BOUNDARY BEHAVIOR OF POTENTIALS

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Let L be an elliptic operator of the form

$$Lu = \sum_{i,k=1}^{m} \frac{\partial}{\partial x^k}\left(a_{ik} \frac{\partial}{\partial x^i} u\right) + \sum_{i=1}^{m} e_i \frac{\partial}{\partial x^i} u + cu$$

with sufficiently smooth coefficients in a domain $\Omega \subset R^m$ $(m > 2)$. It is well known that under certain conditions on L and Ω there exists a fundamental solution $G(x,y)$ on $\Omega \times \Omega$ which is smooth off the diagonal and has a specified singularity at points of the diagonal admitting locally uniform estimates of the type

$$G(x,y) = \mathcal{O}\left(\operatorname{dist}(x,y)^{2-m}\right), \tag{1}$$

$$|dG(x,y)| = \mathcal{O}\left(\operatorname{dist}(x,y)^{1-m}\right) \tag{2}$$

as $\operatorname{dist}(x,y) \to 0+$ (here dist... denotes the distance and d stands for the differential). For compactly supported finite signed Borel measures μ the potentials

$$G\mu(x) = \int_{\Omega} G(x,y)\, d\mu(y) \tag{3}$$

are locally integrable together with their derivatives and are often used to transform boundary value problems for L into integral equations. Various aspects of the method of potentials in the theory of partial differential equations together with ample references to the classical work of E.E.Levi, G.Giraud, M. Gevrey and others may be found in C.Miranda's monograph [1]. As pointed out by W.Feller [2], the leading part of the operator L can (possibly after multiplication by a suitable factor) conveniently be written in the form of the Laplace-Beltrami operator

$$\frac{1}{\sqrt{g}} \sum_{i,k=1}^{m} \frac{\partial}{\partial x^k}\left(g^{ik} \sqrt{g}\, \frac{\partial}{\partial x^i} \cdots\right)$$

corresponding to a Riemannian metric defined by the form

$$\sum_{i,k=1}^{m} g_{ik}\, dx^i\, dx^k \quad .$$

This permits a better insight in some properties of solutions of $Lu = 0$; in particular, the usual conormal derivative associated with L reduces to the ordinary normal derivative corresponding to the Riemannian metric. We wish to indicate here that this point of view has useful applications in connection with investigation of boundary behavior of potentials and, in particular, their weak normal derivatives.

Instead of a domain in R^m we shall thus consider an m-dimensional Riemannian manifold Ω (without boundary) which is smooth (say, of class C^∞) and oriented. On Ω we shall consider an operator L of the form

$$Lu = *(\, d*du + du\wedge E + uC\,)$$

whose leading part is the Laplace-Beltrami operator on Ω ; here $*$ is the Hodge star operator mapping k-forms into (m-k)-forms , d is the exterior derivative, $\wedge$ is the exterior product, E is a differential (m-1)-form and C is a differential m-form. The transpose of L has the form

$$Mv = *(\, d*dv - dv\wedge E + v(C-dE)\,) \,.$$

We shall suppose that we are given a function $G(x,y)$ on $\Omega \times \Omega$ which is smooth off the diagonal and satisfies in the weak sense the equations

$$L_x\, G(x,y) = \varepsilon_y \quad , \quad y \in \Omega \quad ,$$

$$M_y\, G(x,y) = \varepsilon_x \quad , \quad x \in \Omega \quad ,$$

where ε_z denotes the Dirac measure concentrated at z; with the estimates of the form (1),(2) (where now the distance dist...

is derived from the Riemannian metric) we have then for each compactly supported signed Borel measure μ the potential (3) which together with $dG\mu$ is almost everywhere defined (and locally summable).

We shall fix an open set $Q \subset \Omega$ with a compact boundary $B \subset \Omega$ and denote by $C^*(B)$ the Banach space of all signed Borel measures with support contained in B ; the norm in $C^*(B)$ is given by total variation. C_0^1 will denote the class of all continuously differentiable functions with compact support on Ω . If $\mu \in C^*(B)$, then the weak normal derivative of $u = G\mu$ may be defined as the functional Nu over C_0^1 by the formula

$$\langle \varphi, Nu \rangle = \int_Q [d\varphi \wedge * du - \varphi du \wedge E - \varphi uC] .$$

(If the boundary B of Q is a properly oriented hypersurface, then $\langle \varphi, Nu \rangle = \int_B \varphi \wedge * du$ so that Nu is a reasonable weak characterization of the normal derivative .)

With the exception of the compactness requirement we make now no à priori restriction on the boundary B of Q and with each $\mu \in C^*(B)$ we associate the corresponding functional $NG\mu$. It is easily seen that the support of $NG\mu$ is contained in B (in the sense that $\langle \varphi, NG\mu \rangle = 0$ whenever $\varphi \in C_0^1$ has support disjoint with B). In general, $NG\mu$ need not be representable by a (signed) measure. On the other hand, if there is a representing measure ν for $NG\mu$, which means that

$$\langle \varphi, NG\mu \rangle = \int_\Omega \varphi \, d\nu$$

for all $\varphi \in C_0^1$, then necessarily the support of ν is contained in B so that $\nu \in C^*(B)$; in this case we identify $NG\mu = \nu$, as usual.

We thus arrive naturally at the following

Question . What conditions on B guarantee that $NG\mu \in C^*(B)$ for every $\mu \in C^*(B)$?

