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APPLICATION OF BOUNDED OPERATORS AND LYAPUNOV'S MAJORIZING EQUATIONS TO THE ANALYSIS OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

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1. Introduction

Given a system of ordinary differential equations with a small parameter ε $(\varepsilon \geq 0)$

(1) $\dot{z} = F(z,t,\varepsilon)$, $(\dot{} = d/dt)$,

let us consider the problems of existence, of estimating the domain of existence and of the construction of solutions of a certain class, for example, periodic or satisfying some initial conditions. Following the usual methods of small parameter we assume that the solution $z^{0}(t)$ of the system (1) for $\varepsilon = 0$ is known and that the solution $z(t,\varepsilon)$ is continuous at $\varepsilon = 0$. Moreover we assume that the function $F(z,t,\varepsilon)$ is continuous in t, ε and differentiable with respect to z in a neighborhood of $z^{0}(t)$.

A well known method of investigating the problem of existence and uniqueness of a solution consists in proving the possibility of transforming the system (1) into an operator system of the form $x = Sx$ where S is the corresponding operator and x is a new variable, and further in an application of the contractive mapping principle. Our approach which develops further the Lyapunov methods [1], [2] consists in associating the system $x = Sx$ with finite (as a rule, algebraic) equations which will be called Lyapunov's majorizing equations. Constructing these equations, we write the given operator system equivalent to the system (1) on the corresponding set of functions $\mathcal{H}$ in the form

(2) $x = LW(x,t,\varepsilon)$

where L is a linear bounded matrix operator in $\mathcal{H}$ while $W(x,t,\varepsilon)$ is a function continuous in t, ε and differentiable in x in a domain $D(\|x\| \leq R,\ 0 \leq t \leq T,\ 0 \leq \varepsilon \leq \varepsilon_{*})$. The variable x is such that

(3) $W(0,t,0) = 0$, $\partial W(0,t,0)/\partial x = 0$.

Then in the general case, the system of majorizing equations is

(4) $u = \Lambda\phi(u,\varepsilon) = 0$

where Λ is a constant matrix such that the following vector condition of boundedness of the operator L is satisfied:

(5) $\quad (\|L\varphi(t)\|) \le \Lambda(\|\varphi(t)\|)\ , \quad \varphi(t) \in \mathcal{H}$

while $\phi(u, \varepsilon)$ is the so called Lyapunov's majorant with respect to $W(x,t,\varepsilon)$; all components W_i, ϕ_i satisfy the inequalities

(6) $\quad \|W_i(x,t,\varepsilon)\| \le \phi_i(u,\varepsilon)\ , \quad \|\partial W_i(x,t,\varepsilon)/\partial x_j\| \le$
$\le \partial\phi_i(u,\varepsilon)/\partial u_j\ ,$

provided $x,t,\varepsilon \in D$, $\|u\| \le R$, $|x_j| \le u_j$, $j=1,2,\ldots,n$.

By means of the majorizing system (4) it is not only possible to establish the convergence of the iterative process

(7) $\quad x_k = LW(x_{k-1}, t, \varepsilon)\ , \ k=1,2,\ldots, \quad x_0 \equiv 0$

in a certain domain of variation of ε to a unique solution but also to obtain estimates of this domain as well as of the error of the approximative solutions constructed on the basis of (7).

Various modifications of the system (2) and of the majorizing equations (4) may be studied. A number of results based on this approach are given in [3] - [10].

Notation used throughout the paper: (i) The symbol $\|\varphi(t)\|$ means the usual norm $\sup_t |\varphi(t)|$ of a scalar function in the space C^0; (ii) The symbol $\|\varphi(t)\|_*$ stays for the so called trigonometric norm in the space of functions which are expressed by absolutely convergent Fourier series:

$$\|\varphi(t)\|_* = \sum_{|k| \ge 0} |a_k|$$

where a_k are the coefficients of the complex Fourier series (or polynomial) for $\varphi(t)$; (iii) By $(\|x(t)\|)$ we denote the vector whose components are $\|x_1(t)\|,\ldots,\|x_n(t)\|$; (iv) A vector inequality $x \le y$ is considered equivalent to the same inequalities for all the components of the vectors x,y. Vectors which satisfy the inequality $x > 0$ or $x \ge 0$ are called positive or nonnegative, respectively. The other notation is standard.

