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DIFFERENTIAL SUBSPACES ASSOCIATED WITH PAIRS OF ORDINARY DIFFERENTIAL OPERATORS

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1. Introduction. This is an account of some joint work in progress with H.S.V. de Snoo. It represents an attempt to place a study of boundary value and eigenvalue problems, associated with a pair of ordinary differential expressions L, M, in the general framework of two earlier papers by E.A. Coddington and A. Dijksma [7], [8]. In the first of these we showed how to describe very general eigenvalue problems, for the case when M is the identity and L is formally symmetric, and to obtain eigenfunction expansion results for these problems. In the second we described abstractly the adjoints of subspaces (multi-valued operators) in Banach spaces in terms of generalized boundary conditions, and applied these results to a study of boundary value problems with not necessarily formally symmetric differential expressions L.

There is a large literature devoted to problems for two expressions L, M. We mention the recent work by F. Brauer [2], [3], [4], F. Browder [5], [6], Å. Pleijel [9], C. Bennewitz [1]. We deal with systems, not necessarily formally symmetric L, and we do not assume that the order of M is less than the order of L. From the point of view of subspaces, if a subspace S is associated with a right definite M, then S^{-1} is a problem associated with a left definite case. The set of Hilbert spaces which we allow differ from those considered by Bennewitz in [1].

We settle some notation matters. Let $\mathbb{R}$, $\mathbb{C}$ denote the real and complex numbers. We consider an open real interval $\iota = (a,b)$, and the set $F_m(\iota)$ of all vector valued functions $f : \iota \to \mathbb{C}^m$. By $C(\iota)$ we denote the set of all continuous $f \in F_m(\iota)$, and

$$C^k(\iota) = \{f \in F_m(\iota) \mid f^{(k)} \in C(\iota)\} ,$$

$$C_0^k(\iota) = \{f \in C^k(\iota) \mid \text{support of } f \text{ is compact}\} ,$$

$$C_0^\infty(\iota) = \bigcap_k C_0^k(\iota) .$$

By $L^2_{loc}(\iota)$ we mean the set of all $f \in F_m(\iota)$ such that

$$\int_J |f|^2 < \infty , \quad \text{each compact subinterval } J \subset \iota ,$$

where $|f|^2 = f^* f$, and we let

$$L^2(\iota) = \{f \in L^2_{loc}(\iota) \mid \int_\iota |f|^2 < \infty\} ,$$

$$L^2_0(\iota) = \{f \in L^2(\iota) \mid \text{support of } f \text{ is compact}\} .$$

If $f,g \in F_m(\iota)$, we use the notations

$$(f,g)_{2,J} = \int_J g^* f , \qquad (f,g)_2 = \int_\iota g^* f ,$$

if the components of $g^* f$ are integrable on the compact subinterval $J \subset \iota$, or on ι, respectively. Note that we do not assume f, g are in $L^2_{loc}(\iota)$ or $L^2(\iota)$.

2. Hilbert spaces associated with positive differential expressions. Let M be the formal ordinary differential expression of order ν

$$M = \sum_{k=0}^{\nu} Q_k D^k , \qquad D = d/dx ,$$

where the Q_k are $m \times m$ complex matrix-valued functions whose columns are in $C^k(\iota)$, and $Q_\nu(x)$ is invertible for $x \in \iota$. We want to associate an inner product with this M by first defining

$$(2.1) \qquad (\varphi, \psi) = (M\varphi, \psi)_2 , \qquad \varphi, \psi \in C_0^\infty(\iota) .$$

