EQUADIFF 4

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In: Jiří Fábera (ed.): Equadiff IV, Czechoslovak Conference on Differential Equations and Their Applications. Proceedings, Prague, August 22-26, 1977. Springer-Verlag, Berlin, 1979. Lecture Notes in Mathematics, 703. pp. [286]--299.

Persistent URL: http://dml.cz/dmlcz/702229

ON THE REGULARITY OF WEAK SOLUTIONS TO VARIATIONAL EQUATIONS AND INEQUALITIES FOR NONLINEAR SECOND ORDER ELLIPTIC SYSTEMS

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The history of the solution of the 19th Hilbert's problem, i.e. in fact of the problem of regularity, is described in the book by O.A. Ladyženskaja, N.N. Uralceva [1] and in the paper by Ch.B. Morrey [2]. The weak solution is a vector function from the Sobolev space $W^{1,2}(\Omega)$ satisfying the equations in the sense of distributions and it is regular if it belongs to $C^{(1)}(\Omega)$ (interior regularity) or $C^{(1)}(\overline{\Omega})$ (regularity up to the boundary).

The problem of the regularity for the dimension $n=2$ was solved very soon (1937) by Ch.B. Morrey [3], also for systems. The generalization of this result for higher dimensions, but only for a single second-order equation, was done by E. De Giorgi [4] in 1957 and his method, based in fact on the maximum principle, was further developed by several authors, see J. Moser [5], G. Stampacchia [6], the book [1] and others.

If we consider a vector function $u = (u_1, u_2, \dots u_m)$ from $[W^{1,2}(\Omega)]^m$, satisfying the system

$$(1) \qquad -\frac{\partial}{\partial x_i}\left[a_i^r(x,u,\nabla u)\right] + a_0^r(x,u,\nabla u) = f_r(x), \qquad r=1,2,\dots m,$$

if $f_r \in L_2(\Omega)$ and if, for example, $u = u^0$ on $\partial\Omega$, where $u^0 \in [W^{1,2}(\Omega)]^m$ is a prescribed function, then the existence of a weak solution as well as its uniqueness can be proved relatively easily, see for example the book by J.L. Lions [7], under the standard assumptions:

$$(2) \qquad \left|\frac{\partial a_i^r}{\partial \frac{\partial u_s}{\partial x_j}}\right| + \left|\frac{\partial a_i^r}{\partial u_s}\right| + \left|\frac{\partial a_0^r}{\partial \frac{\partial u_s}{\partial x_j}}\right| + \left|\frac{\partial a_0^r}{\partial u_s}\right| \le c,$$

$$(3) \qquad \frac{\partial a_i^r}{\partial \frac{\partial u_s}{\partial x_j}} \eta_i^r \eta_j^s \ge \alpha \eta_i^r \eta_i^r, \qquad \alpha > 0,$$

$$(4) \qquad a_i^r(x,\eta_s,\eta_j^s)\eta_i^r + a_0^r(x,\eta_s,\eta_j^s)\eta_r \ge \beta \eta_i^r \eta_i^r, \quad \beta > 0,$$

provided that the derivatives of a_i^r in (2), (3) satisfy the Caratheodory condition.

Instead of equations, we can study inequalities, if for example on $\partial\Omega$ (or on some part of $\partial\Omega$) a unilateral condition of

Signorini's type

(5) $$b_{rs}u_s \geq \psi_r, \qquad r=1,2,\dots,k \leq m,$$

is given. Writing

(6) $$K \equiv \{v \in [W^{1,2}(\Omega)]^m \mid b_{rs}v_s \geq \psi_r \text{ on } \partial\Omega\},$$

we look for $u \in K$ such that, $\forall v \in K$,

(7) $$\int_\Omega a_i^r(x,u,\nabla u)\left(\frac{\partial v_r}{\partial x_i} - \frac{\partial u_r}{\partial x_i}\right) dx \geq \int_\Omega f_r(v_r-u_r)\, dx.$$

For the existence and other questions, see [7]. The conditions (2) and (3) guarantee the first step to the interior regularity, i.e., the proof of the inclusion $u \in [W^{2,2}(\Omega')]^m$, $\overline{\Omega}' \subset \Omega$. If $\Omega' = \Omega$, we get the first step to the regularity up to the boundary. For the idea of the proof of this step, see also [7]. If, for simplicity, we restrict ourselves in the following to the case $a_i^r(x,u,\nabla u)=a_i^r(\nabla u)$, $a_0^r=0$, then we can immediately see that this first step leeds to an equation in variations obtained through integration by parts of the equation

(8) $$\int_\Omega a_i^r(\nabla u)\frac{\partial \varphi_r}{\partial x_i}\, dx = \int_\Omega f_r \varphi_r\, dx, \qquad \varphi_r \in \mathcal{D}(\Omega);$$

if we denote by u' some derivative, then we get from (8), substituting here φ':

(9) $$\int_\Omega a_{ij}^{rs}\frac{\partial u_s'}{\partial x_j}\frac{\partial \varphi_r}{\partial x_i}\, dx = \int_\Omega f_r' \varphi_r\, dx,$$

where $a_{ij}^{rs} = \dfrac{\partial a_i^r}{\partial \frac{\partial u_s}{\partial x_j}}$. (9) is a linear system in u' with, in general, only measurable, bounded coefficients a_{ij}^{rs}.