2. Fundamental theorems for the systems (2) and (4)

Generally, the function $\Lambda\phi(u,\varepsilon)$ is continuous in ε, continuously differentiable in u and belongs to the class of nonlinear vector functions which are positive for $\varepsilon > 0$, $u > 0$, none of the elements of the matrix $\Lambda\,\partial\phi/\partial u$ is negative and there is at least one element which is an increasing function of at least one component of the vector u . Moreover,

$$\phi(0,0) = 0 , \quad \partial\phi(0,0)/\partial u = 0 .$$

Further we assume that the system (4) is non singular, i.e., it neither splits into separate subsystems nor has a solution for $\varepsilon > 0$ with some components of the vector u equal to zero and the others positive. Then we have the following

Theorem 1.

I. The system (4) has a positive solution $u = u(\varepsilon)$ in and only in the domain $[0, \varepsilon_*]$ whose upper limit ε_* and the corresponding vector $u_* = u(\varepsilon_*)$ satisfy simultaneously the relation (4) and the relation

(8) $$\det [E - \Lambda \partial\phi(u, \varepsilon)/\partial u] = 0$$

where E is the unit matrix.

II. For $\varepsilon \in [0, \varepsilon_*]$ the system (4) has a unique solution $u = \bar{u}(\varepsilon) \in C^0[0, \varepsilon_*]$ such that $\bar{u}(\varepsilon) > 0$ for $\varepsilon > 0$ and $\bar{u}(0) = 0$; for $\varepsilon \in (0, \varepsilon_*)$ and the corresponding $\bar{u}(\varepsilon)$ the determinant (8) as well as all its principal minors are positive.

III. For $\varepsilon \in [0, \varepsilon_*]$ the iterations

(9) $$u_k = \Lambda\phi(u_{k-1}, \varepsilon) , \ k=1,2,\ldots, \quad u_0=0$$

form a nondecreasing sequence and converge to $\bar{u}(\varepsilon)$.

Theorem 2.

For a given system (2) and the corresponding system of inequalities

(10) $$v \le \Lambda\phi(v, \varepsilon)$$

let $u = \bar{u}(\varepsilon)$ be the solution of the system (2) from Theorem 1, and let $v = v(\varepsilon) \in C^0[0, \varepsilon_*]$ satisfy (10), $v(\varepsilon) > 0$ for $\varepsilon > 0$ and $v(0) = 0$.

Then $v(\varepsilon) \le \bar{u}(\varepsilon)$, $\varepsilon \in [0, \varepsilon_*]$.

The proof of the above assertions is first carried out for the case when (2) is a scalar equation ($u = f(u, \varepsilon)$). Simple geometric arguments are used (graphs of the curves $y = f(u, \varepsilon)$ for $\varepsilon < \varepsilon_*$, $\varepsilon = \varepsilon_*$, $\varepsilon > \varepsilon_*$ on the surface (u,y) and the graph of the straight line $y=u$ are considered) together with the monotonicity of $f(u, \varepsilon)$, $f'_u(u, \varepsilon)$ for increasing u, ε and with the boundedness of u, ε from above which is a consequence of an equation analogous to (8). The method of induction allows us to extend the results to systems of arbitrary orders (see [6] , [10]). At the same time it is proved that (2), (8) together form a system of equations with respect to u, ε possessing a unique positive solution $u=u_*$, $\varepsilon = \varepsilon_*$.

Hence to find ε_* is an algebraic problem and it is known that its solution exists and is unique.

The fundamental result concerning the system (2) is the following

Theorem 3.

Let us consider the system (2) in the domain $D(\|x\| \leq R,\ 0 \leq t \leq T,\ 0 \leq \varepsilon \leq \varepsilon^0)$ and let (4) be the corresponding majorizing system in the domain. Let $u = \bar{u}(\varepsilon)$ be the solution of the system (4) and $[0, \varepsilon_*]$ the domain $(\varepsilon_* \leq \varepsilon^0)$ from Theorem 1, $\|\bar{u}(\varepsilon)\| \leq R$ for $\varepsilon \in [0, \varepsilon_*]$.

Then for $\varepsilon \in [0, \varepsilon_*]$ (i) the sequence $\{u_k(\varepsilon)\}$ defined according to (9) is majorizing with respect to the sequence $\{x_k(t, \varepsilon)\}$ defined according to (7); (ii) the sequence $x_k(t, \varepsilon)$ converges on the segment $0 \leq t \leq T$ to the solution $x(t, \varepsilon)$ of the system (2) and this solution is unique in the class of functions belonging to $C^0[0, \varepsilon_*]$ and equal to zero for $\varepsilon = 0$.