If this is to be an inner product on $C_0^\infty(\iota)$ we must have

$$(2.2) \qquad \begin{aligned} M = M^+ &= \sum_{k=0}^{\nu} (-1)^k D^k Q_k^* , \\ (M\varphi, \varphi)_2 &\geq 0 , \qquad \varphi \in C_0^\infty(\iota) , \end{aligned}$$

and we assume this. From this it follows that ν is even, $\nu = 2\mu$, and $(-1)^\mu Q_\nu(x) > 0$, $x \in \iota$, in the sense that

$$\xi^*(-1)^\mu Q_\nu(x)\xi \geq c(x)\xi^*\xi , \qquad \xi \in C^m ,$$

for some $c(x) > 0$. We can write such an M in the form

$$M = \sum_{j=0}^{\mu} \sum_{k=j-1}^{j+1} (-1)^j D^j Q_{jk} D^k ,$$

where $Q_{jk}^* = Q_{kj}$, and $Q_{jj} \in C^j(\iota)$, $Q_{j+1\,j} \in C^{j+1}(\iota)$, $Q_{j\,j+1} \in C^{j+1}(\iota)$. Using this form for M the formula (2.1) can be written as

$$(\varphi, \psi) = (M\varphi, \psi)_2 = \int_\iota \sum_{j=0}^{\mu} \sum_{k=j-1}^{j+1} (D^j\psi^*) Q_{jk} (D^k\varphi) , \qquad \varphi, \psi \in C_0^\infty(\iota) ,$$

and the right side is denoted by $(\varphi, \psi)_D$, the Dirichlet inner product.

The definition (2.1) gives an inner product $(\ , \ )$ on $C_0^\infty(\iota)$ under the assumption (2.2), and $\| \ \| = (\ , \ )^{1/2}$ is a norm on $C_0^\infty(\iota)$. Let $\mathfrak{H}_M$ denote the completion of $C_0^\infty(\iota)$; it is a Hilbert space. In many cases $\mathfrak{H}_M$ can be imbedded into $L^2_{loc}(\iota)$, and this is assured if we assume:

(A_1) for each compact subinterval $J \subset \iota$ there is a $c(J) > 0$ such that

$$\|\varphi\| \geq c(J)\|\varphi\|_{2,J} , \qquad \varphi \in C_0^\infty(\iota) .$$

Then the identity map on $C_0^\infty(\iota)$ has an extension which is an injection of $\mathfrak{H}_M$

into $L^2_{loc}(\iota)$, and we can identify $\mathfrak{H}_M$ as a subset of $L^2_{loc}(\iota)$. We have

$$(f,\varphi) = (f,M\varphi)_2 , \quad f \in \mathfrak{H}_M , \quad \varphi \in C_0^\infty(\iota) ,$$

$$\|f\| \geq c(J)\|f\|_{2,J} , \quad f \in \mathfrak{H}_M ,$$

and the injection $\mathfrak{H}_M \to L^2_{loc}(\iota)$ implies the existence of an injection G_M : $L^2_0(\iota) \to \mathfrak{H}_M$ with the properties:

$$(2.3) \qquad \begin{aligned} (f,G_M h) &= (f,h)_2 , \quad f \in \mathfrak{H}_M , \quad h \in L^2_0(\iota) , \\ G_M M\varphi &= \varphi , \quad \varphi \in C_0^\infty(\iota) , \\ MG_M h &= h , \quad h \in L^2_0(\iota) , \\ (\Re(G_M))^c &= \mathfrak{H}_M , \end{aligned}$$

where A^c denotes the closure of a set A, and $\Re(G_M)$ denotes the range of G_M.

An important special case is obtained if instead of (A_1) we assume

$$(A_1') \qquad \|\varphi\| \geq c\|\varphi\|_2 , \quad \text{for some} \quad c > 0 .$$

Then $\mathfrak{H}_M \subset L^2(\iota)$ and G_M has an extension, call it G_M also, to an injection $G_M : L^2(\iota) \to \mathfrak{H}_M$ such that (2.3) is valid with $L^2_0(\iota)$ replaced by $L^2(\iota)$ everywhere. In fact, assuming (A_1') we can identify G_M more precisely. Let M_0 be the operator in $L^2(\iota)$ with domain $\mathfrak{D}(M_0) = C_0^\infty(\iota)$ given by $M_0\varphi = M\varphi$. It is a symmetric operator which is bounded below by $c > 0$ if (A_1') holds, and thus has a Friedrichs extension which is a selfadjoint operator M_F having the same lower bound c. Its inverse M_F^{-1} exists on all of $L^2(\iota)$ and one can show that $G_M = M_F^{-1}$, and that $\mathfrak{H}_M$ is the domain $\mathfrak{D}(M_F^{1/2})$ of the positive square root $M_F^{1/2}$ of M_F.

