THE RAYLEIGH AND VAN DER POL WAVE EQUATIONS, SOME GENERALIZATIONS*

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Here are two interesting nonlinear partial differential equations. The first we call the Rayleigh wave equation,

$$y_{tt} - y_{xx} = \varepsilon(y_t - y_t^3) \qquad (1.1)$$
$$y(t,0) = y(t,\pi) = 0,$$

and the second is the wave equation of Van der Pol type,

$$\begin{aligned} y_{tt} - y_{xx} &= \varepsilon(1 - y^2)y_t \\ y(t,0) &= y(t,\pi) = 0 \end{aligned} \qquad (1.2)$$

Each of these has been used to model physical phenomena, although they first appeared in the literature as curiosities. For example, in [3] we see (1.1) serving as a model for the large amplitude vibrations of wind-blown, ice-laden power transmission lines. Equation (1.2), on the other hand, can describe plane electromagnetic waves propagating between two parallel planes in a region where the conductivity varies quadratically with the electric field [5].

Just as their counterparts from ordinary differential equations can be transformed one to the other, (1.1) and (1.2) are related. As we shall see, solutions to each can be obtained by simple operations performed on the solution of a certain first order, nonlinear wave equation. In fact, the goal of this note is to show how certain aspects of this particular equation can be studied such as global existence, uniqueness, and the transient and steady state behavior for small $\varepsilon > 0$.

It is perhaps surprising that some second order equations can be solved as first order problems. However, this is strongly suggested by the form of (1.1) where y itself is absent. Also, although two independent initial conditions are required for (1.1) and (1.2), each must be an odd, 2π-periodic function of x. Two such odd functions can always be generated from an arbitrary 2π-periodic function by separating it into its odd and even parts and integrating or differentiating the latter. Obviously, this

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procedure can be reversed, and a single periodic function can be built from two odd functions. Thus it is possible for the initial value of an appropriate first order equation to carry the initial position and velocity for (1.1) and (1.2) in its odd and even parts.

Let us now derive the first order wave equation corresponding to (1.1). Let

$$u = y_t - y_x \tag{1.3}$$

and let P project a periodic function of x to its odd part. Since y and y_t are odd in x, $y_t = Pu$. Hence, from (1.3),

$$u_t + u_x = y_{tt} - y_{xx} = \varepsilon(y_t - y_t{}^3) = \varepsilon(Pu - (Pu)^3) \tag{1.4}$$

Similarly, we can obtain (1.4) from (1.2) by the transformation,

$$y = \sqrt{3}z, \quad u = z - \int^t z_x \tag{1.5}$$

Of course, these derivations are formal, but they strongly suggest that once we have a solution to (1.4), then $\sqrt{3}$ Pu will solve (1.1) and $\int^t$ Pu will be a solution to (1.1). We shall not discuss here the question of whether these equations are equivalent. Rather we simply regard (1.4), or rather

$$u_t + u_x = \varepsilon(Pu - h(Pu) + f(t,x)), \tag{1.6}$$

where h is a suitable monotone increasing function and f is a 2π-periodic forcing term depending on t and x, as the fundamental equation generalizing the previous examples. We mention, however, that the equivalency of (1.4) and (1.1) has been established in [7].

Some names associated with the study of (1.1) and (1.2) are Kurzweil [8], [9], Vejvoda and Štědrý [10], Chikwendu and Kevorkian [2], and Fink, Hall, and Hausrath [4], [5], [7]. Kurzweil's contribution is by far the most important, as he was able to prove the existence of exponentially asymptotically stable integral manifolds of periodic solutions for both (1.1) and a general form of (1.2). Vejvoda and Štědrý showed the existence of periodic solutions to these equations by elementary methods. In [2], a formal analysis of (1.1) appears, using a two-time method from the theory of ordinary differential equations. In [4] a convergent two-time method is developed, and in [7] a rather detailed analysis is made of the Rayleigh equation.

Let us begin the analysis of (1.6) by studying the question of global existence and uniqueness.

