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In: Michal Greguš (ed.): Equadiff 5, Proceedings of the Fifth Czechoslovak Conference on Differential Equations and Their Applications held in Bratislava, August 24-28, 1981. BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982. Teubner-Texte zur Mathematik, Bd. 47. pp. 131--135.

Persistent URL: http://dml.cz/dmlcz/702276

OPTIMIZATION OF THE DOMAIN IN ELLIPTIC UNILATERAL BOUNDARY VALUE PROBLEMS BY FINITE ELEMENT METHOD

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Introduction

In optimal design some problems of technical interest remain open, to the authors' knowledge. Thus in some problems of a unilateral contact between elastic bodies the shape of the boundaries should be optimized to obtain minimal cost functional such as the integral of energy, contact forces or displacements.

It is the aim of the paper to start the analysis of this class of problems on a simplified model with a unilateral problem in R^2 for the Poisson equation and Signorini's boundary conditions. On a given part of the boundary the Dirichlet homogeneous condition is prescribed and the remaining part - with unilateral conditions - has to be determined.

In Section 1 we prove the existence of a solution for 4 different cost functionals and for one common state problem, which is formulated in terms of a variational inequality on a variable domain. In Section 2 finite element approximations are proposed, employing piecewise linear approximations of the unknown part of the boundary and piecewise bilinear finite elements on a uniform mesh in a reference square domain. In Section 3 we study the convergence of the approximations and in Section 4 some numerical methods are discussed.

1. Existence of a solution to the model problems.

We introduce the following model problems. Let $\Omega(v) \in R^2$ be the domain

$$\Omega(v) = \{0 < x_1 < v(x_2),\ 0 < x_2 < 1\},$$

where the function v is to be determined from the problem

$$(P_i) \qquad \mathcal{J}_i(v) = \min_{w \in U_{ad}} \mathcal{J}_i(w).$$

Here

$$U_{ad} = \{w \in C^{(0),1}([0,1]) \text{ (i.e. Lipschitz function)}$$

$$0 < \alpha \leqq w \leqq \beta,\quad |dw/dx_2| \leqq C_1,\quad \int_0^1 w(x_2)\, dx_2 = C_2\},$$

with given constants α, β, C_1, C_2, $i = 1,2,3,4$, $\gamma_i(w) = J_i(y(w))$, z_0 = const is given,

$$J_1(y(w)) = \int_{\Omega(w)} (y(w)-z_0)^2\, dx, \quad J_2(y(w)) = \int_{\Omega(w)} |\nabla y(w)|^2\, dx,$$

$$J_3(y(x)) = \int_0^1 y(w)|_{\Gamma(w)}\, dx_2, \quad J_4(y(w)) = \int_0^1 (y(w)|_{\Gamma(w)} - z_0)^2\, dx_2,$$

where $\Gamma(w)$ is the graph of w and $y(w)$ is the solution of the following unilateral boundary value problem :

$$-\Delta y = f \quad \text{in} \quad \Omega(w), \tag{1.1}$$

$$y \geqq 0, \quad \frac{\partial y}{\partial \nu} \geqq 0, \quad y\frac{\partial y}{\partial \nu} = 0 \quad \text{on} \quad \Gamma(w),$$

$$y \geqq 0 \quad \text{on} \quad \partial\Omega(w) \doteq (w).$$

Here $f \in L^2(\Omega_\beta)$ is given, $\Omega_\beta = (0,\beta) \times (0,1)$ and $\partial y/\partial\nu$ denotes the derivative with respect to the outward normal to $\Gamma(w)$.

It is well known that the state problem (1.1) can be formulated in terms of a variational inequality, as follows:

let $K(w) = \{z \in H^1(\Omega(w)) \mid z \geqq 0 \text{ on } \Gamma(w),\ z=0 \text{ on } \partial\Omega(w) - \Gamma(w)\}$;

find $y \in K(w)$ such that for any $z \in K(w)$

$$\int_{\Omega(w)} \nabla y . \nabla(z-y)\, dx \geqq \int_{\Omega(w)} f(z-y)\, dx. \tag{1.2}$$

Theorem 1.1 The problem (P_i) has at least one solution for $i = 1,2,3,4$.

The proof can be found in [1] .

2. Approximate solutions by finite elements.

Let N be a positive integer and $h = 1/N$. Denote by e_j, $j = 1,\dots,N$ the interval $[(j-1)h, jh]$ and introduce the set

$$U_{ad}^h = \{w_h \in U_{ad} \mid w_h|e_j \in P_1 \ \forall j\},$$

where P_1 is the space of linear polynomials.

We define the reference square domain

$$\hat{\Omega} = (0,1) \times (0,1)$$

and the subsquares $\hat{K}_{ij} = e_i \times e_j$, generating a uniform mesh $\hat{\mathcal{K}}_h = \{\hat{K}_{ij}\}_{i,j=1}^{N}$.

Let Ω_h denote the domain $\Omega(w_h)$. We introduce the mapping

$$(2.1) \qquad F_h : \hat{\Omega} \longrightarrow \Omega_h, \quad F_h = (F_{1h}, F_{2h}),$$

$$F_{1h}(\hat{x}) = \hat{x}_1 w_h(\hat{x}_2), \quad F_{2h}(\hat{x}) = \hat{x}_2$$

and transform the state problem (1.2) on Ω_h into an equivalent problem on the domain $\hat{\Omega}$, by means of the mapping (2.1).

