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On the transformation theory of ordinary second-order linear symmetric differential equations

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1. Introduction.

In this paper we consider some aspects of the transformation theory of the second-order, linear quasi-differential equation

$$M[y] = \lambda S[y] \text{ on } I \tag{1.1}$$

where I is an interval of the real line R, λ is a parameter taking values in the complex plane C, and M and S are symmetric, quasi-differential expressions of the second-order and first-order respectively.

Full details of the results given in this paper will be found in Everitt [4]. The general theory of symmetric quasi-differential expressions is given in the survey paper Everitt and Zettl [7].

An extensive account of the algebraic-analytical theory of transformation of second-order linear differential equations in Jacobian form is given in the treatise [2] of Borůvka. General remarks on transformation theory, with particular reference to unitary (isometric) transformations, are made by Dunford and Schwartz [3, pages 1498-1503].

The spectral theory of differential equations of the form (1.1) is of current interest; see Everitt [5] and the list of references therein.

2. Differential expressions and equations.

The general second-order quasi-differential expression M is of the form, see [7, section 1] and [4, section 2],

$$M[f] = -(p(f'-rf))' - \overline{r}p(f'-rf) + qf \text{ on } I \tag{2.1}$$

where a prime ' denotes classical differentiation on R, the coefficients $p,q : I \to R$, $r : I \to C$ and $p^{-1},q,r \in L_{loc}(I)$; here L denotes Lebesque integration. Similarly the general first-order quasi-differential expression S is of the form, see [4, section 2],

$$S[f] = i(\rho f)' + i\rho f' + wf \text{ on } I \tag{2.2}$$

where i is the complex number $(0,1)$, the coefficients $\rho,w : I \to R$, $\rho \in AC_{loc}(I)$ and $w \in L_{loc}(I)$; here AC denotes absolute continuity.

The existence of solutions of the quasi-differential equation (1.1), i.e.

$$-(p(y'-ry))' - \overline{r}p(y'-ry) + qy = \lambda\{i(\rho y)' + i\rho y' + wy\} \text{ on } I, \tag{2.3}$$

may be determined from the following equivalent system of linear first-order equations, compare with Naimark [9, sections 15 and 16],

$$Y' = AY \text{ on } I \tag{2.4}$$

where $Y = [y_1 y_2]^T$ is a 2×1 column vector and A is the 2×2 function matrix defined by

$$A = \begin{bmatrix} r - i\lambda\rho p^{-1} & p^{-1} \\ q - R & -\bar{r} - i\lambda\rho p^{-1} \end{bmatrix} \tag{2.5}$$

with $R = \lambda w + i\lambda\rho(r-\bar{r}) + \lambda^2\rho^2 p^{-1}$. From the given conditions on the coefficients p,q,r,w and ρ it follows that $R \in L_{loc}(I)$ and, in turn, that the matrix $A \in L_{loc}(I)$. Standard existence theorems for linear differential systems, see [9, sections 15 and 16] and [7, sections 3 and 5], then show that solutions of the differential equation (2.3) exist globally on I and, with suitable initial conditions, are holomorphic functions of the complex variable λ .

It should be noted that the $L_{loc}(I)$ conditions, required for the application of the standard existence theorems to the equation (2.3), are not only sufficient but also necessary for existence; see Everitt and Race [6].

3. Transformation theory. The aim of a transformation theory for a differential equation of the form (1.1) is to determine equivalent forms of the equation, and to study fundamental properties which are invariant under the transformations.

The first fundamental transformation of the equation (1.1), equivalently (2.3), shows that the complex-valued coefficient r may be taken as the null function, and this without loss of generality. Let the independent variable in (2.3) be $x \in I$; if new independent and dependent variables X and Y are determined by

$$X = x \qquad Y(X) = \{\mu(x)\}^{-1}y(x) \qquad (x \in I) \tag{3.1}$$

where, for some point $k \in I$,

$$\mu(x) = \exp\left[\int_k^x r(t)dt\right] (x \in I)$$

then a calculation shows that (2.3) is transformed to

$$-(PY')' + QY = \lambda\{i(RY)' + iRY' + WY\} \text{ on } I \tag{3.2}$$

where the prime ' now denotes differentiation with respect to the variable X . In (3.2) the transformed coefficients are given by, for all $x \in I$,

$$P(X) = |\mu(x)|^2 p(x) \qquad Q(X) = |\mu(x)|^2 q(x) \qquad R(X) = |\mu(x)|^2 \rho(x)$$

$$W(X) = |\mu(x)|^2 w(x) - 2|\mu(x)|^2 \rho(x) \operatorname{im}[r(x)] \; . \tag{3.3}$$

It may be seen that P,Q,R and W are real-valued on I and satisfy the same local integrability conditions, with $R \in AC_{loc}(I)$, as do the original coefficients p,q,ρ and w .