Let us mention the known fact that once being $u \in [C^{(1)}(\Omega)]^m$, we get arbitrary higher regularity of the solution, provided that the coefficients and right-hand sides are regular enough.

The significance of the problem of regularity is underlined by the fact that the regularity up to the boundary, provided that the coefficients are analytic, implies that in the potential case the set of critical values is a sequence, tending to zero, see S. Fučík, J. Nečas, J. Souček, V. Souček [8]. Also the Newton's type methods are convergent only in the space of regular solutions.

For more general systems than discussed in the paper [2], J.

Stará proved the regularity in [9], also for $r=2$, using the method of the papers by J. Nečas [10], [11] concerning higher order single equations.

E. Giusti, M. Miranda constructed in the paper [12], for $n \geq 3$, a regular functional, whose critical point is $u = \frac{x}{|x|}$. This functional is continuous on $[W^{1,2}(\Omega)]^m$, but not differentiable. In the same work, a system with coefficients $a_i^r = A_{ij}^{rs}(u) \frac{\partial u_s}{\partial x_j}$, $s = 1,2,\dots,n$, is constructed with the ellipticity condition

$$A_{ij}^{rs} \eta_i^r \eta_j^s \geq \alpha |\eta|^2 \tag{10}$$

and the same solution $\frac{x}{|x|}$. Some variations of this type of example can be found in the paper by S.A. Arakčejev [13].

Let us start with a more detailed description of the results of the paper by J. Nečas [14] with small complements.

The easiest example of a fourth order system with a non-regular solution is

$$\Delta^2 u_i + \frac{1}{(n+1)^2(n-2)} \frac{\partial^2}{\partial x_j \partial x_k} [\Delta u_i \Delta u_j \Delta u_k] = 0, \tag{11}$$

provided that this system is defined on the set of u's such that $\Delta u_i \Delta u_i \leq (n+1)^2$. The solution of this system is $u_i = x_i |x|$ and the corresponding conditions (3), (4) are satisfied for $n \geq 6$.

K being the unit ball $|x|<1$, let us consider the system (in the weak formulation),

$$\begin{aligned} &\int_K \frac{\partial u_{ij}}{\partial x_k} \frac{\partial \varphi_{ij}}{\partial x_k}\, dx + \lambda_2 \int_K \frac{\partial u_{kk}}{\partial x_i} \frac{\partial \varphi_{\ell\ell}}{\partial x_i}\, dx + \\ &+\lambda_3 \int_K \frac{\partial u_{\alpha j}}{\partial x_\alpha} \frac{\partial \varphi_{\beta j}}{\partial x_\beta}\, dx + \lambda_4 \int_K \frac{\partial u_{\alpha\beta}}{\partial x_\alpha} \frac{\partial \varphi_{ii}}{\partial x_\beta}\, dx + \\ &+\lambda_1 \int_K \frac{\partial u_{\gamma i}}{\partial x_\gamma} \frac{\partial u_{\alpha j}}{\partial x_\alpha} \frac{\partial u_{\beta\ell}}{\partial x_\beta} \frac{\partial \varphi_{ij}}{\partial x_\ell}\, dx = 0, \end{aligned} \tag{12}$$

$i,j = 1,2,\dots n$, where $\lambda_3 = \frac{2(n-1)^3 - 9n}{9(n-1)}$ for $n \geq 5$, $\lambda_3 = \frac{2(n-1)^3 - n}{n-1}$ for $3 \leq n \leq 4$, $\lambda_1 = \frac{n + \lambda_3(n-1)}{(n-1)^4(n+1)^2} n^2$, $\lambda_4 = -\frac{1+\lambda_3(n-1)}{(n-1)^2}$ and λ_2 is large enough. If we put $u_{ij} = \frac{x_i x_j}{|x|} - \frac{1}{n}\delta_{ij}|x|$, then $u_{ii}=0$ and u_{ij} satisfy (12). The coefficients of (12) are defined only for such "ξ_k^{ij}" where

$\frac{\partial u_{\alpha i}}{\partial x_\alpha} \frac{\partial u_{\beta i}}{\partial x_\beta} \leq \left(\frac{n^2-1}{n} + \delta \right)^2$, $\delta > 0$ and small enough.

For $n \geq 5$, we have (3) and (4), for $3 \leq n \leq 4$, we have (4). If we replace the nonlinear term in (12) by

$$(13) \qquad \lambda_1 \left(\varepsilon + \frac{(n^2-1)^2}{n^2} \right) \int_K \frac{\partial u_{\gamma i}}{\partial x_\gamma} \frac{\partial u_{\alpha j}}{\partial x_\alpha} \frac{\partial u_{\beta \ell}}{\partial x_\beta} \cdot$$

$$\cdot \left(\varepsilon + \frac{\partial u_{ab}}{\partial x_a} \frac{\partial u_{cb}}{\partial x_c} \right)^{-1} \frac{\partial \varphi_{ij}}{\partial x_\ell} \, dx$$

with $\varepsilon > 0$ small enough, we get the same result with coefficients defined everywhere.