There exist other Hilbert spaces $\mathfrak{H}$ having the essential properties of $\mathfrak{H}_M$. Let $\mathfrak{H}$ be any Hilbert space with inner product (,) and norm $\| \ \|$ satisfying:

$$(A_2) \qquad \begin{aligned} &C_0^\infty(\iota) \subset \mathfrak{H} \subset L^2_{loc}(\iota) , \\ &(f,\varphi) = (f,M\varphi)_2 , \quad f \in \mathfrak{H} , \quad \varphi \in C_0^\infty(\iota) , \\ &\|f\| \geq c(J)\|f\|_{2,J} , \quad f \in \mathfrak{H} , \quad c(J) > 0 , \end{aligned}$$

for each compact subinterval $J \subset \iota$. We have $(C_0^\infty(\iota))^c = \mathfrak{H}_M$, and in fact

$$\mathfrak{H} = \mathfrak{H}_M \oplus \mathfrak{N}_M ,$$

an orthogonal sum, where

$$\mathfrak{N}_M = \{f \in C^\nu(\iota) \cap \mathfrak{H} \mid Mf = 0\} .$$

Clearly $\dim \mathfrak{N}_M \leq \nu m$. As before there exists an injection $G : L^2_0(\iota) \to \mathfrak{H}$ such that:

$$
(2.4)\qquad
\begin{aligned}
(f,Gh) &= (f,h)_2\ , \quad f \in \mathfrak{H}\ , \quad h \in L^2_0(\iota)\ ,\\
GM\varphi &= \varphi\ , \quad \varphi \in C^\infty_0(\iota)\ ,\\
MGh &= h\ , \quad h \in L^2_0(\iota)\ ,\\
(\mathfrak{R}(G))^c &= \mathfrak{H}\ ,\\
G_M &= P_M G\ ,
\end{aligned}
$$

where P_M is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_M$. If instead of (A_2) we have

$$
(A_2')\qquad
\begin{aligned}
&C^\infty_0(\iota) \subset \mathfrak{H} \subset L^2(\iota)\ ,\\
&(f,\varphi) = (f,M\varphi)_2\ , \quad f \in \mathfrak{H}\ , \quad \varphi \in C^\infty_0(\iota)\ ,\\
&\|f\| \geq c\|f\|_2\ , \quad f \in \mathfrak{H}\ , \quad c > 0\ ,
\end{aligned}
$$

then G has an extension to all of $L^2(\iota)$ satisfying (2.4) with $L^2_0(\iota)$ replaced by $L^2(\iota)$.

3. Examples. Let H be a positive selfadjoint extension of M_0 in $L^2(\iota)$ such that

$$
(3.1)\qquad (Hf,f)_2 = (Mf,f)_2 \geq (c(J))^2(f,f)_{2,J}\ , \quad f \in \mathfrak{D}(H)\ , \quad c(J) > 0\ ,
$$

for each compact subinterval $J \subset \iota$, and let $\mathfrak{H}_H$ be the completion of $\mathfrak{D}(H)$ with

$$
(f,g) = (Mf,g)_2\ , \quad f,g \in \mathfrak{D}(H)\ .
$$

This is a Hilbert space, and it will be in $L^2_{loc}(\iota)$ if the following is assumed:

(A_3) $f_n \in \mathfrak{D}(H)$, $\|f_n - f_m\| \to 0$, $\|f_n\|_{2,J} \to 0$ for each compact subinterval $J \subset \iota$, implies $\|f_n\| \to 0$.