The transformation (3.1) is unitary (isometric) with respect to the Hilbert function spaces associated in the spectral theory of the original differential equation (2.3); the spectral properties of the two equations (2.3) and (3.2) are equivalent.

Re-writing the original equation (2.3) with $r = 0$ gives

$$-(py')' + qy = \lambda\{i(\rho y)' + i\rho y' + wy\} \text{ on } I . \tag{3.4}$$

The second fundamental transformation shows that if the additional condition

$$p(x) \geq 0 \ \underline{\text{for almost all}}\ x \in I \tag{3.5}$$

(or similarly $p(x) \leq 0$) is now imposed on the coefficient p then the equation (3.4) is transformed by

$$X = K + \int_k^x \{p(t)\}^{-1} dt \qquad Y(X) = y(x) \qquad (x \in I) , \tag{3.6}$$

where $k \in I$ and $K \in R$, to

$$-Y'' + QY = \lambda\{i(RY)' + i\,RY' + WY\} \text{ on } J . \tag{3.7}$$

Here the prime ' on Y denotes differentiation with respect to X , the interval J is the transformation of I under (3.6), and

$$Q(X) = p(x)q(x) \quad W(X) = p(x)w(x) \quad R(X) = \rho(x) \quad (x \in I). \tag{3.8}$$

As before Q,R and W are real-valued and satisfy local integrability conditions on the interval J , with $R \in AC_{loc}(J)$.

The transformation (3.6) is also unitary and the spectral properties of the two equations (3.4) and (3.7) are equivalent.

In general no further reduction of the original quasi-differential equation (2.3) is possible under the basic Lebesgue local integrability conditions on the coefficients.

It should be noted that for the generalized Sturm-Liouville equation, i.e. (2.3) with $\rho = 0$ and $w \geq 0$ on I , a reduction to the form

$$-y'' + qy = \lambda wy \text{ on } I \tag{3.9}$$

is possible, if (3.5) is satisfied; however, the weight coefficient w cannot be transformed out except in special circumstances (see the next section) and the spectral classification depends essentially on w . An example of this dependence is given by

$$-y''(x) = \lambda x^{\alpha} y(x) \qquad (x \in [0,\infty))$$

with $\alpha > -1$; it is known that certain spectral properties of this equation are essentially different for different values of the parameter α .

4. The Liouville transformation. If additional positivity and smoothness constraints are placed on the coefficients p,q,ρ and w then the differential equation (3.4) can be transformed by

$$X = K + \int_k^x \{w(t)/p(t)\}^{1/2} dt \qquad Y(X) = \{p(x)w(x)\}^{1/4} y(x) \quad (x \in I) , \tag{4.1}$$

where $k \in I$ and $K \in R$, to the Liouville normal form, here J is the transformed interval I,

$$- Y'' + QY = \lambda\{i(RY)' + iRY' + Y\} \text{ on } J \tag{4.2}$$

and, particularly, in the case when $\rho = 0$ to

$$- Y'' + QY = \lambda Y \text{ on } J . \tag{4.3}$$

For details and properties of this transformation see Birkhoff and Rota [1, chapter X, sections 1 and 5], [4, sections 4 and 5], Ince [8, section 11.4] and, in particular, Neuman [10].

However the Liouville transformation, in addition to requiring the positivity and smoothness conditions on the coefficients, enjoys only relatively few of the spectral invariant properties of the two transformations given in section 3 above. The Liouville transformation is a unitary (isometric) transformation but has to be counted as relatively unimportant in the general problem of the spectral classification of the quasi-differential equation (1.1). Further details and examples are given in [4, sections 4.3, 4.4 and 5.3].

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