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SOLUTION SET PROPERTIES FOR SOME NONLINEAR PARABOLIC DIFFERENTIAL EQUATIONS

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1. Introduction. This paper is concerned mainly with reporting some solution set properties for various classes or problems for nonlinear parabolic equations. Most of this work was done jointly with K. Schmitt. The example in section 6 is a special case of a class of problems being studied jointly with K.-N. Chueh and W. Fulks.

During the past few decades much work has been devoted to the problem of characterizing sets which are invariant with respect to a given ordinary differential equation. More recently several papers ([3], [5], [11], [13]) have considered the same question for nonlinear parabolic differential equations. In [5], the relationship between invariant sets and traveling wave solutions is noted. This relationship can be used to study the Fitzhugh-Nagumo and Hodgkin-Huxley equations, for example.

The assumptions which are sufficient for a given set to be invariant also yield existence of solutions for initial boundary value problems and yield the classical Kneser-Hukuhawa property, i.e., the set of solutions is a continuum in an appropriate function space. For scalar-valued problems, conditions sufficient for invariance give existence of maximal and minimal solutions [4].

2. Definitions and Notation. Let $\mathbb{R}^n$ denote n-dimensional real Euclidean space and let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is an (n-1) - dimensional manifold of class $C^{2+\alpha}$, $\alpha \in (0,1)$. Let $\pi = \Omega \times (0,T)$ and $\Gamma = (\partial\Omega \times [0,T)) \cup (\Omega \times \{0\})$. For $u\colon \bar{\pi} \to \mathbb{R}$, define the differential operators $L_k u$ by:

$$L_k u(x,t) = \sum_{i,j=1}^{n} a_{ij}^k(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1} b_i^k(x,t) \frac{\partial u}{\partial x_i} + c^k(x,t)u - \frac{\partial u}{\partial t}$$

where $a_{ij}^k, b_i^k, c^k \in C^{\alpha,\alpha/2}(\bar{\Omega} \times [0,T])$, $0 < \alpha < 1$, $1 \le i, j \le n$, $1 \le k \le m$, and for all k, $c^k \le 0$. Here $C^{\alpha,\alpha/2}(\cdot)$ denotes the usual Hölder spaces of functions $u(x,t)$. For $u\colon \bar{\pi} \to \mathbb{R}^m$, let $L = (L_1,\dots,L_m)$ be defined by $Lu = (L_1 u_1,\dots,L_m u_m)$. Assume that L is uniformly parabolic. Let $f\colon \bar{\pi} \times \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}^m$, defined by $(x,t,u,p) \to f(x,t,u,p)$ with $(x,t) \in \bar{\pi}$, $u = (u_1,\dots,u_m) \in \mathbb{R}^m$, $p = (p_1,\dots,p_m) \in \mathbb{R}^{nm}$, $p_i \in \mathbb{R}^n$ be a locally Hölder continuous function with Hölder exponents $\alpha, \alpha/2, \alpha, \alpha$ in the respective variables x,t,u,p.

Given $\psi\colon \Gamma \to \mathbb{R}^m$, consider the first initial boundary value problem $(IBVP)_1$:

(1) $\quad Lu = f(x,t,u,\nabla u) \quad , \quad (x,t) \in \pi$

(2) $\quad u = \psi \quad , \quad (x,t) \in \Gamma$

where ψ is continuous on Γ, may be extended to $\bar{\pi}$ so as to belong to $C^{2+\alpha,\,1+\alpha/2}(\bar{\pi})$, and satisfies compatibility conditions appropriate to (1) and (2).

For $\varphi: \bar{\Omega} \to \mathbb{R}^m$, the second initial boundary value problem $(\text{IBVP})_2$: (1) with

(3) $\quad u(x,0) = \varphi(x) \quad , \quad x \in \bar{D}$

(4) $\quad \dfrac{\partial u}{\partial \nu} = 0 \quad , \qquad (x,t) \in \partial D \times [0,T)$

where $\dfrac{\partial u(x,t)}{\partial \nu(x)} = (\nu \cdot \nabla u_1, \ldots, \nu \cdot \nabla u_m)$, $\nu(x)$ is an outer normal to Ω at x, and $\nu(x) \in C^{\alpha+1}(\partial\Omega)$ will also be considered.