Then $\mathfrak{H} = \mathfrak{H}_H$ satisfies (A_2). As an example consider $M = -D^2$, $m = 1$, $\iota = (0,\infty)$. The maximal operator M_{max} for M in $L^2(\iota)$ has a domain $\mathfrak{D}_{max}$ consisting of all $f \in L^2(\iota)$ such that f' is absolutely continuous on each compact subinterval $J \subset [0,\infty)$, and $Mf \in L^2(\iota)$. The selfadjoint extensions of M_0 are obtained from M_{max} by imposing a homogeneous boundary condition at 0. Let H_h be the selfadjoint extension of M_0 given by

$$
\begin{aligned}
\mathfrak{D}(H_h) &= \{f \in \mathfrak{D}_{max} \mid f'(0) = hf(0)\}\ , && h \in \mathbb{R}\ ,\\
&= \{f \in \mathfrak{D}_{max} \mid f(0) = 0\}\ , && h = \infty\ .
\end{aligned}
$$

We have for $f,g \in \mathfrak{D}(H_h)$

$$(H_h f,g)_2 = hf(0)\overline{g}(0) + (f',g')_2^2 , \qquad h \in \mathbb{R},$$
$$= (f',g')_2^2 , \qquad h = \infty .$$

Only for $0 \leq h \leq \infty$ will H_h satisfy $(H_h f,f)_2 \geq 0$ for $f \in \mathfrak{D}(H_h)$. In case $0 < h \leq \infty$ we can show that for each compact subinterval $J \subset [0,\infty)$ there is a $c(J) > 0$ such that

$$(H_h f,f)_2^{1/2} = \|f\| \geq c(J)\|f\|_{2,J} , \qquad f \in \mathfrak{D}(H_h) ,$$

and (A_3) is valid. Then the Hilbert space completion $\mathfrak{H}_h$ of $\mathfrak{D}(H_h)$ is in $L^2_{loc}(\iota)$ and the form of the inner product persists, that is,

$$(f,g) = hf(0)\overline{g}(0) + (f',g')_2 , \qquad f,g \in \mathfrak{H}_h , \quad 0 < h < \infty ,$$
$$(f,g) = (f',g')_2 , \qquad f,g \in \mathfrak{H}_h , \quad h = \infty .$$

Moreover it can be shown that $\mathfrak{N}_M = \text{span}\{1\}$ if $0 < h < \infty$ and $\mathfrak{N}_M = \{0\}$ if $h = \infty$. None of these $\mathfrak{H}_h$ are contained in $L^2(\iota)$, for there exists a sequence $\varphi_n \in C_0^2(\iota) \subset \mathfrak{D}(H_h)$ such that $\|\varphi_n\|^2 = (\varphi_n', \varphi_n') \to 0$ but $\|\varphi_n\|_2 \to +\infty$. In case $h = 0$ we get an inner product $(f,g) = (f',g')_2$ on $\mathfrak{D}(H_0)$, but the completion $\mathfrak{H}_0$ of $\mathfrak{D}(H_0)$ is not contained in $L^2_{loc}(\iota)$. There exists a sequence $\varphi_n \in \mathfrak{D}(H_0)$ such that $\|\varphi_n\| \to 0$ but $\|\varphi_n\|_{2,J} \to \infty$ on each proper compact subinterval $J \subset [0,\infty)$.

There may exist positive selfadjoint extensions H of M_0 in $L^2(\iota)$ satisfying a global inequality:

$$(Hf,f)_2 = (Mf,f)_2 \geq c^2(f,f)_2 , \qquad f \in \mathfrak{D}(H) , \quad c > 0 .$$

If $\mathfrak{H}_H$ is the completion of $\mathfrak{D}(H)$ with $(f,g) = (Mf,g)_2$, $f,g \in \mathfrak{D}(H)$, then $\mathfrak{H}_H \subset L^2(\iota)$ and $\mathfrak{H} = \mathfrak{H}_H$ satisfies (A_2'). In fact $\mathfrak{H}_H = \mathfrak{D}(H^{1/2})$ and $G = H^{-1}$ in this case.