If we consider the functional of the total potential energy in finite elasticity under the incompressibility constraint then there exists a universal, isotropic body, see C. Truesdell [16], and its deformation from the so-called 5th class, a critical point of the functional under the constraint, which is not regular. But the set of irregular points is a segment, so it is not possible to get in this way immediately an example of an irregular solution without a constraint because the Hausdorff measure of irregular points must be less than 1, see E. Giusti [17].

The example (12), (13) is a vector function $|x| f(\frac{x}{|x|})$ and the functions $f_i(\xi)$ are not linear in ξ. If we write such an example in polar coördinates $r, \vartheta_1, \vartheta_2, \dots \vartheta_{n-1}$, $0 < r < \varepsilon$, $0 < \vartheta_j < \pi$, $j=1,2,\dots,n-2$, $0 < \vartheta_{n-1} < 2\pi$, putting

$$(14) \qquad x_1 = r \cos \vartheta_1, \; x_2 = r \sin \vartheta_1 \cos \vartheta_2, \dots, x_{n-1} =$$
$$= r \sin \vartheta_1 \sin \vartheta_2 \dots \sin \vartheta_{n-2} \cos \vartheta_{n-1},$$
$$x_n = r \sin \vartheta_1 \sin \vartheta_2 \dots \sin \vartheta_{n-2} \sin \vartheta_{n-1},$$

$$(15) \qquad \partial_1 v = \frac{\partial v}{\partial r}, \; \frac{1}{r} \frac{\partial v}{\partial \vartheta_1} = \partial_2 v, \; \frac{1}{r \sin \vartheta_1} \frac{\partial v}{\partial \vartheta_2} = \partial_3 v, \dots,$$
$$\frac{1}{r \sin \vartheta_1 \dots \sin \vartheta_{n-2}} \frac{\partial v}{\partial \vartheta_{n-1}} = \partial_n v,$$

we first get $\partial_i v = a_{ij} \frac{\partial v}{\partial x_j}$, where a_{ij} is an orthonormal matrix. Let us define the elementary differential operators

$$(16) \qquad \bar{\partial}_2 h = \frac{\partial h}{\partial \vartheta_1}, \; \bar{\partial}_3 h = \frac{1}{\sin \vartheta_1} \frac{\partial h}{\partial \vartheta_2}, \dots, \; \bar{\partial} h = (\bar{\partial}_2 h, \dots \bar{\partial}_n h).$$

We introduce the space $W^{1,2}(S)$, S being the unit sphere, as the closure of infinitely differentiable functions in the norm

(17) $$\left(\int_S [f^2 + \bar{\partial}_{j'} f \, \bar{\partial}_{j'} f] \, dS\right)^{1/2}$$

where the indices with primes are summed from 2 up to n. Starting from the system (8), we get for f another system on the unit sphere

(18) $$\int_S [-(n+1)A_1^r(\vartheta, f, \bar{\partial} f)\tilde{f}_r + A_{j'}^r(\vartheta, f, \bar{\partial} f)\bar{\partial}_{j'}\tilde{f}_r] \, dS = 0,$$

where $f, \tilde{f} \in [W^{1,2}(S)]^m$.

We get immediately

(19) $$\left|\frac{\partial A_j^r}{\partial f_s}\right| + \left|\frac{\partial A_j^r}{\partial \, \partial_{j'} f_s}\right| \leq c,$$

(20) $$\frac{\partial A_j^r}{\partial(\bar{\partial}_{i'} f_s)} \xi_{j'}^r \xi_{i'}^s \geq \alpha \xi_{j'}^r \xi_{j'}^r, \qquad \alpha > 0.$$

Let J be the kernel of (18), i.e., the set of all the solutions from $[W^{1,2}(S)]^m$. We introduce $J_0 \subset J$, the trivial subset of J, consisting of the linear combinations of the coordinate functions $\cos\vartheta_1$, $\sin\vartheta_1 \cos\vartheta_2$, ... $\sin\vartheta_1 \sin\vartheta_2 \ldots \sin\vartheta_{n-2} \sin\vartheta_{n-1}$.

Let us consider a weak solution to the system $(K_\varepsilon \equiv \{|x| \leq \varepsilon\})$

(21) $$\int_{K_\varepsilon} a_i^r(\nabla u) \frac{\partial \varphi_r}{\partial x_i} \, dx = 0.$$

We easily get

<u>Theorem 1.</u> The necessary condition for the regularity of every weak solution to (21) is $J = J_0$.

<u>Proof:</u> Let us suppose the contrary and let us take $f \in J \setminus J_0$. Put $u = rf(\vartheta)$; u satisfies (21) and so

$$\frac{\partial u_r}{\partial x_i}(0) = \lim_{r \to 0} \frac{\partial u_r}{\partial x_i}(x) = \lim_{r \to 0} [a_{1i}(\vartheta) f_r(\vartheta) + a_{j',i}(\vartheta) \bar{\partial}_{j'} f_r(\vartheta)] =$$
$$= \frac{\partial u_r}{\partial x_i}(x);$$ hence $u_r(x)$ is a linear function and, therefore,

$f \in J_0$, which is impossible, q.e.d.