2. A Global Existence Theorem

Let H be the space of 2π-periodic, square integrable functions of x with inner product $\langle u, v\rangle$ and norm $|u|$. Write (1.6) in the form,

$$\dot{u} - Au + \varepsilon Nu = \varepsilon u + \varepsilon f \qquad (2.1)$$

where $Au = -du/dx$ and $Nu = h(Pu) + (I - P)u$. Assume $\varepsilon > 0$ and h is a continuous, monotone increasing, odd function defined for all real u. Then $Ph(Pu) = h(Pu)$, and so on the domain of h,

$$\langle h(Pu) - h(Pv), u - v\rangle = \langle h(Pu) - h(Pv), Pu - Pv\rangle \geq 0.$$

since $\langle u, Pv\rangle = \langle Pu, v\rangle$ for all u and v in H. Thus N is monotone on H. On its domain H_1 of elements having square integrable derivatives, $-A$ is trivially monotone. Hence $B = -A + \varepsilon N$ is monotone as well.

Now suppose B is maximal monotone and let f in $L_1(0, 2\pi; H)$. By Theorem 3.17 of Brezis [1], p. 105, for each u_0 in H, (1.6) has a unique weak solution on $[0, 2\pi]$. This means that there are sequences u_n in $C(0, 2\pi; H)$ converging uniformly to u and $f_n \to f$ in $L_1(0, 2\pi; H)$ such that each pair (u_n, f_n) strongly satisfies (1.6).

If f has some additional smoothness, then u is also differentiable in t. For example, if f is in $W_{1,1}(0, 2\pi; H)$ and u_0 in H_1,

$$\left|\frac{du(t)}{dt}\right| \leq e^{\varepsilon t}\left|\varepsilon f(0+) - u_{0x} + \varepsilon Pu_0 - \varepsilon h(Pu_0)\right| + \varepsilon\int_0^t (t-s)\left|\frac{df(s)}{ds}\right| ds$$

and when $f \equiv 0$, $u(t, u_0) = S(t)u_0$ is a semi-group on $[0, \infty)$ with expansion constant $e^{\varepsilon t}$.

The main condition that must be verified, therefore, is that B is maximal. First, note that it is enough to check that $-A + \varepsilon hP$ is maximal because then B is of the form maximal monotone plus monotone Lipschitzian defined on all of H. By [1], p. 34, B is also maximal monotone. According to a result of Minty [1], p. 23, $-A + \varepsilon hP$ is maximal monotone if it is monotone (already verified) and if the range of $I + \varepsilon hP - A$ is all of H. So we consider the equation,

$$\frac{du}{dx} + \varepsilon h(Pu) + u = f \qquad (2.2)$$

where f is in H. Splitting u into its odd and even parts v, w we get the system

$$\frac{dw}{dx} + \varepsilon h(v) + v = g\ , \quad \frac{dv}{dx} + w = k \qquad (2.3)$$

where $g = Pf$ and $k = (I - P)f$. But it is easy to see that a solution in H_1 to (2.3) exists if and only if v is an odd function in H_1 satisfying

$$-Cv = \frac{d^2v}{dx^2} - \varepsilon h(v) - v = k_x - g \qquad (2.4)$$

where k_x and d^2v/dx^2 are now in H_{-1}, the dual space of H_1 under $\langle\cdot,\cdot\rangle$. It is however, quite easy to verify that $C\colon H_1 \to H_{-1}$ is strongly monotone and hemicontinuous. Hence by a now classic result, C is bijective and we are done.

3. A Perturbation Analysis Using Averaging

In this section we shall give some of the results of applying a modified method of averaging to (1.6). Suppose $u(t,\varepsilon)$ is one of its solutions. Let $v(\tau)$ solve the associated averaged equation,

$$\frac{dv}{d\tau} = F(v) = \frac{1}{2\pi}\int_0^{2\pi} e^{-At}\, F(t,\, e^{At}v)dt \qquad (3.1)$$

where $F(t,u) = Pu - h(Pu) + f(t,x)$ is the perturbing part of (1.6), and $\{e^{At}\}$ is the group generated by $A = -d/dx$. Let X be a suitable space of initial values with norm $|\cdot|$. According to the theory we can expect the following:

(1) Suppose $\eta > 0$ is given and $u(0, \varepsilon) = v(0) = u_0$ is in X. Then there is a constant $L > 0$ such that for all t in $[0, L/\varepsilon]$, $|u(t, \varepsilon) - e^{At}v(\varepsilon t)| < \eta$.