The proof is based on comparing the expressions (7), (9) for $x_k(t, \varepsilon)$, $u_k(\varepsilon)$ with regard to the inequalities (5), (6). We obtain

$$(11) \qquad (\|x_k(t, \varepsilon)\|) \leq u_k(\varepsilon), \quad (\|x_{k+1}(t, \varepsilon) - x_k(t, \varepsilon)\|) \leq \leq u_{k+1}(\varepsilon) - u_k(\varepsilon)$$

and this implies in virtue of Theorem 1 the uniform convergence of the sequence $\{x_k(t, \varepsilon)\}$ to the solution $x(t, \varepsilon)$ of the system (2) on the segment $[0, T]$ for $\varepsilon \in [0, \varepsilon_*]$ and moreover, $x(t, \varepsilon) \in C^0[0, \varepsilon_*]$, $x(t, 0) = 0$. The uniqueness of solution is established via Theorem 2. Assuming the existence of a solution $\tilde{x}(t, \varepsilon) \in C^0[0, \varepsilon_*]$ $(\tilde{x}(t, 0) \equiv 0)$ of the system (2) we obtain that the vector $v(\varepsilon) = (\|\tilde{x}(t, \varepsilon)\|)$ satisfies the system of inequalities (10). In virtue of Theorem 2 we have $v(\varepsilon) \leq \bar{u}(\varepsilon)$. Further we establish the inequalities

$$(\|\tilde{x}(t, \varepsilon) - x_k(t, \varepsilon)\|) \leq \bar{u}(\varepsilon) - u_k(\varepsilon), \quad k = 1, 2, \ldots .$$

Hence we conclude that for $\varepsilon \in [0, \varepsilon_*]$ the system (2) has no solution different from the limit of the sequence $\{x_k(t, \varepsilon)\}$ which belongs to the class $C^0[0, \varepsilon_*]$ and equal to zero for $\varepsilon = 0$.

Remark 1. Theorem 3 offers directions how to construct the number ε_* which gives a lower estimate of the values ε for which the desired solution $x(t, \varepsilon)$ of the system (2) exists and is unique. Furthermore, we obtain an estimate for the solution itself and for the error of the k-th approximation $x_k(t, \varepsilon)$ since

$$(12) \qquad (\|x(t, \varepsilon)\|) \leq \bar{u}(\varepsilon), \quad (\|x(t, \varepsilon) - x_k(t, \varepsilon)\|) \leq \bar{u}(\varepsilon) - u_k(\varepsilon).$$

Remark 2. The majorizing equations (4) may be simplified by replacing them in the simplest case by a single equation. At the same time these equations and consequently, also the established estimates may be improved by immediate estimation of a certain number of approximations $x_k(t, \varepsilon)$. In the case of periodic solutions and analytic in x right hand sides of (2) this improvement can be achieved by taking into account the structure of the desired solution and using the norm $\|\cdot\|_*$ in the conditions (5), as well as owing to the possibility of distinguishing the main harmonics in the solution (cf. [10]).

3. Analytic case

If the function $W(x,t,\varepsilon)$ in (2) is an analytic function of x, ε in the domain D and if an analytic with respect to $W(x,t,\varepsilon)$ majorant of Lyapunov is used in the construction of the system (4), then the number ε_* from Theorem 3 gives an estimate of the domain of convergence of the power series in ε, in the form of which the solution of the system (2) can be sought. This series always converges for $|\varepsilon| \leq \varepsilon_*$. This follows from the fact that in this case all $x_k(t, \varepsilon)$ are expressed by series (polynomials) in ε and they are majorized by the corresponding series (polynomials) for $u_k(\varepsilon)$ with nonnegative coefficients; the sequence $\{u_k(\varepsilon)\}$ converges for $0 \leq \varepsilon \leq \varepsilon_*$ to a function which can be expressed by a series of the same character.

4. Connection with the condition of contractivity of the mapping

The system (2) being written in the form $x = Sx$, Theorem 3 may be viewed as a theorem on existence and uniqueness of a fixed point of the operator (mapping) S on the segment $0 \leq t \leq T$ for $\varepsilon \in [0, \varepsilon_*]$ in the class of functions $C^0[0, \varepsilon_*]$ equal to zero for $\varepsilon = 0$. Nonetheless the proof of Theorem 3 proceeds without using the notion of contractivity of the operator.