Another method of constructing an $\mathfrak{H}$ satisfying (A_2) is as follows. Let $\mathfrak{N}_M$ be any linear subset of $N_M = \{f \in C^\nu(\iota) \mid Mf = 0\}$ with any inner product $(\ , \ )_0$ such that

$$\|f_0\|_0 \geq c_0(J)\|f_0\|_{2,J} , \qquad f_0 \in \mathfrak{N}_M ,$$

for some $c_0(J) > 0$ and each compact subinterval $J \subset \iota$. Let $(\ , \ )_1$, for the moment, denote the inner product on $\mathfrak{H}_M$. Define $\mathfrak{H} = \mathfrak{H}_M \oplus \mathfrak{N}_M$ with the inner product

$$(f,g) = (f_1,g_1)_1 + (f_0,g_0)_0 ,$$
$$f = f_1 + f_0 , \quad g = g_1 + g_0 , \quad f_1,g_1 \in \mathfrak{H}_M , \quad f_0,g_0 \in \mathfrak{N}_M .$$

Then (A_2) is valid. As an example we could use $(f,g)_0 = (f,g)_2$, or $(f,g)_0 = (f,g)_D$.

4. Maximal and minimal subspaces. Let M be as before, and let L be another formal differential operator

$$L = \sum_{k=0}^{n} P_k D^k ,$$

where the P_k are $m \times m$ complex matrix-valued functions on ι whose columns are in $C^k(\iota)$, and $P_n(x)$ is invertible for $x \in \iota$ if $n > \nu$. We consider any Hilbert space $\mathfrak{H}$ satisfying (A_2). In $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ we define the maximal linear manifolds

$$T = \{\{f,g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), \quad g \in C^\nu(\iota), \quad Lf = Mg\} ,$$
$$T^+ = \{\{f,g\} \in \mathfrak{H}^2 \mid f \in C^r(\iota), \quad g \in C^\nu(\iota), \quad L^+f = Mg\} ,$$

where $r = \max(n,\nu)$, and the minimal linear manifolds

$$S = \{\{\varphi, GL\varphi\} \mid \varphi \in C_0^\infty(\iota)\} ,$$
$$S^+ = \{\{\varphi, GL^+\varphi\} \mid \varphi \in C_0^\infty(\iota)\} .$$

Now S, S^+ are (the graphs of) operators, whereas T, T^+ need not be operators. In fact,

$$T(0) = \{g \in \mathfrak{H} \mid \{0,g\} \in T\} = T^+(0) = \mathfrak{N}_M ,$$

and this implies S, S^+ are densely defined if and only if $\mathfrak{N}_M = \{0\}$. It is clear that $S \subset T$, $S^+ \subset T^+$, and if we put $T_0 = S^c$, $T_1 = T^c$, $T_0^+ = (S^+)^c$, $T_1^+ = (T^+)^c$, we have $T_0 \subset T_1$, $T_0^+ \subset T_1^+$ and these are subspaces (closed linear manifolds) in $\mathfrak{H}^2$.

On $\mathfrak{H}^2 \times \mathfrak{H}^2$ we introduce the form $\langle\ ,\ \rangle$ given by

$$\langle u,v\rangle = (g,h) - (f,k) , \qquad u=\{f,g\},\ v=\{h,k\} \in \mathfrak{H}^2 .$$

If $Ju = \{g,-f\}$ then $\langle u,v\rangle = (Ju,v) = -(u,Jv)$. If A is any linear manifold in $\mathfrak{H}^2$ its adjoint A^* is the subspace defined by

$$A^* = \{v \in \mathfrak{H}^2 \mid \langle u,v\rangle = 0, \quad \text{all } u \in A\} .$$

The following result describes the adjoints of S, T, S^+, T^+ and their properties.

THEOREM. We have

(i) $S^* = \{\{f,g\} \in \mathfrak{H}^2 \mid (g,M\varphi)_2 = (f,L\varphi)_2, \ \underline{\text{all}}\ \varphi \in C_0^\infty(\iota)\} = T_1^+$,

(ii) $T_1^+ \ominus T_0^+ = T^+ \cap JT$,

(iii) $(S^+)^* = \{\{f,g\} \in \mathfrak{H}^2 \mid (g,M\varphi)_2 = (f,L^+\varphi)_2, \ \underline{\text{all}}\ \varphi \in C_0^\infty(\iota)\} = T_1$,

