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DUAL FINITE ELEMENT ANALYSIS FOR SOME UNILATERAL BOUNDARY VALUE PROBLEMS

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A great number of papers of both technical and mathematical character has been devoted to the numerical solution of variational inequalities. For instance, in the book [1] more than 300 titles are quoted.

The boundary value problems with inequalities contain two important classes:

(i) problems with inequalities on the boundary of the domain under consideration,

(ii) problems with inequalities in the domain.

In general, the problems of both classes can be solved approximately by means of finite differences or finite elements. For problems of the class (i), however, where the inequalities are concentrated on the boundary, only the finite element method is applied in case of a general boundary. In the following, we restrict ourselves to the problems of the class (i) for elliptic equations. A survey of some recent results on the dual variational approach applied to several problems of the second order will be presented.

1. An equation of the second order

To point out the main idea, let us consider the following model problem in a bounded polygonal domain $G \subset R^2$:

$$(1) \qquad \begin{aligned} -\Delta u &= f \quad \text{in } G, \\ u &= 0 \quad \text{on } \Gamma_u, \\ u \geq 0, \quad \frac{\partial u}{\partial n} &\geq 0, \quad u\frac{\partial u}{\partial n} = 0 \qquad \text{on } \Gamma_a = \Gamma - \Gamma_u, \end{aligned}$$

where Γ_u and Γ_a are parts of the boundary $\Gamma = \partial G$, $\partial u/\partial n$ denotes the derivative with respect to the outward normal $\underline{n}$.

We shall denote $H^k(G) = W_2^{(k)}(G)$ the Sobolev spaces with the usual norm $\|\cdot\|_k$ and $H^0(G) = L_2(G)$. The semi-norm consisting of all derivatives of the k-th order only, will be denoted by $|\cdot|_k$.

Assume that the right-hand side of the equation (1) $f \in L_2(G)$.

1.1. Primary variational formulation

Let us introduce the functional of the potential energy

$$L(v) = \frac{1}{2} \int_G |\operatorname{grad} v|^2 dx - \int_G fv dx$$

and the set of admissible functions

$$K = \{ v \in H^1(G) \mid \gamma v|_{\Gamma_u} = 0, \quad \gamma v|_{\Gamma_a} \geq 0 \},$$

(where γv denote the traces of the function v).

The problem (1) corresponds to the following variational (primary) problem:

(2) to find a function $u \in K$ such that

$$L(u) \leq L(v) \;\forall\; v \in K .$$

The problem (2) has a unique solution. It is not difficult to prove that (i) any solution of the problem (1) satisfies the condition (2) and (ii) any solution of (2) satisfies the equation (1_1) in the sense of distributions and the boundary conditions (1_3) in a functional sense, i.e. in the space $H^{-1/2}(\Gamma)$.

For the approximations to the primary problem - see e.g. [3],[5] .

1.2. Dual variational formulation

We often have problems when the gradient (or cogradient) of the solution u is more interesting than the solution itself. In physical problems grad u represents the vector of fluxes, in elasticity it corresponds to the stress tensor.

Therefore it may be useful to formulate the problem directly in terms of the unknown vector-function of the gradient. To this end let us introduce the set

$$Q = \{ q \in [L_2(G)]^2 \mid \operatorname{div} q \in L_2(G) \} .$$

For $q \in Q$ we may define the functional (outward flux) $q.n \in H^{-1/2}(\Gamma)$ as follows

$$\langle q.n, w \rangle = \int_G (q.\operatorname{grad} v + v \operatorname{div} q)dx \;\forall\; w \in H^{1/2}(\Gamma) ,$$

where $v \in H^1(G)$ is an extension of the function $w = \gamma v$.