So the study of the kernel J leads to the construction of an irregular solution in 3 and 4 dimensions. If $J = J_0$, we can hope that this condition is sufficient for the regularity of every weak solution of this equation.

Let us consider some sufficient conditions for the regularity in 3 dimensions in more detail. We refer to the papers [14] and [15].

Let us consider the Euler equation

$$(22) \qquad \int_{\Omega} \frac{\partial F}{\partial \xi_i^r} (\nabla u) \frac{\partial \varphi_r}{\partial x_i} \, dx = \int_{\Omega} f_i^r \frac{\partial \varphi_r}{\partial x_i} \, dx,$$

where the Lagrangian $F(\xi)$ is defined and continuous together with its 4 derivatives in the cube $\overline{K}_a = \{ \xi \mid |\xi|_i^r \leq a \}$. Let $\partial\Omega$ be smooth enough, let $f_i^r \in W^{2,2}(\Omega)$, $u_r^0 \in W^{3,2}(\Omega)$, and let us look for a solution u of (22) such that $u \in [W^{3,2}(\Omega)]^m$, $u = u^0$ on $\partial\Omega$, $\| u \|_{1,\infty} = \max_{r,i} \left(\max_{x \in \overline{\Omega}} \left| \frac{\partial u_r}{\partial x_i}(x) \right| \right) < a$. We shall suppose the ellipticity condition

$$(23) \qquad \frac{\partial^2 F}{\partial \xi_i^r \partial \xi_j^s} (\xi) \eta_i^r \eta_j^s \geq c_1 |\eta|^2, \quad c_1 > 0$$

and the regularity condition

$$(24) \qquad c_1 - 3a^2 T > 0,$$

where

$$(25) \qquad \frac{\partial^4 F}{\partial \xi_i^r \partial \xi_j^s \partial \xi_k^t \partial \xi_e^v} (\xi) \eta_i^r \eta_j^s \eta_k^t \eta_e^v \leq T \sum_{r,i} (\eta_i^r)^4.$$

<u>Theorem 2.</u> (A priori estimate.) Let (23), (24) be satisfied. If u is the solution in question, then

$$(26) \qquad \| u \|_{3,2} \leq c(1 + \| f \|_{2,2}^2 + \| u^0 \|_{3,2}^2).$$

For the half space R_3^+ we get

<u>Theorem 3.</u> Let $\Omega = R_3^+$, $u^0 = 0$, let (23), (24) be satisfied. Then

$$(27) \qquad \| u \|'_{3,2} \leq c(\| f \|_{2,2} + \| f \|_{2,2}^2)$$

$$\left(\| u \|'_{k,2} \equiv \left(\int_{R_3^+} \sum_{1 \leq \alpha \leq k} (D^\alpha u)^2 \, dx \right)^{1/2} \right).$$

<u>Main idea of the proof:</u> Let $'$ be the derivative $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$. We have

$$(28)\qquad \int_{R_3^+} (f_i^r)'' \frac{\partial \varphi_r''}{\partial x_i}\,dx = \int_{R_3^+} \frac{\partial^2 F}{\partial \xi_i^r \partial \xi_j^s} \frac{\partial u_s''}{\partial x_j} \frac{\partial \varphi_r''}{\partial x_i}\,dx +$$

$$+ \int_{R_3^+} \frac{\partial^3 F}{\partial \xi_i^r \partial \xi_j^s \partial \xi_k^t} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial \varphi_r''}{\partial x_i}\,dx \equiv I_1 + I_2 .$$

Substituting the function u for φ in (28), we get

$$(29)\qquad I_2 = -\frac{1}{3} \int_{R_3^+} \frac{\partial^4 F}{\partial \xi_i^r \partial \xi_j^s \partial \xi_k^t \partial \xi_\ell^v} \frac{\partial u_r'}{\partial x_i} \frac{\partial u_s'}{\partial x_j} \frac{\partial u_t'}{\partial x_k} \frac{\partial u_v'}{\partial x_\ell}\,dx .$$

Through integration by parts, we obtain $\forall\, v \in \mathcal{D}(R_1)$:

$$(30)\qquad \int_{-\infty}^{\infty} (v')^4\,dx \leq 9 \max_{x \in R_1} [v(x)]^2 \cdot \int_{-\infty}^{\infty} (v'')^2\,dx .$$

From (28) - (30) we obtain

$$(31)\qquad (c_1 - 3\,T\,a^2)\, \| u'' \|'_{1,2} \leq \| f \|_{1,2} \,.$$

(29), (30) imply also

$$(32)\qquad \int_{R_3^+} \sum_{r,i} \left(\frac{\partial u_r'}{\partial x_i} \right)^4 dx \leq \frac{9a^2}{(c_1 - 3Ta^2)^2} \| f \|_{2,2}^2 \,.$$

