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# Necessary conditions for the multiple integral problem and elliptic variational inequalities

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NECESSARY CONDITIONS FOR THE MULTIPLE

INTEGRAL PROBLEM AND ELLIPTIC

VARIATIONAL INEQUALITIES

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We are here concerned with the optimization problem

(1) $$\min\left\{\int_\Omega L(y(x),\nabla y(x))dx;\ y\in K\right\}$$

where $K=\{y\in W_0^{1,p}(\Omega);\ y(x)\geq\psi(x)\ \text{a.e.}\ x\in\Omega\}$ (the case $K=W_0^{1,p}(\Omega)$ is allowed ). Here $\Omega$ is a bounded, open subset of $R^n$ with a sufficiently smooth boundary $\Gamma$ and $\psi\in C(\bar{\Omega})$ is a given function such that $\psi\leq o$ on $\Gamma$. The integrand $L:R\times R^n\longrightarrow R$ is assumed to satisfy the condition:

(i) $L(y,z)\geq o$ *for all* $(y,z)\in R\times R^n$ *and for some positive* M.

(2) $$L(y+u,z+v)\leq\exp M(|(u,v)|\,(L(y,z)+M|(u,v)|\,(1+|(y,z)|\,))$$

*for all* $(y,z)$ *and* $(u,v)$ *in* $R\times R^n$.

Given a locally Lipschitzian function $\varphi:R^m\longrightarrow R$ we shall denote by $\nabla\varphi$ the gradient of $\varphi$ and by $D\varphi$ the mapping

(3) $$D\varphi(y)=\bigcap_{\delta>o}\ \bigcap_{\nu(N)=o}\overline{\text{conv}}\,\nabla\varphi(S(y,\delta)\setminus N)$$

where $\nu$ is the Lebesgue measure and $S(y,\delta)$ is the ball of radius $\delta$ and center y.

If $\partial\varphi$ is the generalized gradient in the sense of Clarke of $\varphi$ then as is readily seen we have: $D\varphi\subset\partial\varphi$.

THEOREM 1 *Assume that condition* (i) *holds*. *If* $y^*\in W_0^{1,p}(\Omega)$, $1\leq p<\infty$ *is optimal in problem* (1) *then there exists a function* $\eta\in L^1(\Omega;R^n)$ *and a Radon measure* $\mu$ *on* $\Omega$

such that $\operatorname{div}\eta - \mu \in L^1(\Omega)$ and

(4) $$(\operatorname{div}\eta - \mu, \eta) \in DL(y^*, \nabla y^*) \quad \text{a.e. on } \Omega$$

(5) $$\mu \leq 0 \text{ on } \Omega ; \mu = 0 \text{ on } \operatorname{int}\{x \in \Omega ; y^*(x) > \psi(x)\}.$$

The special case $K = W_0^{1,p}(\Omega)$ of this theorem has been proved via a minimax theorem by Clarke [3].

We give a brief outline of the proof. The detailed proof may be found in [2]. For $\lambda > 0$ consider the problem

(6) $$\min\left\{ \int_\Omega (L^\lambda(y, \nabla y) + \varkappa_\lambda(x,y))dx + \frac{1}{2}\| y - y\|_m^2 ; y \in W_0^{1,p}(\Omega) \right\}$$

wherein $\|\cdot\|_m$ is the norm of $H_0^m(\Omega)$, $m > n+2$, $\varkappa_\lambda(x,y) = (2\lambda)^{-1}|(y-\psi(x))^-|^2$ and

$$L^\lambda(y,z) = \int_{R \times R^n} L(y-\lambda\theta, z-\lambda\tau)\,\rho(\theta,\tau)\,d\theta\,d\tau$$

($\rho$ is a mollifier on $R^{n+1}$ ). Let $y_\lambda$ be optimal in problem (6). Since $L^\lambda$ and $\varkappa_\lambda$ are differentiable it follows that there exists $\eta_\lambda \in L^1(\Omega ; R^n)$ such that $\operatorname{div}\eta_\lambda \in L^1(\Omega) + H^{-m}(\Omega)$ and

(7) $$\eta_\lambda = \nabla_2 L^\lambda(y_\lambda, \nabla y_\lambda),\ \operatorname{div}\eta_\lambda = \nabla_1 L^\lambda(y_\lambda, \nabla y_\lambda) + \nabla\varkappa_\lambda(x, y_\lambda) + A(y_\lambda - y^*)$$

where $\nabla L^\lambda = (\nabla_1 L^\lambda, \nabla_2 L^\lambda)$ and $A$ is the canonical isomorphism of $H_0^m(\Omega)$ onto $H^{-m}(\Omega)$.