3. Positive Invariance. A set $S \subset \mathbb{R}^m$ is positively invariant relative to $(\text{IBVP})_1$ ($(\text{IBVP})_2$) in case, given $\psi: \Gamma \to S$ ($\varphi: \Omega \to S$), every solution $u \in C^{(2,1)}(\bar{\pi})$ of $(\text{IBVP})_1$ ($(\text{IBVP})_2$) is such that $u: \bar{\pi} \to S$. A set $S \subset \mathbb{R}^m$ is weakly positively invariant relative to $(\text{IBVP})_1$ ($(\text{IBVP})_2$) in case, given any $\psi: \Gamma \to S$ ($\varphi: \Omega \to S$), there exists at least one solution $u \in C^{2,1}(\bar{\pi})$ of $(\text{IBVP})_1$ ($(\text{IBVP})_2$) such that $u: \bar{\pi} \to S$.

Theorem 1. Let $L_i = L_1$, $i = 1,\ldots,m$. Let $S \subset \mathbb{R}^m$ be a nonempty open bounded convex neighborhood of 0 such that for each $u \in \partial S$, there exists an outer normal $n(u)$ to S at u with

(5) $\quad n(u) \cdot f(x,t,u,p) > 0$

for all $p = (p_1,\ldots,p_n)$, $p_i \in \mathbb{R}^m$, $1 \le i \le n$, such that $n(u) \cdot p_i = 0$, $i = 1,\ldots,n$, $(x,t) \in \bar{\pi}$ ($\nu(x) \cdot (p_1^j,\ldots,p_n^j) = 0$, $j = 1,\ldots,m$, $x \in \partial\Omega$, $t \in [0,T]$ or $p_i \cdot n(u) = 0$, $i = 1,\ldots,n$, $(x,t) \in \pi$). Then S is positively invariant relative to $(\text{IBVP})_1$ ($(\text{IBVP})_2$).

This theorem is easily proven by standard maximum principal arguments.

Using the above theorem on positive invariance and assuming a Nagumo growth condition on f with respect to p, a weak invariance result, i.e., existence of a solution which lies in the set, can be proven.

Theorem 2. Let $L_i = L_1$, $i = 1,\ldots,m$. Let $S \subset \mathbb{R}^m$ be a nonempty convex set such that for each $u \in \partial S$ and every out normal $n(u)$ to S at u

(6) $\quad n(u) \cdot f(x,t,u,p) \ge 0$

for all $p = (p_1,\ldots,p_n)$, $p_i \in \mathbb{R}^m$, $1 \le i \le n$, such that $n(u) \cdot p_i = 0$, $i = 1,\ldots,n$, $(x,t) \in \bar{\pi}$ ($\nu(x) \cdot (p_1^j,\ldots,p_n^j) = 0$, $j = 1,\ldots,m$, $x \in \partial\Omega$, $t \in [0,T]$ or $p_i \cdot n(u) = 0$, $i = 1,\ldots,n$, $(x,t) \in \pi$). Furthermore, let there exist a positive, continuous, nondecreasing function $\varphi(s)$ satisfying $s^2/\varphi(s) \to \infty$ as $s \to \infty$ and $|f(x,t,u,p)| \le \varphi(|p|)$, $u \in S$, $(x,t) \in \pi$.

Then S is weakly positively invariant relative to $(\text{IBVP})_1$ ($(\text{IBVP})_2$).

The growth condition imposed on f is the Nagumo condition.

The details of the proof of this theorem can be found in [3]. To convey the idea of the proof for $(IBVP)_1$, first assume S is an open convex neighborhood of 0 and that the strict outer normal condition (5) is satisfied. Let $F: C^{1,0}(\bar{\pi}) \to C(\bar{\pi})$ be the continuous map taking bounded sets into bounded sets defined by

(7) $\quad (Fu)(x,t) = f(x,t,u(x,t) , \nabla u(x,t))$.

Let $K: C(\bar{\pi}) \to C^{1,0}(\bar{\pi})$ be the compact bounded linear extension of the linear map $K: C^{\alpha,\alpha/2}(\bar{\pi}) \to C^{\alpha+2,1+\alpha/2}(\bar{\pi})$ defined as follows: for $v \in C^{\alpha,\alpha/2}(\bar{\pi})$, Kv is the unique solution of

(8) $\quad LKv = v$

$\quad\quad Kv = 0$.

Let $g \in C^{\alpha+2,1+\alpha/2}(\bar{\pi})$ be the unique solution to

(9) $\quad Lg = 0$

$\quad\quad g = \psi$.

For any $\lambda \in [0,1]$, $\lambda KF: C^{1,0}(\bar{\pi}) \to C^{1,0}(\bar{\pi})$ is a completely continuous map. For $\lambda \in [0,1]$, $g \in C^{2+\alpha,1+\alpha/2}(\bar{\pi})$ a solution of (9) , $u \in C^{1,0}(\bar{\pi})$ is a solution of

(10) $\quad u = \lambda KFu + \lambda g$

if and only if $u \in C^{2+\alpha,1+\alpha/2}(\bar{\pi})$ is a solution of

(1_λ) $\quad Lu = \lambda f(x,t,u,\nabla u)$, $(x,t) \in \pi$

(2_λ) $\quad u(x,t) = \lambda\psi(x,t)$, $(x,t) \in \psi$.