Dealing with (2), let us choose arbitrary consistent norms of vectors and matrices, formulate the conditions (5) of boundedness of the operator L and construct the system (4). Let the latter be written in the form

$$(13) \qquad u = Qu , \quad (Qu \equiv \Lambda \phi(u, \varepsilon))$$

where Q is a finite functional operator. This operator majorizes the operator S so that the contractivity of Q is a sufficient condition of contractivity of S. However, it may happen that the operator Q does not satisfy the condition of contractivity on the whole segment $[0, \varepsilon_*]$ from Theorem 1, if the above chosen norm is considered. It is not difficult to find examples of such systems of the form (4) (see [10]). In these cases it cannot be guaranteed that the operator S fulfils the condition of contractivity in the given norm for all $0 \le \varepsilon \le \varepsilon_*$. In other words, the requirement of contractivity of the operator with respect to the given norm may prove more restrictive than that which follows from the analysis of the majorizing system.

At the same time there exists such a vector norm that the condition of contractivity of the operator Q is fulfilled for all $0 < \varepsilon < \varepsilon_*$. Indeed, let us consider the matrix $Q'(u) = \Lambda \partial \phi(u, \varepsilon)/ / \partial u$ and its spectral radius $\rho(\varepsilon)$ for a fixed $u = \tilde{u} > 0$. The matrix $Q'(\tilde{u})$ is nonnegative and the assumptions of Theorem 1 imply that for $0 < \varepsilon < \varepsilon_*$ and $u = \bar{u}(\varepsilon)$, ρ remains less than one, while $\rho = 1$ for $\varepsilon = \varepsilon_*$. According to [11] there is a vector norm such that the corresponding norm $\|Q'(\bar{u})\|$ is greater than $\rho(\varepsilon)$ by an arbitrarily small number. Hence for every ε from the interior of the segment $[0, \varepsilon_*]$ there is a vector norm such that $\|Q'(\bar{u})\| < \delta < 1$. This is sufficient for the validity of the assertion on contractivity of the operator Q, and hence also of the majorizing operator S. However, for $\varepsilon = \varepsilon_*$ none of the norms $\|Q'(\bar{u})\|$ can be less than one so that the operator Q fails to be contractive. Consequently, for $\varepsilon = \varepsilon_*$ the operator S can also be non contractive. However, this does not influence our proof of Theorem 3.

5. Examples

1. Let us consider the periodic solution of the equation

$$(14) \qquad \ddot{z} + \omega^2 z = \varepsilon z^3 + \sin t , \qquad \omega^2 = 5 .$$

We have $z^0 = q \sin t$, $q = \frac{1}{4}$ and putting $z = z^0 + x$ we obtain the equation

$$(15) \qquad \ddot{x} + \omega^2 x = \varepsilon f(x,t) = \\ = \varepsilon [q^3 \sin^3 t + 3q^2 \sin^2 tx + 3q \sin tx^2 + x^3] .$$

The equivalent operator equation is $x = \varepsilon Lf(x,t)$, where L is considered in the class of functions representable by trigonometric polynomials and

$$(16) \qquad Le^{imt} = e^{imt}/(\omega^2 - m^2) .$$

All approximations $x_k(t, \varepsilon)$ are Fourier polynomials which include only odd harmonics. An estimate for L on the family of such functions with $\omega^2 = 5$ with respect to the norm $\|\cdot\|_*$ is

(17) $\|L\varphi(t)\|_* \le \|\varphi(t)\|_*/(\omega^2-1) = \frac{1}{4}\|\varphi(t)\|_*$

while the majorizing Lyapunov's equation (being scalar) has the form

(18) $u = \frac{1}{4}(q^3 + 3q^2u + 3qu^2 + u^3)\ , \quad q = \frac{1}{4}\ .$

This equation yields $\varepsilon_* = 9.48$, $u_* = 0.12$ (with an error not greater than 0.01) so that the convergence of the sequence $\{x_k(t,\varepsilon)\}$ to the unique solution is guaranteed for rather great ε. The solution $\bar{u}(\varepsilon)$ of the equation (18) yields an estimate with respect to the norm $\|\cdot\|_*$ of the deviation of the periodic solution $z(t,\varepsilon)$ of the equation (14) from the generating solution $z^0(t)$. The quantity $\delta_1 = \bar{u}(\varepsilon) - u_1(\varepsilon) = \bar{u}(\varepsilon) - \frac{1}{4}\varepsilon q^3$ represents an estimate of the first approximation $z_1(t,\varepsilon) = z^0(t) + x_1(t,\varepsilon)$. For example, for $\varepsilon = 8$ we obtain (with an error not greater than 0.001):
$\bar{u}(\varepsilon) = 0.059$, $u_1 = 0.031$, $\delta_1 = 0.028$.