Because the derivatives $\dfrac{\partial^2 u_s}{\partial x_3^2}$ can be calculated from the elliptic system

$$(33)\qquad -\frac{\partial^2 F}{\partial \xi_i^r \partial \xi_j^s} \frac{\partial^2 u_s}{\partial x_i \partial x_j} = -\frac{\partial f_i^r}{\partial x_i}, \qquad r = 1,2,\dots m,$$

we get from (32) and (33)

$$(34)\qquad \| u \|_{2,4} \leq c_2 c_1^{-1} [c_1 - 3Ta^2]^{-1/2} [\| f \|_{2,2}^{1/2} + \| f \|_{2,2}] \,.$$

Differentiating (33) first with respect to x_1 and x_2, we get

$$(35)\qquad \| u' \|_{2,2} \leq c_3 c_1^{-2} [c_1 - 3Ta^2]^{-1} (\| f \|_{2,2} + \| f \|_{2,2}^2).$$

Finally, differentiating (33) with respect to x_3, we get (27)

q.e.d.

The existence and uniqueness of a solution path, i.e., of u

from $C([0,t_{cr}], [W^{3,2}(\Omega)]^m)$, can be easily proved by the implicit function theorem and (26), provided that $f_i^r \in C([0,T], W^{2,2}(\Omega))$, $u_r^0 \in C([0,T], W^{3,2}(\Omega))$, see [14] and [15]. $t_{cr} \le T$ and is maximal, i.e., if $t_{cr} < T$, then $\max_{r,i} \left(\max_{x\in\bar{\Omega}} \frac{\partial u_r}{\partial x_i}(x) \right) = a$.

The papers [3], [9], [10], [11] and the papers by J. Nečas [18], J. Kadlec, J. Nečas [19] contain, in fact, estimates of the condition number, i.e. the estimate

$$\frac{c_1}{c_2} > h(n) \ge 0, \tag{36}$$

implying the regularity. Here, as before,

$$c_1 |\eta|^2 \le a_{ij}^{rs}(\xi)\eta_i^r \eta_j^s \le c_2 |\eta|^2, \tag{37}$$

where, for simplicity, we suppose $a_{ij}^{rs} = a_{ji}^{sr}$. In all the mentioned papers, $h(n)$ is not evaluated, each time only $h(2) = 0$. A precise evaluation is done by A.I. Koševev, see [20], where for systems a generalization of H.O. Cordes's condition is given, see [21]. Koševev's condition implies that the weak solution belongs to $[C^{(0),\mu}(\bar{\Omega})]^m$ which follows also from the fact that the weak solution belongs to $\bigcap_{p>1}[W^{1,p}(\Omega)]^m$, provided that some asymptote type conditions are valid for the functions $a_i^r(\xi)$, see J. Nečas [22].

We shall sketch the proof of

Theorem 4. Let

$$\frac{c_1}{c_2} > \frac{\sqrt{1 + \frac{(n-2)^2}{n-1}} - 1}{\sqrt{1 + \frac{(n-2)^2}{n-1}} + 1} . \tag{38}$$

Then the weak solution u to the equation

$$\int_\Omega a_i^r(\nabla u) \frac{\partial \varphi_r}{\partial x_i}\, dx = \int_\Omega f_i^r \frac{\partial \varphi_r}{\partial x_i}\, dx \tag{39}$$

lies in $[C^{(1),\mu}(\Omega)]^m$, provided that $f_i^r \in W^{1,p}(\Omega)$, $p > n$, and $\mu = 1 - \frac{n}{p}$. We have for $\bar{\Omega}' \subset \Omega$:

$$\|u\|_{[C^{1,\mu}(\bar{\Omega}')]^m} \le c(\Omega')\left[\|f\|_{1,p} + \|u\|_{1,2}\right], \tag{40}$$

n	2	3	4	5
$\frac{c_1}{c_2}$	0	0,101	0,209	0,286

.

<u>Lemma 1.</u> Let $u \in W^{1,2}(K_\varepsilon) \cap C^{(0),\gamma}(K_\varepsilon)$ be the solution of

$$-\Delta u = -\frac{\partial g_i}{\partial x_i}\,. \tag{41}$$

Let $n-2 < \lambda < n-2+2\gamma$. Then

$$\int_{K_\varepsilon} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i} r^{-\lambda}\,dx \le \alpha(\lambda)\left(1+\frac{(n-2)^2}{n-1}\right)\cdot \tag{42}$$

$$\Big[\int_{K_\varepsilon} g_i g_i r^{-\lambda}\,dx + c_3\int_{K_\varepsilon} g_i g_i\,dx +$$

$$+ c_3\int_{K_\varepsilon}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\,dx + c_3|u(0)|^2\Big],$$

where $\alpha(\lambda) \to 1$ as $\lambda \to (n-2)$.