(2) Let v_0 be an equilibrium point for (3.1) and suppose $F'(v_0)$ has a bounded inverse on X. Then (1.6) has a 2π-periodic solution given approximately by $e^{At}v_0$

(3) If the variational equation of (3.1) is exponentially asymptotically stable then so is the corresponding periodic solution to (1.6).

For a discussion of the averaging method and its application to various types of partial differential equations see [4] and [7]. A simplified explanation can be found in [6].

Statements (1), (2), and (3) above tell us simply that for times of order ε^{-1}, a solution to (3.1) for arbitrary initial value is asymptotic to the actual solution of (1.6), whereas the equilibrium points correspond to the periodic steady states. Thus we must analyze (3.1) for its behavior first as a differential equation with assigned initial value and secondly for the existence and nature of its constant solutions.

Before continuing in this direction, let us say something about the space X. The averaging method in its present form does not apply when the nonlinear term h is unbounded. As we would certainly like to admit polynomials, the L_p spaces for $1 \leq p < \infty$ are ruled out. On the other hand, we need a set large enough to accommodate nonclassical steady states. For these reasons, we have taken X to be the 2π-periodic essentially bounded functions of x with no mean value. This choice results in a series of interesting but sometimes bizarre consequences. The most notable is that the solution itself, which is of the form $u(t, \varepsilon) = e^{At}z(t, \varepsilon)$ (with z strongly continuous in t) is only weak* continuous since $\{e^{At}\}$ is simply right translation of the space variable x. Hence translations of solutions $u(t+h, \varepsilon)$, while certainly solutions in the autonomous case $f(t,x) \equiv 0$, are distinct and isolated from each other for each value of h in the norm topology of L_∞. Thus even when (1.6) is autonomous, the periodic solutions can be exponentially asymptotically stable.

From (3.1) and the definitions of e^{At} and $F(t,u)$,

$$F(v)(\tau,x) = \frac{1}{2}v(\tau,x) - \frac{1}{2\pi}\int_0^{2\pi} h\left(\frac{v(\tau,x) - v(\tau,s)}{2}\right) ds + f_0(x) \tag{3.2}$$
$$f_0(x) = \frac{1}{2\pi}\int_0^{2\pi} f(s, x+s)\, ds$$

When $h(u) = u^3$, it is possible to solve (3.1) up to a quadrature depending only on the initial value v_0. The details are in [7]. When h is more general we can still say something about the behavior of the averaged equation. For example, if h is odd and monotone increasing, then we can prove solutions exist on $[0, \infty)$ but may not be bounded. In fact, if p is odd,

$$Nu = \frac{1}{2\pi}\int_0^{2\pi} h\left(\frac{u(x) - u(s)}{2}\right) ds,$$

and $\langle u,v \rangle$ is the usual inner product, then

$$\langle Nu,u^p \rangle = \frac{1}{2\pi}\int_0^{2\pi} ds \int_0^{2\pi} dx\, h\left(\frac{u(x) - u(s)}{2}\right)(u^p(x) - u^p(s))ds - \langle Nu, u^p \rangle$$

Thus, $2 \langle Nu, u^p \rangle \geq 0$ since $h(u)u \geq 0$.

Now let $z(\tau) = |v|_{p+1}$ be the L_{p+1} norm of v.

From (3.1), (3.2), and the inequality on N,

$$\frac{dz}{d\tau} \le \frac{1}{2}z + |f_0|,$$

and this gives $z(\tau) \le \{|v_0| + 2|f_0|\}e^{\tau/2}$.

A better result can be obtained if we suppose $h(u)u \ge c_r u^{r+1}$ where $r > 1$ is odd, and if X restricted to the Π-antiperiodic elements $u(x+\pi) = -u(x)$. In this case we can show

$$\langle Nu, u^p\rangle \ge \frac{c_r}{2^r} |u|_{p+r}^{p+r},$$

and from the Holder inequality we get $\langle Nu, u^p\rangle \ge a_r |u|_{p+1}^{p+r}$ where a_r is independent of p.