We write $s|_{\Gamma_a} \geq 0$ for a functional $s \in H^{-1/2}(\Gamma)$ if

$$\langle s, \gamma v \rangle \geq 0 \;\forall\; v \in K .$$

Let us introduce the set of admissible functions

$$U = \{ q \in Q \mid \operatorname{div} q + f = 0 \quad \text{in } G, \; q.n|_{\Gamma_a} \geq 0 \} ,$$

the functional of complementary energy

$$S(q) = \frac{1}{2} \| q \|_0^2$$

and the dual variational problem:

to find $q^0 \in U$ such that

(3) $$S(q^0) \leq S(q) \;\forall\; q \in U .$$

The problem (3) has a unique solution. Moreover, there is a connection between the solution u of the primary problem and the solution q^0 of the dual problem as follows:

(4) $$q^0 = \operatorname{grad} u ,$$

(5) $$L(u) + S(q^0) = 0 .$$

The proof of (4) and (5) can be based on the saddle point theory (cf. [5]).

1.3. Approximations to the dual problem

Assume that we have a vector $\bar{q} \in Q$ such that $\operatorname{div} \bar{q} + f = 0$ in the domain G. (We can set e.g. $\bar{q} = (\bar{q}_1, 0)$, where
$$\bar{q}_1(x_1,x_2) = - \int_0^{x_1} f(t,x_2)dt \quad .)$$

Then it is readily seen that $q \in U$ if and only if $p = q - \bar{q} \in U_0$, where
$$U_0 = \left\{ p \in Q \,\middle|\, \operatorname{div} p = 0, \quad (p.n + \bar{q}.n)|_{\Gamma_a} \geq 0 \right\} .$$

Thus we are led to the following problem, which is equivalent with the problem (3):

to find $p^0 \in U_0$ such that

(6) $$J(p^0) \leq J(p) \;\forall\; p \in U_0 ,$$

where
$$J(p) = \frac{1}{2} \|p\|_0^2 + (\bar{q},p)_0 .$$

With respect to the definition of the set U_0, instead of the standard finite element spaces V_h we have to employ the spaces of the so called equilibrium (solenoidal) elements, which satisfy the equation $\operatorname{div} p = 0$ in the domain G at least in the sense of distributions. To this end we apply the spaces N_h of piecewise linear triangular elements, which were proposed by Veubeke and Hogge in [6] and studied in the paper [7]. The latter study yields the following approximation property of the equilibrium spaces N_h (on any regular family of triangulations):

(7) $$\forall\; p \in [H^2(G)]^2 , \operatorname{div} p = 0, \quad \exists\, \pi^h \in N_h :$$
$$\|p - \pi^h\|_0 \leq C\, h^2 |p|_2 .$$

In the following, we shall use the strongly regular family of triangulations $\{T_h\}$ (i.e., the minimal angle in T_h is bounded from below and the ratio of any two sides in T_h is bounded from above, the bounds being independent of the parameter h - the maximal side in the triangulation T_h).

Assume that a function $F \in [H^2(G)]^2$ exists, such that

$$\operatorname{div} F = 0 \quad \text{in} \quad G, \quad F.n|_{\Gamma_a} = -\bar{q}.n|_{\Gamma_a} .$$

Let us denote $-\bar{q}.n = g$ and construct a function $g_h \in L_2(\Gamma_a)$ such that its restriction $g_h|_{S_j}$ onto any side $S_j \subset \bar{\Gamma}_a$ of the triangulation T_h coincides with the $L_2(S_j)$-projection of g into the subspace of linear polynomials $P_1(S_j)$.

Defining

$$U_{0h} = \{ p \in N_h \mid p.n|_{\Gamma_a} \geq g_h \} ,$$

we say that a vector $p^h \in U_{0h}$ is a finite element approximation to the dual problem, if

(8) $$J(p^h) \leq J(p) \;\forall\; p \in U_{0h} .$$

(Note that $U_{0h} \not\subset U_0$ unless $g_h \geq g$ holds everywhere on Γ_a.)

Theorem 1. Let the boundary Γ consist of a finite number of non-intersecting polygons ∂G_j and let

$$\operatorname{meas}(\Gamma_u \cap \partial G_j) > 0 \;\forall\; j .$$

Assume that

$$p^0 \in [H^2(G)]^2 , \quad (p^0 - F).n \in H^2(\Gamma_a \cap \Gamma_m)$$

holds for any side Γ_m of the polygonal boundary.