<u>Proof.</u> Put $v(x) \equiv (u(x)-u(0))\,\psi(x)$, $\psi \in \mathcal{D}(K_\varepsilon)$, $\psi(x) = 1$ for $|x| \le \frac{\varepsilon}{2}$, $0 \le \psi \le 1$. We have

$$\int_{R_n}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}\,dx = \int_{R_n} h_i\frac{\partial u}{\partial x_i}\,dx, \tag{43}$$

with

$$\Big[\int_{R_n} h_i h_i(1+r^{-\lambda})\,dx\Big]^{1/2} \le \Big[\int_{K_\varepsilon} g_i g_i r^{-\lambda}\,dx\Big]^{1/2} + \tag{44}$$

$$+ c_4\Big[\int_{K_a} g_i g_i\,dx\Big]^{1/2}.$$

In polar coordinates (employing our notation) we get for v:

$$\int_0^\infty\int_S \partial_i v\,\partial_i\varphi\, r^{n-1}dr\,dS = \int_0^\infty\int_S m_i\,\partial_i\varphi\, r^{n-1}\,dr\,dS, \tag{45}$$

where $m_i = a_{ij}h_j$. Putting $p' = p-(n-3)$, $p = x'+iy$, $x = -\frac{1}{2}\,[\lambda-(n-2)]$, $y \in R_1$, $x < x' < 0$, $x \ge -\gamma$, $\varphi = w(\vartheta)r^{p'-1}$, $w \in \mathcal{E}(S)$ and denoting by

$$(46)\quad V(p,\vartheta) \equiv \int_0^\infty r^{p-1} v(r,\vartheta)\, dr, \quad M_i(p,\vartheta) \equiv \int_0^\infty r^p m_i(r,\vartheta)\, dr,$$

Mellin's transforms of v and m_i, we obtain from (45)

$$(47)\quad \int_S \left[-p(p'-1)Vw + \overline{\partial}_i, V \overline{\partial}_i, w\right] dS = \int_S \left[M_1(p'-1)w + M_i, \overline{\partial}_i, w\right] dS.$$

Let us decompose $W^{1,2}(S) = W_1 + W_2$, $W_1 \equiv \{const\}$, let $V = V_1 + V_2$, $M = M_1^1 + M_1^2$ be orthogonal decompositions, the latter in $L_2(S)$. Put $w = -\frac{\overline{p}}{p'-1}\overline{V}_1$ in (47). We obtain

$$(48)\quad \int_S |p|^2 |V_1|^2\, dS \leq \int_S |M_1|^2\, dS.$$

Put $w = \overline{V}_2$. Because we have

$$(49)\quad (n-1)\int_S |w|^2\, dS \leq \int_S \overline{\partial}_i, w \overline{\partial}_i, \overline{w}\, dS,$$

for w from W_2 we get in virtue of (48):

$$(50)\quad \int_S \left[|p|^2 |V|^2 + \overline{\partial}_i, V \overline{\partial}_i, \overline{V}\right] dS \leq$$

$$\leq \left[1 + \frac{(n-2)^2}{n-1} - 2x' \frac{n-2}{n-1}\right] \int_S M_i \overline{M}_i\, dS.$$

Parseval's identities

$$(51)\quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[|p|^2|V|^2 + \overline{\partial}_i, V\overline{\partial}_i, \overline{V}\right] dy = \int_0^\infty (\partial_i v \partial_i v) r^{-\lambda'+n-1}\, dr,$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} M_i \overline{M}_i\, dy = \int_0^\infty m_i m_i r^{-\lambda'+n-1}\, dr,$$

together with (50) imply the result, q.e.d.

<u>Lemma 2.</u> Let $v \in [W^{1,2}(K_a)]^m$ be a weak solution to the linear system

$$(52)\quad \int_{K_\varepsilon} a_{ij}^{rs} \frac{\partial v_r}{\partial x_i} \frac{\partial \varphi_s}{\partial x_j}\, dx = \int_{K_\varepsilon} h_i^r \frac{\partial \varphi_r}{\partial x_i}\, dx,$$

$a_{ij}^{rs} = a_{ji}^{sr} \in L_\infty(K_a)$,

(53) $$c_1|\xi|^2 \leq a_{ij}^{rs}\xi_i^r\xi_j^s \leq c_2|\xi|^2.$$

Let λ be chosen such that (see Lemma 1)

(54) $$\frac{c_1}{c_2} > \frac{\alpha(\lambda)^{1/2}\sqrt{1+\frac{(n-2)^2}{n-1}}-1}{\alpha(\lambda)^{1/2}\sqrt{1+\frac{(n-2)^2}{n-1}}+1}.$$

Then

(55) $$\sup_{x_0\in K_{\varepsilon/2}}\int_{K_\varepsilon}\frac{\partial v_r}{\partial x_i}\frac{\partial v_r}{\partial x_i}|x-x_0|^{-\lambda}dx \leq$$

$$\leq c_3\left[\sup_{x_0\in K_\varepsilon}\int_{K_\varepsilon}h_i^r h_i^r|x-x_0|^{-\lambda}dx+\int_{K_\varepsilon}\frac{\partial v_r}{\partial x_i}\frac{\partial v_r}{\partial x_i}dx\right].$$

<u>Proof.</u> Smoothing h_i^r and a_{ij}^{rs} by a positive mollifier, we can suppose $v\in[C^{(1)}(K_\varepsilon)]^m$. Let us remark that (53) remains true after mollifying. Put $\gamma=\frac{2}{c_1+c_2}$. Using the equation (where $K(x_0,\delta)\equiv$ $\equiv|x-x_0|<\delta$)