Now from the averaged equation we obtain

$$\frac{dz}{d\tau} \le g(z) = \frac{1}{2}z + |f_0| - a_r z^r$$

Eventually, $g(x) \to -\infty$ as $\alpha \to +\infty$ monotonically. So for large α, $g(\alpha) < -\delta$ and $\alpha_2 > \alpha_1$ implies $g(\alpha_2) < g(\alpha_1)$, For such an α let $|v(\tau)| > \alpha$. Since $|v(\tau)|_{p+1}$ is continuous in τ and converges monotonically to $|v(\tau)|$ we can apply Dini's Theorem to obtain a P_0 such that $|v(\tau)|_{p+1} > \alpha$ when $p > P_0$ uniformly for τ in $[0,L]$. Thus on $(0,L)$

$$\frac{dz}{d\tau} \le g(z) < g(\alpha) < -\delta$$

and hence z, and therefore $|v(\tau)|$ itself is nonincreasing outside some large ball. Hence not only do we have global existence, but we also know all solutions are bounded. Of course, we cannot conclude the same is true of (1.6) since we are assured only that its solutions follow those of (3.1) on some long but finite time interval.

The constant solutions of (3.1) correspond to the periodic solutions of (1.6) provided the conditions of the implicit function theorem can be met. So let us examine the roots of F(v). In most circumstances, the term $f_0(x)$ is absent even if f is not. In fact, unless f has a right traveling wave component, $f_0(x) \equiv 0$. So let us begin with this case.

If $f_0 = 0$ we expect that the most general solution to $F(v_0) = 0$ will be constant on subsets of $[0, 2\pi)$. Let v_0 be the 2π-periodic extensions of

$$v_0(x) = \begin{cases} 0 & x \in A_0 \\ \alpha & \in A_1 \\ -\alpha & \in A_2 \end{cases} \qquad (3.3)$$

where A_i, $i = 0,1,2$ are disjoint measurable sets whose union is $[0, 2\pi)$. In order that v_0 have no mean value we need $\text{mes}\, A_1 = \text{mes}\, A_2$.

We thus obtain

$$F(v_0) = \begin{cases} -a_1\ \{h(-a/2) + h(\alpha/2)\}, & x \in A_0 \\ F(\alpha) = \frac{1}{2}\alpha - a_1 h(\alpha) - a_0 h(\frac{\alpha}{2}), & x \in A_1 \cup A_2 \end{cases}$$

where $a_i = \text{mes}\, A_i/2\pi$, $i = 0,1$. If h is odd, then F vanishes on A_0. Let us also suppose that $h'(0) = 0$, $h'(u) \to +\infty$ and $h''(u) > 0$ for $u > 0$. (For example, $h(u) = u^p$, $p \geq 3$ and odd, and $h(u) = \sinh u - u$ both satisfy this requirement). Then $F(\alpha)$ is concave on $\alpha > 0$, $F'(0) = \frac{1}{2}$ and $F'(\alpha)$ decreases steadily to $-\infty$ as $\alpha \to +\infty$. Hence there is a unique $\alpha > 0$ where $F(\alpha) = 0$.

When $\text{mes}\, A_0 = 0$, then α is simply the root of $\alpha = h(\alpha)$. Using techniques developed in [7] we can prove that $F'(v_0)$ is boundedly invertible when

$$\lambda = 1 - h'(\alpha) \neq 0 \qquad\qquad \mu = 1 - \tfrac{1}{2}h'(\alpha) \neq 0.$$

The solution to the variational equation is

$$\xi = e^{\lambda\tau}a + b(e^{\mu\tau} - 1)$$

where a and b are linear functionals of the initial value. Hence we have exponential decay and thus stability of the periodic solution if λ and μ are both negative at the root of $F(\alpha)$.