Then for any strongly regular family of triangulations it holds

(9) $$\| p - p^h \|_0 \leq C(p^0) h^{3/2} .$$

Proof. Let us define the mapping $\Pi_T \in \mathcal{L}([H^1(T)]^2; [P_1(T)]^2)$ for all $T \in T_h$ by the following conditions: the $L_2(S_j)$-projection of the flux $q.n|_{S_j}$ is equal to the flux $(\Pi_T q).n|_{S_j}$ for each side S_j of the triangle T.

Define also the set

$$R(G) = \{ q \mid q \in [H^1(G)]^2 , \operatorname{div} q = 0 \}$$

and the mapping $r_h \in \mathcal{L}(R(G); N_h)$ by the conditions

$$r_h q|_T = \Pi_T q \;\forall\; T \in T_h .$$

(Then $\pi^h = r_h p$ can be taken in (7).)

Let us introduce the cones

$$C = \{ q \in Q \mid \operatorname{div} q = 0, \quad q.n|_{\Gamma_a} \geq 0 \} , \quad C_h = C \cap N_h .$$

Lemma 1. Denote $U = p^0 - F \in C$ and assume that a $W_h \in C_h$ exists such that $2U - W_h \in C$. Then it holds

(10) $$\| p^0 - p^h \|_0 \leq \| U - W_h \|_0 + \| F - r_h F \|_0 .$$

(For the proof of (10) one employs the variational inequalities characterizing p^0 and p^h, respectively.)

Thus it suffices to find a suitable element $W_h \in C_h$ close to U. To this end we (i) construct a one-sided piecewise linear approximation ψ_h of the boundary flux $U.n$ and (ii) define $W_h = r_h U$ in the interior elements $T \in T_h$, (iii) correct $r_h U$ in the boundary strip to obtain $W_h.n = \psi_h$ on Γ.

Let us present the approach in detail:

Lemma 2. If the assumptions of Theorem 1 hold, then there exists a piecewise linear function ψ_h on Γ (with the nodes determined by the triangulation T_h) and such that

$$\int_{\partial G_j} \psi_h ds = \int_{\partial G_j} (r_h U).n ds \;\forall j , \tag{11}$$

$$0 \le \psi_h \le U.n \quad \text{on} \quad \Gamma_a , \tag{12}$$

$$\|(r_h U).n - \psi_h\|_{0,\Gamma} \le c\, h^2 \sum_m |U.n|_{2,\, \Gamma_a \cap \Gamma_m} . \tag{13}$$

Proof. Denote $U.n = t$, $(r_h U).n = t_h$ and consider a side $S_i \subset \subset \overline{\Gamma}_a$. Let t_I be the linear interpolate of t over S_i. First we construct the function $\psi_h|_{S_i} \equiv \psi_h^i$.

1^0 If $t \ge t_I$ on S_i, setting $\psi_h^i = t_I$ we obtain

$$\|\psi_h^i - t_h\|_{0,S_i} \le \|t - t_I\|_{0,S_i} + \|t - t_h\|_{0,S_i} \le \le c\, h^2 |t|_{2,S_i} . \tag{14}$$

2^0 Let $t < t_I$ at some point $P \in \overline{S}_i$ such that the tangent to the graph of t at P lies under the graph of t and, if ψ_h^i corresponds to the tangent, $\psi_h^i \ge 0$ on $\overline{S}_i$. Then the estimate (14) holds. Thus we construct ψ_h over Γ_a. On $\partial G_j - \Gamma_a$ we define

$$\psi_h = t_h + a_j , \tag{15}$$

where a_j is a constant such that (11) takes place. The estimate (13) follows from (14) and (15).

Lemma 3. Let a piecewise linear (discontinuous, in general) function φ on Γ be given such that

$$\int_{\partial G_j} \varphi\, ds = 0 \;\forall j . \tag{16}$$

Then there exists a vector-function $w^h \in N_h$ such that $w^h.n = \varphi$ on Γ and