(56) $$\int_{K(x_0,\frac{\varepsilon}{2})}\frac{\partial v_r}{\partial x_i}\frac{\partial\varphi_r}{\partial x_i}dx=\int_{K(x_0,\frac{\varepsilon}{2})}\left(\frac{\partial v_r}{\partial x_i}-\gamma a_{ij}^{rs}\frac{\partial v_s}{\partial x_j}\right)\cdot$$

$$\cdot\frac{\partial\varphi_r}{\partial x_i}dx+\gamma\int_{K(x_0,\frac{\varepsilon}{2})}g_i^r\frac{\partial\varphi_r}{\partial x_i}dx$$

and the relation

(57) $$\int_{K(x_0,\frac{\varepsilon}{2})}\left(\frac{\partial v_r}{\partial x_i}-\gamma a_{ij}^{rs}\frac{\partial v_s}{\partial x_j}\right)\left(\frac{\partial v_r}{\partial x_i}-\gamma a_{ij}^{rs}\frac{\partial v_s}{\partial x_j}\right)\cdot$$

$$\cdot\ x-x_0{}^{-\lambda}dx\leq\left(\frac{c_2-c_1}{c_2+c_1}\right)^2\int_{K(x_0,\frac{\varepsilon}{2})}\frac{\partial v_r}{\partial x_i}\frac{\partial v_r}{\partial x_i}|x-x_0|^{-\lambda}dx,$$

we get the result (if we let the mollifier's parameter converge to zero), taking into account that, $\forall\,\delta>0$,

(58) $$v_r(x_0)v_r(x_0)\leq\delta\sup_{x_0'\in K_{\frac{\varepsilon}{2}}}\int_{K(x_0',\frac{}{2})}.$$

$$\frac{\partial v_r}{\partial x_i}\frac{\partial v_r}{\partial x_i}\,|x-x_0'|^{-\lambda}\,dx + \mu(\delta)\int_{K_\varepsilon}\left(\frac{\partial v_r}{\partial x_i}\frac{\partial v_r}{\partial x_i} + v_r v_r\right) dx\ ,$$

see for example J. Nečas [23] , q.e.d.

Theorem 4 can be proved by the standard method, starting from the equations (9) in variations, using Lemmas 1 and 2, and finally the standard results for the regularity of the weak solution to the linear systems with Hölder-continuous coefficients, see for example S. Agmon, A. Douglis, L. Nirenberg [24].

We conclude our explanation with a regularity theorem for systems of variational inequalities and Signorini's type unilateral conditions, see G. Fichera [25], J. Nečas [26], J. Frehse [28], [29].

Let Ω be a domain with the Lipschitz boundary $\partial\Omega$, let $\Gamma \subset \partial\Omega$ be a part of $\partial\Omega$ smooth enough. We suppose on Γ

(59) $$b_{rs}u_s \geq \psi_r, \quad r = 1,2,\dots k \leq m,$$

with b_{rs}, ψ_r regular enough. Let $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}$, where Γ_1, Γ_2, Γ are disjoint open sets in $\partial\Omega$. We suppose that there exists $u_1 \in [W^{1,2}(\Omega)]^m$ such that $b_{rs}u_s^1 = \psi_r$. Let $u^0 \in$ $\in [W^{1,2}(\Omega)]^m$, $g \in [L_2(\Gamma_2 \cup \Gamma)]^m$, $f \in [L_2(\Omega)]^m$. Let the rank of $b_{rs}(x) = k$. Put

(60) $$K \equiv \{v \mid v = u^0 \text{ on } \Gamma_1,\ b_{rs}u_s \geq \psi_r,\ r = 1,2,\dots k \text{ on } \Gamma\}.$$

We suppose that $u_0 \in K$ and we look for $u \in K$ such that, for $\forall v \in K$, we have

(61) $$\int_\Omega a_i^r(\nabla u)\left(\frac{\partial v_r}{\partial x_i} - \frac{\partial u_r}{\partial x_i}\right) + \int_\Omega u_r(v_r-u_r)\,dx \geq$$
$$\geq \int_\Omega f_r(v_r-u_r)\,dx + \int_{\Gamma_2\cup\Gamma} g_r(v_r-u_r)\,dS.$$

We can answer the regularity question by the penalty method. We put

(62) $$(\beta(u),v) = -\int_\Gamma (b_{rs}u_s-\psi_r)^- b_{rt}v_t\,dS$$

and look for the solutions u^ε of the equations with penalty.