It is interesting to note that the stability result above is independent of f if it has no right traveling wave component. Also, if $\text{mes}\, A_0 \neq 0$, then we can prove that ξ is actually increasing exponentially for x in A_0. This indicates, (but does not prove, of course) that those equilibrium points with $\text{mes}\, A_0 \neq 0$ are unstable.

We shall conclude by looking at a particular case when $f_0(x) \neq 0$ Specifically, we take $h(u) = u^3$ and we restrict X to the π-anti-periodic elements so that the odd powers have no mean value. Then

$$F(v) = \tfrac{1}{2}v - \tfrac{1}{8}(v^3 + 3 < v^2 > v) + f_0(x)$$

$$<v^2> = \frac{1}{2\pi}\int_0^{2\pi} v^2(x)dx$$

Let us try to solve $F(v) = 0$ with $v = m\cos\theta$. This gives

$$m^3\cos^3\theta - (4 - 3m^2 < \cos^2\theta >)\, m\cos\theta - 8f_0(x) = 0 \qquad (3.4)$$

Replace $\langle\cos^2\theta\rangle$ by β and consider

$$m^3\cos^3\theta - (4 - 3m^2\beta)\, m\cos\theta - 8f_0(x) = 0. \quad (3.5)$$

Since $4\cos^3\theta - 3\cos\theta - \cos 3\theta = 0$ we will have a solution if

$$\frac{m^3}{4} = \frac{m(4-3m^2\beta)}{3} = \frac{8f_0(x)}{\cos 3\theta}$$

Hence

$$\cos 3\theta = 32f_0(x)/m^3 \text{ and } m = 4/\sqrt{3(1+4\beta)},$$

and θ will be real if $-m^3 \leq 32f_0(x) \leq m^3$. Since $\beta \in [0,1]$, we can continue this procedure if

$$-2/3^{3/2} \leq f_0(x) \leq 2/15^{3/2} \qquad \text{a.e.}$$

Let $\theta_1(x,\beta)$ be a root of (3.5) in $0 \leq \theta \leq \pi$. Then $\theta_2 = \theta_1+2\pi/3$ and $\theta_0 = \theta_1 + 4\pi/3$ are also roots, and we can construct a candidate for an equilibrium point. Let B_0, B_1, B_2 be three mutually disjoint measurable sets whose union is $[0,\pi)$. Define

$$\tilde{v}_0(x,\beta) = \begin{cases} m\cos\theta_1(x,\beta) & x \in B_1 \\ m\cos\theta_2(x,\beta) & x \in B_2 \\ m\cos\theta_0(x,\beta) & x \in B_0 \end{cases}$$

and let v_0 be its π-antiperiodic extension. Then

$$\langle v^2\rangle = \frac{1}{\pi}\int_0^\pi v_0^2(x)dx$$

and we have a solution to (3.4) if we can find β in [0.1] such that

$$\beta = \frac{1}{\pi}\left\{\int_{B_1}\cos^2\theta_1(x,\beta)dx + \int_{B_2}\cos^2\theta_2(x,\beta)dx + \int_{B_3}\cos^2\theta_0(x,\beta)dx\right\}$$

But the right side of this equation is just a continuous map of the unit interval to itself. Hence we have at least one fixed point

It can be proved that if mes $B_0 = 0$, then the condition of the implicit function theorem and the requirement on the exponential decay of the variational equation are met. We remark that the sets B_0, B_1, B_2 play rolls analogous to those of A_0, A_1, and A_2 of the previous example and the solutions given here reduce to those given earlier as $f_0 \to 0$.

We conclude with the following observation. When $f \equiv 0$, the approximate steady states $e^{At}v_0$ are the actual solutions in a generalized sense. In particular, when $h(u) = u^3$, and $v_0(x) = 1$ on A_1, -1 on A_2 (where $A_1 \cup A_2 = [0, 2\pi)$ and mes A_1 = mes A_2) then a family of stable steady states for the Van der Pol wave equation is given by

$$\frac{\sqrt{3}}{2}\left\{v_0(x-t) - v_0(-x-t)\right\} ,$$

and for the Rayleigh equation by

$$\frac{1}{2}\int^{t}_{-} \left\{v_0(x-s) - v_0(-x-s)\right\} ds \quad .$$

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