2. Let us consider the Mathieu equation

(19) $\ddot{z} + (a + \varepsilon\cos 2t)z = 0\ .$

The Mathieu functions as well as the corresponding eigenvalues a_{cn} , a_{sn} can be found in the form of power series in ε. The literature offers only few facts concerning the radii of convergence of these series.

Let us discuss e.g. the case $ce_1(t)$, setting $a = 1 + \varepsilon h$, $z = ce_1(t) = \cos t + x$. The equation obtained for x is

(20) $$\ddot{x} + x = \varepsilon f(x,t,h) = \varepsilon\left[-(h + \tfrac{1}{2})\cos t - \tfrac{1}{2}\cos 3t - (h + \cos 2t)x\right].$$

We seek for the approximations $x_k(t,\varepsilon)$, $h_k(\varepsilon)$, $k=1,2,\ldots$ from the equations

(21) $\ddot{x}_k + x_k = \varepsilon f(x_{k-1}, t, h_k)\ , \quad x_0 \equiv 0$

having determined h_k from the periodicity conditions for $x_k(t,\varepsilon)$. The operator equations corresponding to the given iteration process have the form

(22) $$x = \varepsilon L_1\left[-\tfrac{1}{2}\cos 3t - (h + \cos 2t)x\right]\ ,$$
$$h = -\tfrac{1}{2} - L_2(x\cos 2t)\ .$$

Since all $x_k(t,\varepsilon)$ are expressed in the form of Fourier polynomials in cosines jt , $j = 3,5,7,\ldots$, L_1 and L_2 are considered to be in the same class of functions and hence

(23) $\|L_1\varphi(t)\|_* \le \frac{1}{8}\|\varphi(t)\|_*\ , \ \|L_2\varphi(t)\|_* = |L_2\varphi(t)| \le \frac{1}{2}\|\varphi(t)\|_*.$

Using these estimates we write the majorizing equation

(24) $$u = \frac{1}{8}\varepsilon\left[\frac{1}{2} + (v+1)u\right], \quad v = \frac{1}{2} + \frac{1}{2}u$$

where u and v majorize $x(t,\varepsilon)$ and $h(\varepsilon)$, respectively. For these equations we obtain $u_* = 1$, $v_* = 1$, $\varepsilon_* = 0.2$. Since the analytic case is considered, the convergence of the sequence $\{x_k(t,\varepsilon)\}$ to a function $x(t,\varepsilon)$ analytic in ε is guaranteed for $|\varepsilon| \le 0.2$.

Let us note that the majorizing equations (24) can be improved so that an estimate $|\varepsilon| \le 0.35$ is obtained [10]. Further, let us mention that the given algorithm for the construction of $ce_1(t)$ was used for machine computation. Practical convergence occurs for $0 < \varepsilon < 0.50$. For $\varepsilon = 0.55$ the process proved to be divergent.

3. Let us consider a system with an additional small parameter $\mu > 0$ at the derivative

$$\mu\dot{x} = Ax + \varepsilon F(x,t)$$

and its solution which vanishes for $\varepsilon = \mu = 0$. The matrix A is assumed to be constant and to satisfy the so called stability condition: the real parts of all its eigenvalues are negative. The function $F(x,t)$ is assumed to be continuous in t and differentiable in x in a domain. The equivalent operator system reads

(25) $$x = \varepsilon L_\mu F(x,t), \quad L_\mu \varphi(t) = \frac{1}{\mu}\int_0^t \exp\left[\frac{A}{\mu}(t-s)\right]\varphi(s)ds.$$

In virtue of the stability condition we have

$$\|\exp(At)\| \le c\exp(-\alpha t), \quad c > 0, \quad \alpha > 0$$

so that L_μ is a bounded operator on the whole half-axis $t \ge 0$ for any $\mu > 0$ and the estimate $\sup_t \|L\varphi(t)\| \le \varrho \sup_t \|\varphi(t)\|$, $\varrho = \frac{c}{\alpha}$ is independent of μ. Hence the system (25) has the same character as the system (2). Its solution may be constructed by an iterative process of the form (7) whose convergence is guaranteed on a segment $[0, \varepsilon_*]$ and for all $\mu > 0$. For example, for the scalar equation

$$\mu\dot{x} = -2x + \varepsilon(\cos t + 2x + x^2)$$

we have an estimate of the operator $\|L\varphi(t)\| \le \frac{1}{2}\|\varphi(t)\|$ and the majorizing equation $u = \frac{1}{2}\varepsilon(1 + 2u + u^2)$ for which $u_* = 1$, $\varepsilon_* = \frac{1}{2}$.

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