Next it follows that $y_\lambda - y^* \longrightarrow 0$ strongly in $H_0^m(\Omega)$ and by (2) we deduce that $\{\nabla L^\lambda(y_\lambda, \nabla y_\lambda)\}$ is a weakly compact subset of $L^1(\Omega ; R^{n+1})$. Thus we may assume that

(8) $$\begin{aligned} \eta_\lambda &\longrightarrow \eta \quad \text{weakly in } L^1(\Omega ; R^n) \\ \nabla_1 L^\lambda(y_\lambda, \nabla y_\lambda) &\longrightarrow \zeta \quad \text{weakly in } L^1(\Omega). \end{aligned}$$

Then arguing as in the proof of Lemma 3 in [1] we find that $(\zeta(x), \eta(x)) \in DL(y^*(x), \nabla y^*(x))$ a.e. $x \in \Omega$. Finally, there is $\mu \in \mathcal{D}'(\Omega)$ such that $\varkappa_\lambda(x, y_\lambda) \longrightarrow \mu$ weakly in $L^1(\Omega) + H^{-m}(\Omega)$. Since $\varkappa_\lambda(x, y_\lambda) = -\lambda^{-1}(y_\lambda - \psi(x))^- \leq 0$ a.e. $x \in \Omega$ we may infer that $\mu$ is a negative measure on $\Omega$. Then letting $\lambda$ tend to zero in (7) we find (4) as claimed.

We continue with the following consequence of Theorem 1.

THEOREM 2 <u>In Theorem 1 assume in addition that</u> $L(y,\cdot)$ <u>is convex for every</u> $y \in R$ <u>and for each</u> $k>0$ <u>there exists</u> $C_k$ <u>such that</u>

(9) $$L(y,z) \geq k|z|^p - C|y|^p - C_k \quad \text{for all} \quad (y,z) \in R \times R^n$$

<u>where</u> $C$ <u>is independent of</u> $k,y,z$ <u>and</u> $1 \leq p < \infty$. <u>Then there exist the functions</u> $y^* \in K$, $\eta \in L^1(\Omega;R^n)$ <u>and a Radon measure</u> $\mu$ <u>on</u> $\Omega$ <u>such that</u> $\operatorname{div}\eta - \mu \in L^1(\Omega)$ <u>and satisfying</u> Eqs. (4), (5).

To prove the theorem consider on the space $W_0^{1,p}(\Omega)$ the functional $I(y) = \int_\Omega L(y, \nabla y)dx$ if $y \in K$, $I(y) = +\infty$ if $y \bar{\in} K$. According to a general result given in [5], the functional I is sequentially weakly lower semicontinuous. On the other hand, (9) implies that every level subset $\{y \in W_0^{1,p}(\Omega);\ I(y) \leq \lambda\}$ is weakly compact. Hence the functional I has at least one minimum point which by Theorem 1 is a solution to (4), (5).

Given a function $f \in L^\infty_{loc}(R)$ we set (see[4])

$$\tilde{f}(y) = \bigcap_{\delta>0} \bigcap_{\nu(N)=0} \overline{\text{conv}}\ f([y-\delta, y+\delta]\setminus N) =$$
$$= [m(f(y)), M(f(y))], \quad y \in R$$

where

(10) $$M(f(y)) = \lim_{\delta \to 0} \operatorname*{ess\,sup}_{u \in [y-\delta, y+\delta]} f(u); m(f(y)) = $$
$$= \lim_{\delta \to 0} \operatorname*{ess\,inf}_{u \in [y-\delta, y+\delta]} f(u).$$

Consider the variational inequality (the "obstacle problem")

(11) $$\begin{aligned} &\sum_{i=1}^n (a_i(y_{x_i}))_{x_i} - f(y) \leq 0 && \text{on} \quad \Omega, \\ &\sum_{i=1}^n (a_i(y_{x_i}))_{x_i} - f(y) = 0 && \text{on} \quad \{y > \psi\}, \\ &y \geq \psi \ \text{on}\ \Omega;\ y = 0 && \text{on} \quad \Gamma. \end{aligned}$$

where $a_i$, $i = 1,\dots n$ are continuous monotone increasing functions satisfying the conditions

(12) $$a_i(o)=o;|a_i(r)| \leq M(\int_o^r a_i(s)ds+|r|+1) \quad \text{for all } r\in R,$$

(13) $$\lim_{|r|\to\infty} r^{-1}\int_o^r a_i(s)ds = +\infty,$$

while f is a $L^\infty_{loc}$ function on R which satisfies

(14) $$f(r)(r-c)\geq o;|f(r)| \leq M(\int_o^r f(y)dy+|r|+1) \text{ a.e. } r\in R.$$

THEOREM 3 Under the above assumptions Eq.(11) has at least one solution $y\in W_o^{1,1}(\Omega)$ in the following sense: there exists a Radon measure $\mu$ on $\Omega$ and $q\in L^1(\Omega)$ such that

$$\sum_{i=1}^n (a_i(y_{x_i}))_{x_i}-q = \mu \quad \text{a.e. } x\in\Omega$$

$$\mu\leq o \text{ on } \Omega;\ \mu = o \text{ on int}\{y>\psi\}$$

$$q(x)\in\tilde{f}(y(x)) \text{ a.e. } x\in\Omega;\ a_i(y_{x_i})\in L^1(\Omega);i = 1,\dots n.$$

To prove the theorem it suffices to apply Theorem 2 where

$$L(y,z) = \int_c^y f(r)dr+\sum_{i=1}^n \int_o^{z_i} a_i(s)ds \ ;\ (y,z)\in R\times R^n.$$

Theorem 3 extends some recent results in [6], [7].

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