In order to answer this question in geometric terms it appears useful to generalize the concept of a hit introduced in [3] , [4] in connection with investigation of Newtonian potentials in m-space. Let us denote by H^1 the length (= 1-dimensional Hausdorff measure) derived in the usual way from the metric in Ω . If Γ is a simple arc and $P \subset \Omega$ is a Borel set, we call $\eta \in \Gamma$ a hit of Γ on P (and say that Γ hits P at η) provided, for every neighborhood U of η ,

$$H^1(U \cap \Gamma \cap P) > 0 \quad \text{and} \quad H^1((U \setminus P) \cap \Gamma) > 0 .$$

Let us now fix a point $y \in \Omega$ and consider the tangent space $T\Omega_y$ of Ω at y ; let $S_y = \{ \theta \in T\Omega_y ; |\theta| = 1 \}$ denote the sphere of unit vectors and $d\sigma$ the element of the surface measure in S_y (induced by the metric in $T\Omega_y$), $A = \int_{S_y} d\sigma$. If $r > 0$ is sufficiently small, then the exponential map at y

$$\exp_y : T\Omega_y \longrightarrow \Omega$$

is well defined and 1-1 on the set

$$\{ \rho\theta ; \theta \in S_y , 0 \leq \rho < r \}$$

and we may consider the geodesic arcs

$$\Gamma_r(y,\theta) = \{ \exp_y \rho\theta ; 0 < \rho < r \} , \quad \theta \in S_y .$$

We shall denote by $n_r^Q(y,\theta)$ the total number of all hits of $\Gamma_r(y,\theta)$ on Q ($0 \leq n_r^Q(y,\theta) \leq +\infty$). It can be shown that the function

$$\theta \longmapsto n_r^Q(y,\theta)$$

is Borel measurable so that we may define

$$v_r^Q(y) = \frac{1}{A} \int_{S_y} n_r^Q(y,\theta)\, d\,\sigma(\theta) .$$

Thus $v_r^Q(y)$ is just the average number of points at which the open geodesic arcs of length r starting at y hit Q. It is also useful to adopt the following notation. Let $K \subset \Omega$ be a compact set. Then, for sufficiently small $r > 0$, $v_r^Q(y)$ is defined for all $y \in K$ and we put

$$V_o^Q(K) = \lim_{r \downarrow 0} \sup \left\{ v_r^Q(y);\ y \in K \right\} .$$

With this notation we have the following answer to the above question.

Theorem 1 . If $NG\mu \in C^*(B)$ for every $\mu \in C^*(B)$, then necessarily

$$V_o^Q(B) < +\infty . \tag{4}$$

Conversely, if (4) holds, then $NG\mu \in C^*(B)$ whenever $\mu \in C^*(B)$ and the operator

$$NG : \mu \longmapsto NG\mu \tag{5}$$

is bounded on $C^*(B)$.

The basic ideas of the proof of this theorem are similar to those employed in section 1 in [3] .

If we assume (4) and denote by $C(B)$ the Banach space of all continuous functions on B (equipped with the maximum norm), then

$$Wf(y) = \langle f,\ NG\,\varepsilon_y \rangle$$

represents a continuous function of the variable $y \in B$ for every $f \in C(B)$ and the operator (5) is dual to the operator

$$W : f \longmapsto Wf \tag{6}$$

acting on $C(B)$.

The operator W, which is closely connected with the classical double layer potentials, admits various concrete integral representations analoguous to those obtained in section 2 in [3] for Newtonian potentials. They are partly based on the fact that (4) implies that Q has finite perimeter

$$P(Q) = \sup \left\{ \int_Q d\psi \ ; \ |*\psi| \leq 1 \right\},$$

where ψ ranges over differential (m-1)-forms with compact support in Ω, and on some results concerning sets with finite perimeter (compare [5]-[7]).

The operator (6) is more easily treated than (5) and its analytic properties are closely tied with geometric structure of B. As an illustration we shall evaluate the quantity

$$\omega(\alpha) = \inf_T \| W + \alpha I - T \| ,$$

where T ranges over all compact operators on C(B), $\alpha \in R^1$ and I is the identity operator. For simplicity we shall state the formula under a mild simplifying restriction requiring $\mathrm{vol}(U_y \setminus Q) > 0$ for every neighborhood U_y of any $y \in B$ (vol... denotes the volume in Ω). We have

Theorem 2 . If (4) holds, then the density

$$D_Q(y) = \lim_{r\downarrow 0} \frac{\mathrm{vol}(\{ z \in Q;\ \mathrm{dist}(z,y) < r\})}{\mathrm{vol}(\{ z \in \Omega\ ;\mathrm{dist}(z,y) < r\})}$$

exists for all $y \in B$ and the following equality holds for any $\alpha \in R^1$

$$\omega(\alpha) = \lim_{r\downarrow 0} \sup_{y \in B} (|\alpha - D_Q(y)| + v_r^Q(y)).$$

Moreover,

$$\min\left\{ \frac{\omega(\alpha)}{\alpha} \ ; \ \alpha \in R^1 \right\} = 2\,\omega(\tfrac{1}{2}) = 2\ V_0^Q(B) .$$

Results analoguous to theorems 1,2 were originally established for logarithmic and Newtonian potentials and proved to be useful in connection with the Radon scheme [8] for treating the Dirichlet and the Neumann problem as well as related problems in potential theory (compare [3],[4],[9] - [11] including further references). The above results permit similar applications in a more general setting . In distinction to local results we have described here, however, some of these applications depend on global behavior of the kernel G. These considerations remain beyond the scope of the present lecture.

Finally we wish to mention that the quantity $v_r^Q(.)$ permits also to obtain necessary and sufficient conditions for the existence of angular limits of potentials analoguous to those known for logarithmic or Newtonian potentials (cf. [12],[13]) and admits further generalizations useful in various investigations (cf.[14] - [17]).

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