(iv) $T_1 \ominus T_0 = T \cap JT^+$,

(v) $T_1(0) = T_1^+(0) = T(0) = T^+(0) = \mathfrak{N}_M$,

(vi) $\nu(T_1^+ - \ell I) = \nu(T^+ - \ell I) = \{f \in \mathfrak{H} \cap C^r(\iota) \mid L^+f = \ell Mf\}$,

where $\ell \in \mathbb{C}$, $n > 2\mu$,

$\ell \in \mathbb{C}\setminus\{0\}$, $n < 2\mu$,

$$\ell \in \mathbb{C} \setminus \bigcup_{x\in\iota} \sigma(Q_{2\mu}^{-1}(x)P_{2\mu}^{*}(x)) , \quad n = 2\mu ,$$

(vii) $\nu(T_1 - \ell I) = \nu(T - \ell I) = \{f \in \mathfrak{H} \mid f \in C^r(\iota),\ \ Lf = \ell Mf\}$,

<u>where</u> $\ell \in \mathbb{C}$, $n > 2\mu$,

$\ell \in \mathbb{C} \setminus \{0\}$, $n < 2\mu$

$\ell \in \mathbb{C} \setminus \bigcup_{x\in\iota} \sigma(Q_{2\mu}^{-1}(x)P_{2\mu}(x))$, $n = 2\mu$.

In the above theorem, I denotes the identity operator, $\nu(A)$ represents the null space of a linear manifold A,

$$\nu(A) = \{f \in \mathfrak{H} \mid \{f,0\} \in A\} ,$$

and $\sigma(B)$ is the spectrum of a matrix B, that is, the set of its eigenvalues. This result shows that T, T^+ can be regarded as smooth versions of $(S^+)^*$, S^*, respectively, and that the only nonsmooth elements in the latter subspaces come from $T_0^+ \setminus S^+$ and $T_0 \setminus S$, respectively. Although S, S^+ are operators their closures T_0, T_0^+ need not be; they are operators if and only if $\mathfrak{D}(T^+)$, $\mathfrak{D}(T)$ are dense in $\mathfrak{H}$, respectively.

5. <u>Boundary value problems</u>. We are now in a position to apply the results in [8] to describe the subspaces A, A^+ satisfying

$$T_0 \subset A \subset T_1 , \quad T_0^+ \subset A^+ \subset T_1^+ .$$

Let $\dim(T_1 \ominus T_0) = \dim(T_1^+ \ominus T_0^+) = t \leq 2mr$. Then a sample result is the following.

THEOREM. <u>Let</u> A <u>be a subspace satisfying</u>

(i) $$T_0 \subset A \subset T_1 , \quad \dim(A/T_0) = d .$$

<u>Then</u>

(ii) $$T_0^+ \subset A^* \subset T_1^+ , \quad \dim(A^*/T_0^+) = t - d ,$$

<u>and there exist subspaces</u> M_1, M_1^+ <u>such that</u>

(iii) $$M_1 \subset T_1 \ominus T_0 , \quad M_1^+ \subset T_1^+ \ominus T_0^+ ,$$
$$\dim M_1 = d , \quad \dim M_1^+ = t - d ,$$
$$M_1^+ \subset M_1^* ,$$

<u>and</u>

(iv) $$A = T_0 \oplus M_1 , \quad A^* = T_0^+ \oplus M_1^+ ,$$
$$A = T_1 \cap (M_1^+)^* , \quad A^* = T_1^+ \cap M_1^* .$$

<u>Conversely, if</u> M_1, M_1^+ <u>satisfy</u> (iii) <u>then</u> $A = T_0 \oplus M_1$ <u>satisfies</u> (i), <u>and</u> (ii), (iv) <u>are valid</u>.

The descriptions of A, A^* given via $A = T_1 \cap (M_1^+)^*$, $A^* = (T_1^+ \cap M_1^*)$ show how A, A^* are obtained from T_1, T_1^+ by the imposition of generalized boundary conditions. For example, we have

$$A = T_1 \cap (M_1^+)^* = \{w \in T_1 \mid \langle w, m_1^+ \rangle = 0\}$$

where $m_1^+ = (m_1^+, \cdots, m_{1\,t-d}^+)$ is a $1 \times (t - d)$ matrix whose elements form a basis for M_1^+.