By the Nagumo growth condition imposed on f in hypotheses of the theorem, if $u \in C^{\alpha+2,\alpha/2+1}(\bar{\pi})$ is a solution of (1_λ) - (2_λ) for any $\lambda \in [0,1]$ with $u: \bar{\pi} \to \bar{S}$, then there exists $M > 0$ such that $|\nabla u| \le M$.

The crux of the proof is to show that the continuous compact perturbation of the identity given by $I - \lambda(KF + g): \mathcal{O} \subset C^{1,0}(\bar{\pi}) \to C^{1,0}(\bar{\pi})$, $\lambda \in [0,1]$, where $\mathcal{O} = \{u \in C^{1,0}(\bar{\pi}) \mid u: \bar{\pi} \to S , |\nabla u(x,t)| < M+1\}$ is a nonempty bounded open subset in $C^{1,0}(\bar{\pi})$, has nonzero Leray-Schander degree at 0 relative to $\mathcal{O}$. This can be accomplished by a homotopy argument. In this way, the existence of a solution for $(IBVP)_1$ is established for a strict outer normal condition on an open convex neighborhood of zero. By a perturbation argument, the weak outer normal condition (6) suffices. Finally, one shows that if S is a compact convex set, then the result holds for S_ε , ε - neighborhoods of S . By an approximating argument, the weak invariance of S follows.

For certain sets $\Lambda \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ which have compact convex cross sections in $\mathbb{R}^m$ depending on x and t , similar invariance results hold. For example, let $\alpha,\beta \in C^{2,1}(\bar{\pi})$ be given with $\alpha_i(x,t) < \beta_i(x,t)$ on $\bar{\pi}$ and define

$$(\alpha,\beta) = \{u \in \mathbb{R}^m\colon \ \alpha_i \le u_i \le \beta_i \ , \ i = 1,\ldots,m\} \ .$$

Theorem 3. Assume that

$$(11)\ \begin{cases} L_k\alpha_k - f_k(x,t,u_1,\ldots,u_{k-1},\alpha_k,u_{k+1},\ldots,u_m,\nabla u_1,\ldots,\nabla u_{k-1},\nabla\alpha_k,\nabla u_{k+1},\ldots,\nabla u_m) \\ \qquad \ge 0 \ge \\ L_k\beta_k - f_k(x,t,u_1,\ldots,u_{k-1},\beta_k,\beta_{k+1},\ldots,u_m,\nabla u_1,\ldots,\nabla u_{k-1},\nabla\beta_k,\nabla u_{k+1},\ldots,\nabla u_m) \end{cases}$$

for all $(x,t) \in \pi$, $k = 1,\ldots,m$, and $\alpha_j \le u_j \le \beta_j$, $k \neq j$.

Furthermore, assume the Nagumo growth condition of theorem 2 relative to (α,β). Then (α,β) is weakly positively invariant relative to $(\mathrm{IBVP})_1((\mathrm{IBVP})_2)$.

4. Funnel Properties. The classical Hukuhara-Kneser property for ordinary differential equations in $\mathbb{R}^n$ states that if all solutions of a given initial value problem exist on $[t_0,t_0+\delta]$, then the set of solutions is a continuum (a compact connected set) in $C[t_0,t_0+\delta]$. Krasnosel'skii and Sobolevskii [7] very elegently proved an abstracted version of this result for the set of fixed points of completely continuous operators defined in a normed linear space which also satisfy a certain approximation property. By using a modification of this result, the following theorem can be proven.

Theorem 4. Assume the hypotheses of Theorem 2, then the set $Q = \{u \in \mathcal{O}\colon u = KFu + g\}$ is a continuum in $C^{1,0}(\bar{\pi})$.

Here K , f , g , and $\mathcal{O}$ are as in Section 2.

5. Maximal and Minimal Solutions. When $m = 1$, the invariance result given by theorem 3 can be used to establish the existence of maximal and minimal solutions for the scalar version of $(\mathrm{IBVP})_1$ and the Cauchy initial value problem for (1). In this section we report on the main result in [4].

