cap S \subset Z \cap S$ . The mapping $S: Z \cap S \to Z \cap S$ is defined by $\phi(P) = -P$ .

Theorem 2: (Second form) Assume that there exist sets $S \subset \partial\omega$ and $Z \subset \omega \cup S$ , $Z \neq \emptyset$ , satisfying the conditions:

(i) $S = S^{*}$.

(ii) $Z$ is closed bounded convex.

(iii) $Z \cap S$ is a retract of $S$.

(iv) There is a continuous mapping $\phi: Z \cap S \to Z \cap S$ such that $\phi(P) \neq P$ for every $P \in Z \cap S$.

(v) $U$ is compact.

Then there is at least one point $P_0 = (\sigma, x) \in Z \cap \omega$ such that the trajectory $\mathcal{T}^{+}(\sigma, x)$ is contained in $\omega$ , that is, the trajectory through $P_0$ is asymptotic with respect to $\omega$ .

The proof follows as in Theorem 1. Since $U$ is compact the mapping $K$ that takes $P_0$ into $C(P_0, \omega)$ is compact. $r$ is a retraction and then $R = r.K: Z \cap \omega \to Z \cap S$ is compact. The mapping $\phi.rK: P_0 \to C'(P_0, \omega)$ is also compact and never has a fixed point. This is in contradiction with Schauder fixed point Theorem since $\phi.r.K: Z \to Z$ and $Z$ is closed convex, bounded.

Then the trajectory through $P_0$ is asymptotic with respect to $\omega$ and the theorem is proved.

Theorem 2 can be extended by using the fixed point index theory or Leray-Schauder degree in the following way.

Theorem 3: Assume that there exists sets $S_1 \subset S$ and $Z \subset \omega \cup S_1$ , $Z$ closed convex, $Z \neq \emptyset$ satisfying the conditions:

(i) $S = S^{*}$.

(ii) There exists a continuous mapping $\phi : S_1 \to S_1$ such that $\phi(P) \neq P$ for every $P \in S_1$.

(iii) U is compact.

(iv) $\deg(I - \phi U, Z \cap \omega) \neq 0$.

Then there exists at least one point $P_0 = (\sigma, x) \in Z \cap \omega$ such that either $C(P_0) \in S - S_1$ or $C(P_0)$ does not exist, that is, the trajectory $\mathscr{C}^+(\sigma, P_0)$ is asymptotic with respect to $\omega$.

Proof: Assume that the theorem is not true. Then $C(P_0) \in S_1$ for every $P_0 \in Z \cap \omega$ and then $Z \cap \omega \subset G$. Then $Z = (Z \cap S_1) \cup Z \cap \omega \subset S \cup G$. From Lemma 1 the map K is continuous and the restriction of K to $Z \cup S_1$ that we note by K, is also continuous since U is compact the map K that takes $P_0$ into $C(P_0)$ is compact. The transformation $\phi U$ is also compact and $\phi U(P) \neq P$ for every $P \in Z \cup S$. This implies that $\deg(I - \phi U, Z \cup \omega) = 0$ what is a contradiction and the theorem is proved.

A less general formulation of theorem 3 that is more similar to theorem 2 is the following.

<u>Theorem 4</u>: Assume that there exist sets $S_1 \subset S$ and $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, Z closed convex satisfying the conditions:

(i) $S = S^*$.

(ii) $Z \cap S_1$ is a retract of $S_1$, that is, there exists a retraction $r: S_1 \to Z \cap S_1$.

(iii) U is compact.

(iv) There exists a continuous map $\phi : Z \quad S_1 \to Z \cap S_1$ such that $\phi(P) \neq P$ for every $P \in Z \cap S_1$.

(v) $\deg(I - \phi U, Z \cap \omega) \neq 0$.

Then there exists at least one point $P_0 = (\sigma, x) \in Z \cap \omega$ such that either $C(P_0) \in S - S_1$ or $C(P_0)$

does not exist.

The proof is similar to theorem 2 and 3. Nussbaum [12] defined a fixed point index and consequently a degree for $\alpha$-set contradictions and condensing maps.