$$\|w^h\|_0 \le c\, h^{-1/2} \|\varphi\|_{0,\Gamma} . \tag{17}$$

Proof. Let G_h be the union of all triangles $T \in T_h$ such that $T \cap \Gamma \neq \emptyset$ (a boundary strip of G). We determine $w^h \in N_h$ by means of properly chosen flux parameters on the sides of T_h, such that $\operatorname{supp} w^h \subset G_h$. In particular, we choose the flux parameters β equal to the corresponding values of φ on ∂G_j and equal to zero on $\partial G_h^j - \partial G_j$. As the sides connecting ∂G_j and $\partial G_h^j - \partial G_j$ are concerned, we set $\beta = 0$ at the vertices of $\partial G_h^j - \partial G_j$ but the parameters at the vertices of ∂G_j remain to be determined from the conditions (i) of the vanishing divergence and (ii) of the continuity of the fluxes along the interelementary boundaries. Using (16) the linear system for the remaining parameters can be solved and the solution β estimated from above, making use of the strong regularity of the triangulations, as follows

$$|\beta_i| \leq C\, h^{-1} \|\varphi\|_{0,\Gamma} .$$

Then the estimate (17) follows easily.

To finish the proof of Theorem 1, let us set

$$\varphi = (r_h U).n - \psi_h ,$$

where ψ_h is the one-sided approximation from Lemma 2, and consider the extension w^h of φ from Lemma 3. Then the function $W_h = r_h U - w^h$ satisfies the conditions of Lemma 1. The assertion (9) follows from (10), (7), (17) and (13).

Algorithm for the dual approximation

The problem (8) belongs to quadratic programming. It can be solved by various procedures, e.g. by the method of Uzawa (see [4]) or by a method of feasible directions (cf. [9]).

1.4. A posteriori error estimates and two-sided bounds for the energy

Having approximations of the primary and of the dual problem, we are able to calculate a posteriori error estimates and two-sided bounds for the energy of the solution u, as follows.

Theorem 2. Let $u_h \in K$ and $\tilde{q}^h = \bar{q} + \tilde{p}^h \in U$. Then it holds

$$|u - u_h|_1^2 \leq \|\tilde{q}^h - \operatorname{grad} u_h\|_0^2 + 2 \int_{\Gamma_a} \tilde{q}^h.n u_h \, ds \equiv E ,$$

$$\|\tilde{q}^h - \operatorname{grad} u\|_0^2 \leq E ,$$

$$-2L(u_h) \leq |u|_1^2 = (f,u)_0 \leq 2S(\tilde{q}^h) .$$

Proof. One can easily obtain

$$|v - u|_1^2 \le 2[L(v) - L(u)] \; \forall \; v \in K \; .$$

From (5) and (3) it follows that

$$-L(u) = S(q^0) \le S(q) \; \forall \; q \in U \; .$$

Hence for any $v \in K$ and $q \in U$, $q - \bar{q} \equiv p \in U_{0h}$ we may write

$$|v - u|_1^2 \le 2L(v) + 2S(q) = |v|_1^2 - 2(f,v)_0 + \|q\|_0^2 =$$

$$= \|q - q(v)\|_0^2 + 2(q,q(v)) - 2(f,v)_0 \; ,$$

where $q(v) = \text{grad } v$. Moreover, we have

$$(q,q(v)) - (f,v)_0 = \int_G (q.\text{grad } v) + v \text{ div } q)dx =$$

$$= \langle q.n, \gamma v \rangle = \int_{\Gamma_a} q.nvds \; .$$

2. Some other unilateral problems

Recently, the following boundary value problems have been solved by the dual finite element method with analogous results as above:

(i) equations with an "absolute" term

(18) $$-\Delta u + u = f \quad \text{in} \quad G$$

with the conditions (1_3) on the whole boundary. Here the standard piecewise linear elements can be applied for both the primary and the dual approximations (see [5]);

(ii) problems with a non-homogeneous obstacle on the boundary, i.e. an equation (18) with the boundary conditions

$$u \ge g \; , \quad \frac{\partial u}{\partial n} \ge 0 \; , \quad (u - g)\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma,$$

where g is a given function (see [10]);

(iii) semi-coercive problems of the type (1), i.e. the equation (1_1) with the conditions (1_3) on the whole boundary. The proof of convergence requires a different approach, because the one-sided approximations of the flux cannot be used (see [12]);

(iv) Signorini's problem in plane elastostatics and contact problems for two elastic bodies (see [14]). The triangular piecewise linear block-elements, which were proposed by Watwood and Hartz in [15] and studied in [16] , [17] , have been used for the dual approximations.

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