By a standard difference method, see [26] in detail, we get

<u>Theorem 5.</u> Let $F = \overline{F} \subset U(F) \cap \overline{\Omega} \subset F^* \subset \overline{F}^* \subset \Omega \cup \Gamma$, where $U(F)$ is a neighbourhood of F. Under our assumptions, provided that

$u^1 \in [W^{3,2}(F^*)]^m$, $g \in [W^{1,2}(\Gamma)]^m$, we obtain

(63) $$\|u^\varepsilon\|_{[W^{2,2}(F)]^m} \leq c(F)\,\Big[1 + \|u^\varepsilon\|_{[W^{1,2}(\Omega)]^m} + \|f\|_{[L_2(\Omega)]^m} + \|g\|_{[W^{1,2}(\Gamma)]^m} + \|u^1\|_{[W^{3,2}(F^*)]^m}\Big].$$

Bibliography

[1] O.A. Ladyženskaja, N.N. Uralceva: Linějnyje i kvazilinějnyje uravněnija eliptičeskovo tipa, Moskva 1973 (the second edition),

[2] Ch.B. Morrey: Differentiability theorems for weak solutions of nonlinear elliptic differential equations, BAMS, Vol. 75, n.4 (1969), 684-705,

[3] Ch.B. Morrey: Existence and differentiability theorems for the solutions of variational problems for multiple integrals, BAMS 46 (1940), 439-458,

[4] E. Giorgi: Sulla differenziabilita e analiticita delle estremali degli integrali multipli regolari, Mem. Acad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3) 3 (1957), 25-43,

[5] J. Moser: A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations, Comm. Pure Appl. Math. 13 (1960), 457-468,

[6] G. Stampacchia: Problemi al contorno ellitici, con dali discontinui, dotati di soluzioni hölderiane, Ann. Mat. Pura ed Appl. 51 (1960), 1-38,

[7] J.L. Lions: Quelques méthodes de résolution des problèmes aux limites non linéaires, Paris 1969,

[8] S. Fučík, J. Nečas, J. Souček, V. Souček: Spectral analysis of nonlinear operators, lecture notes in Mathematics 346, Springer,

[9] J. Stará: Regularity Results for Nonlinear Elliptic Systems in two Dimensions, Annali Scuola Norm. Sup. Pisa Vol. XXV, Fase I (1971) 163-190,

[10] J. Nečas: Sur la régularité des solutions variationelles des équations elliptiques non-linéaires d'ordre 2k en deux dimensions, Annali Scuola. Norm. Sup. Pisa, fasc. III (1967) 427-457,

[11] J. Nečas: Sur la régularité des solutions des équations elliptiques non linéaires, Comm. Math. Univ. Cavol. 9,3 (1968), 365-413,

[12] E. Giusti, M. Miranda: Un esempio di soluzioni discontinue per un problema di minimo relatio ad un integrale regolare del calcolo delle variazioni, Boll. Un. Mat. Ital. (4) 2 (1968), 1-8,

[13] S.A. Arakčejev: O gladkosti obobščonnych rešenij někotorovo klasa kvazilinějnych ellipitičeskich uravněnij, Věstník Mosk. Univ. n. 1, 1975, 49-57,

[14] J. Nečas: Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity, to appear in Beiträge Zur Analysis, 1977,

[15] J. Nečas, J. Stará, R. Švarc: Classical solution to a second order nonlinear elliptic systems in R_3, to appear in Annali Scuola Norm. Sup. Pisa, 1978,

[16] C. Truesdell: A first course in rational continuum mechanics, 1972,

[17] E. Giusti: Un' aggiunta alla mia nota: Regollarità parziale delle soluzioni di sistemi ellittici quasi lineari di ordine arbitrario, Ann. Scuola Norm. Sup. Pisa Ser. 3, 27, n.2, 1973,

[18] J. Nečas: On the existence and regularity of solutions of non-linear elliptic equations, Acta fac. rer. nat. Univ. Com. Math. XVII, 1967, 101-119,

[19] J. Kadlec, J. Nečas: Sulla regularità delle soluzioni di equazioni ellitiche negli spazi $H^{k,\lambda}$, Annali Scuola Norm. Sup. Pisa, Vol. XXI, Fasc. IV (1967), 527-545,

[20] A.I. Košelev: O gladkosti rešenij kvazilinějnych elliptičeskich sistěm vtorovo porjadka, DAN 1976, T. 228, n.4, 787-789,

[21] H.O. Cordes: Über die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen, Math. Annalen, Bd. 131, 278-312 (1956),

[22] J. Nečas: On the demiregularity of weak solutions of nonlinear elliptic equations, BAMS, Vol. 77, n. 1, (1971),

[23] J. Nečas: Les méthodes directes en théorie des équations elliptiques, 1967,

[24] S. Agmon, A. Dougles, L. Nirenberg: Estimates near boundary for solutions of elliptic partial diff. equat. satisfying general boundary conditions, I, II, Comm. Pure Appl. Math. 12 (1959), 623-727; 17(1964), 35-92,

[25] G. Fichera: Boundary value problems of elasticity with unilateral constraints, Handbuch der Physik Vol. VII a/2, 1972,

[26] J. Nečas: On regularity of solutions to nonlinear variational inequalities for second-order elliptic systems, Rendiconti di Matematica (2), Vol. 8, Ser. VI, 481-498,

[27] E. Shamir: Regularity of mixed second-order elliptic problems, Israel Math. J. 6 (1968),

[28] J. Frehse: On Variational Inequalities with Lower Dimensional Obstacles, Universität Bonn, Preprint n. 114,

[29] J. Frehse: On Signorini's Problem and Variational Problems with Thin Obstacles, Universität Bonn, Preprint n. 117.

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