It is important to note that A, A^* will contain nonsmooth elements in general, for T_0, T_0^+ contain such elements. This even occurs in cases when $\mathfrak{H}$ satisfies the more stringent assumption (A_2'). However, there exist smooth versions of A, A^*, for we can show that if

$$\tilde{A} = T \cap (M_1^+)^*, \quad \tilde{A}^+ = T^+ \cap M_1^* ,$$

then $(\tilde{A})^c = A$, $(\tilde{A}^+)^c = A^*$. Now $\tilde{A}$, $\tilde{A}^+ \subset C^r(\iota) \times C^\nu(\iota)$ and are obtained by restrictions defined by elements in $C^r(\iota) \times C^r(\iota)$. In case (A_2') holds the boundary conditions, in some cases, can be reduced to conditions of the usual type for L in $L^2(\iota) \times L^2(\iota)$.

More general problems can be treated. Let B, B^+ be subspaces in $\mathfrak{H}^2$ such that

$$\dim B = p < \infty , \quad \dim B^+ = p^+ < \infty ,$$

and consider

$$A_0 = T_0 \cap (B^+)^* , \quad A_0^+ = T_0^+ \cap B^* ,$$

where

$$A_0^* = T_1^+ \dot{+} B^+ , \quad (A_0^+)^* = T_1 \dot{+} B$$

are algebraic direct sums. If $A_1^+ = A_0^*$, $A_1 = (A_0^+)^*$, then we have $A_0 \subset A_1$, $A_0^+ \subset A_1^+$, and we can characterize those A, A^* satisfying

$$A_0 \subset A \subset A_1 , \quad A_0^+ \subset A^* \subset A_1^+ ,$$

via generalized boundary conditions; see [8]. The major problem remaining is to see what these conditions reduce to in significant special cases.

6. <u>The symmetric case</u>. The minimal linear manifold S is symmetric $(S \subset S^*)$ if and only if $L = L^+$, and we now assume this. Then S has selfadjoint extensions $H = H^*$ in $\mathfrak{H}^2$ if and only if

$$\dim \nu(T - \ell I) = \dim \nu(T - \bar{\ell} I) , \quad \text{some } \ell \in \mathbb{C}\setminus\mathbb{R} .$$

More generally, if $A_0 = T_0 \cap B^*$, $\dim B = p < \infty$, $B \subset \mathfrak{H}^2$, where $A_0^* = T_1 \dot{+} B$ is a direct sum, then A_0 is symmetric and has selfadjoint extensions in $\mathfrak{H}^2$ if and only if S does. Now A_0 always has selfadjoint extensions H in a larger space $\mathfrak{K}^2 \supset \mathfrak{H}^2$, $\mathfrak{K}$ a Hilbert space. If P is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$, then $R(\ell)$ defined by

$$R(\ell)f = P(H - \ell I)^{-1} f , \quad f \in \mathfrak{H} , \quad \ell \in \mathbb{C}\setminus\mathbb{R} ,$$

is called a generalized resolvent of A_0 associated with the extension H.

We have

$$\{R(\ell)f,\ \ell R(\ell)f + f\} \in A_0^* = T_1 \dotplus B\ , \qquad f \in \mathfrak{H}\ ,$$

and we can show that $R(\ell)$ is an integral operator on $\mathfrak{R}(G)$:

$$R(\ell)Gh(x) = \int_\iota K(x,y,\ell)h(y)\ dy\ , \qquad h \in L_0^2(\iota)\ .$$

In the case $Mf = f$ this fact has been used to obtain an eigenfunction expansion result and Titchmarsh-Kodaira formula for the extension H. The carrying over of this method to the present case seems to require a special choice of basis for the solutions of $(L - \ell M)f = 0$. A second method for obtaining the eigenfunction expansion result in the case $Mf = f$ was presented in [7], and A. Dijksma and H.S.V. de Snoo have carried out this program in the present case, but a regularity result is required to complete the argument. We hope that both of these programs will be completed soon.

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