Theorem 5: Assume that there exist sets $S_1 \subset S$ and $Z \subset \omega \cup S_1$, $Z \neq \emptyset$, $Z$ closed convex satisfying the conditions:

(i) $S = S^*$.

(ii) There exists a continuous map $\phi : S_1 \to S_1$ such that $\phi(P) \neq P$ for every $P \in S_1$.

(iii) $\phi U$ is condensing.

(iv) $\deg(I - \phi U, Z \cap \omega) \neq 0$.

Then there exists at least one point $P \in S - S_1$ such that either $C(P_0) \subset S - S_1$ or $C(P_0)$ does not exist, that is, the trajectory $\tau^+(\sigma, x)$ through $(\sigma, x)$ is contained in $\omega$.

REFERENCES


[1] BROWDER, F. E. - On the fixed point indes for continuous mapping of locally connected spaces - Summa Brasiliensis Mathematica, vol. 4, fasc. 7 (1960).

[2] CONLEY, C. C. - "Isolated Invariant Sets and the Morse Index" CBMS 38, Amer. Math. Soc., Providence, R. I.

[3] CONLEY, C. C. - An application of Wažewski method to a nonlinear boundary value problem which arises in population genetics, J. Math. Viol. 2, (1975), 241-249.

[4] DAFERMOS, C. M. - An invariance principle for compact processes, J. Diff. Eqs. 9, 239-252, (1971).

[5] DRIVER, R. - Existence and continuous dependence of a neutral functional differential equation. Arch. Rat. Mech. Anal., 19, 149-166 (1965).

[6] FRIEDMAN, A. - Partial differential equations - Holt, Rinehart and Winston Inc., (1969).

[7] HALE, J. K. - Theory of functional differential equations, Springer Verlag, (1974).

[8] HENRY, D. - Geometric theory of semilinear parabolic equations, Lecture Notes, University of Kentucky, Springer Verlag.

[9] IZÉ, A. F. - Asymptotic behavior of ratio between the components of solutions of a linear system of ordinary differential equations and its applications to determination of Lyapunov characteristic numbers. Annali di Matematica Pura ed Applicata, Serie IV, tomo LXXXV, p. 1, (1970).

[10] KAPLAN, J. L.; LASOTA, A.; YORKE, J. A. - An application of of the Ważewski retract method to boundary value problems.

[11] MIKOLAJSKA, Z. - Sur les mouvements asymptotiques d'un point matériel mobile dans le champ des forces repousants. Bull. Acad. Polon. Sci. Vol. I, No. 1, 2 (1953).

[12] NUSSBAUM, R. D. - "The fixed point index for local condensing maps", Ann. Mat. Pura Appl. 89, 217-258, (1971).

[13] OLECH, C. - On the asymptotic behavior of the solutions of a system of ordinary differential equations. Bull. Acad. Pol. Sc. vol. IV, No 9, p. 555, (1956).

[14] ONUCHIN, N. - On the Asymptotic Behavior of the Solutions of Functional Differential Equations, In "Differential Equations and Dynamical Systems" (J. K. Hale and J. P. LaSalle, Eds.), Academic Press, New York, (1967).

[15] PELCZAR, A. - Some generalizations of the retract theorem of Wažewski, with applications to ordinary and partial differential equations of the first order. Annales Polonici Mathematici, XXIX, (1974).

[16] PLISS, A. - On a topological method for studying the behavior of the integrals of ordinary differential equations Bull. Acad. Polon. Sci. c-1, III, p. 2, (1954).

[17] RYBAKOWSKI, K. - Wažewski principle for retarded functional differential equations, J. Differential Equations 36 , 117-138, (1980).

[18] RYBAKOWSKI, K. P. - On the homotopy index for infinite dimensional semiflows - Trans. Amer. Math. Soc. to appear.

[19] WAŽEWSKI, T. - Sur un principle ropologique de l'exament de l'allure asymptotique des intégrales des équations differentielles ordinaires. Ann. Soc. Polon. Math. 20, (1947).