Using also some simplifications we define the approximate inequality on the reference domain $\hat{\Omega}$

$$(2.2) \qquad a_h(w_h;\, \hat{y}_h, \hat{z}_h - \hat{y}_h) \geqq L_h(w_h;\, \hat{z}_h - \hat{y}_h) \qquad \hat{z}_h \in \hat{K}_h,$$

where

$$\hat{K}_h = \{\hat{z}_h \in C^0(\bar{\hat{\Omega}}) \mid \hat{z}_h|_{\hat{K}_{ij}} \in Q_1 \quad \forall\, i,j\,,$$

$$\hat{z}_h \geqq 0 \quad \text{on} \quad \hat{\Gamma}\,, \quad \hat{z}_h = 0 \quad \text{on} \quad \partial\Omega - \hat{\Gamma}\}\,,$$

$$\hat{\Gamma} = \{(\hat{x}_1, \hat{x}_2) \mid \hat{x}_1 = 1, \quad 0 \leqq \hat{x}_2 \leqq 1\}\,,$$

Q_1 denotes the set of bilinear polynomials in $\hat{x}_1$ and $\hat{x}_2$,

$$a_h(w_h;\, \hat{y}_h, \hat{t}_h) = \sum_{i,j=1}^{N} \int_{\hat{K}_{ij}} \Big[\frac{1}{w_h^2(\xi_j)} \frac{\partial \hat{y}_h}{\partial \hat{x}_1} \frac{\partial \hat{t}_h}{\partial \hat{x}_1} +$$

$$+ \Big(\xi_i \frac{w_h'}{w_h(\xi_j)} \frac{\partial \hat{y}_h}{\partial \hat{x}_1} - \frac{\partial \hat{y}_h}{\partial \hat{x}_2}\Big)\Big(\xi_i \frac{w_h'}{w_h(\xi_j)} \frac{\partial \hat{t}_h}{\partial \hat{x}_1} - \frac{\partial \hat{t}_h}{\partial \hat{x}_2}\Big)\Big] w_h(\xi_j)\, d\hat{x},$$

$$\xi_i = (i - \tfrac{1}{2})h, \quad \xi_j = (j - \tfrac{1}{2})h, \quad w_h' = d\, w_h / d\, x_2$$

$$L_h(w_h;\ \hat{t}_h) = \frac{h^2}{4} \sum_{i,j=1}^{N} w_h(\xi_j) \sum_{k=1}^{N} \hat{f}(P_{ij}^k)\hat{t}_h(P_{ij}^k),$$

P_{ij}^k, $k = 1,2,3,4$, are the vertices of $\hat{K}_{ij}$.

In the same way, we introduce approximate cost functionals $\mathcal{J}_{ih}(w_h) \equiv J_{ih}(\hat{y}_h(w_h))$ and solve the approximate problems

$$(P_{ih}) \qquad \mathcal{J}_{ih}(u_h) = \min_{U_{ad}^h} \mathcal{J}_{ih}(w_h),$$

where $\hat{y}_h(w_k) \in \hat{K}_h$ are solutions of (2.2).

Remark. The same approach has been employed by Begis and Glowinski [2] in case of the state problem with classical boundary conditions - of Dirichlet's and Neumann's type - and of the cost functional $\mathcal{J}_1$.

3. Convergence of the finite element approximations.

We can prove a convergence of approximate solutions in some sense for the case of a smooth right-hand side f and for the cost functionals $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$.

Theorem 3.1 Assume that $f \in C^1(\bar{\Omega}_\beta)$. Let $\{u_h\}$, $h \to 0$, be a sequence of solutions of the approximate problems (P_{ih}), $i = 1,2,3$, and let $\hat{y}_h = \hat{y}_h(u_h)$ be the corresponding solutions of (2.2) (with $w_h \equiv u_h$), $y_h = \hat{y}_h \circ F_h^{-1}$.

Then a subsequence of $\{u_h\}$ exists such that for $h \to 0$

$$(3.1) \qquad u_h \to u \text{ uniformly},$$

$$(3.2) \qquad y_h \rightharpoonup y(u) \text{ weakly in } H^1(G_m) \quad \forall m > 1/\alpha,$$

where u and $y(u)$ is a solution of the problem (P_i) and of the inequality (1.2) with $w = u$, respectively, m is an integer,

$$G_m = \left\{(x_1,x_2) \mid 0 < x_1 < u(x_2) - \tfrac{1}{m},\ 0 < x_2 < 1\right\}.$$

Any uniformly convergent subsequence of $\{u_h\}$ has the properties mentioned above. For the proof - see [1].

Remark on numerical solution of the approximate problems

Since the functionals $\mathcal{J}_{ih}$ are not differentiable (cf. [3]), we are forced to apply methods of nonlinear programming, which do not employ the gradient of the functional. Moreover, each evaluation of $\mathcal{J}_{ih}(w_h)$ requires to solve the nonlinear state problem (2.2). To this end we apply the method of successive over-relaxation (SOR) with additional projection (see e.g. [4]).

Several methods of nonlinear programming are being tested on an example.

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