In recent years a considerable amount of study has been devoted to establishing the existence of solutions for elliptic and parabolic problems provided upper and lower solutions of such problems exist. Much of this work has its basis in the fundamental paper of Nagumo [8] as carried further by Akô [1]. Keller [6] and Amann [2] constructed solutions between upper and lower solutions of elliptic problems using a monotone iteration scheme which was possible because of certain one sided Lipschitz continuity assumptions on the nonlinear terms and because the nonlinearities are assumed gradient independent. Sattinger [12], Pao [9], and Puel [10] extended Amann's results to parabolic initial boundary value problems using either monotone iteration techniques on the theory of monotone operators. While these procedures have certain computational advantages the permissible class of nonlinearities is quite restrictive.

Using a different approach patterned after methods employed by Akô, the existence of maximal and minimal solutions for the Cauchy initial value problem and the initial value problem for parabolic equations can be proven for a much larger class of nonlinearities.

A continuous function $v: \pi \to \mathbb{R}$ is called a <u>lower solution</u> of (1), (2) in case

(12) $\quad v(x,t) \le \psi(x,t) \quad , \quad (x,t) \in \Gamma$

and if for every $(x_0,t_0) \in \pi$ there exists an open neighborhood U of (x_0,t_0) and a finite set of functions $\{v_r\}_{1\le r\le s} \subset C^{2,1}(\bar{U} \cap \pi)$ such that

(13) $\quad Lv_r \ge f(x,t,v_r,\nabla v_r) \quad , \quad (x,t) \in \bar{U} \cap \pi \quad , \quad 1 \le r \le s \; ,$

and

(14) $\quad v(x,t) = \max\limits_{1\le r\le s} v_r(x,t) \quad , \quad (x,t) \in \bar{U} \cap \pi \quad .$

If in the above definition the inequality signs in (12) and (13) are reversed and in (14) max is replaced by min, then v is called an upper solution of (1), (2).

For such upper and lower solutions β,α of (1) - (2) respectively with $\alpha(x,t) \le \beta(x,t)$, $(x,t) \in \pi$, theorem 3 holds and hence $(IBVP)_1$ has a solution $u \in C^{2,1}(\bar{\pi})$ with $\alpha(x,t) \le u(x,t) \le \beta(x,t)$ for $(x,t) \in \bar{\pi}$.

A solution $\bar{u}$ of the $(IBVP)_1$ with $f = f_1$ $(m = 1)$ is a <u>maximal solution</u> relative to a given pair of lower and upper solutions α and β with $\alpha(x,t) \le \beta(x,t)$, $(x,t) \in \bar{\pi}$ if $\alpha(x,t) \le \bar{u}(x,t) \le \beta(x,t)$ and if u is any other such solution then $u(x,t) \le \bar{u}(x,t)$ for $(x,t) \in \bar{\pi}$. Minimal solutions are defined analogously.

<u>Theorem 5</u>. Assume the hypotheses of theorem 3 for a given pair of upper and lower solutions β and α with $\beta(x,t) \ge \alpha(x,t)$, $(x,t) \in \bar{\pi}$. Then $(IBVP)_1$ has a maximal and a minimal solution.

The proof of the existence of a maximal solution is obtained by considering the collection $\mathcal{L}$ of all lower solutions of $(IBVP)_1$ where $\mathcal{L} = \{v: \bar{\pi} \to \mathbb{R}: \alpha(x,t) \le v(x,t) \le \beta(x,t) \; , \; (x,t) \in \bar{\pi} \; , \; v$ is a lower solution of $(IBVP)_1\}$ and showing that

$$u_{max}(x,t) = \sup\{v(x,t): v \in \mathcal{L} \; , \; \alpha \le v \le \beta\}$$

so defined is the maximal solution using theorem 3.

This same result is true for $(IBVP)_2$ and for the Cauchy initial value problem.

<u>6. An Example</u>. To illustrate how invariance can be used to analyze a problem, consider

$$(15)\begin{cases} L_1 u \equiv a_1 u_{xx} - u_t = -uv^{\gamma} \\ L_2 v \equiv a_2 v_{xx} - v_t = +uv^{\gamma} \end{cases}$$

for $(x,t) \in \pi = (0,1) \times (0,\infty)$ where $\gamma > 0$, together with the initial-boundary conditions

$$(16)\quad \begin{aligned} &u(x,0) = u_0(x) \ge 0 \; , \; v(x,0) = v_0(x) \ge 0 \quad \text{for} \quad x \in [0,1] \\ &u(t,0) = 0 = u(t,1) \; , \; v(t,0) = 0 = v(t,1) \quad , \quad t \in (0,\infty) \; , \end{aligned}$$

where $u_0(x)$, $v_0(x) \in C[0,1]$.

Set $V = \max_{[0,1]} v_0(x)$ and $U = \max_{[0,1]} u_0(x)$, then $(\alpha,\beta) \subset \mathbb{R}^2$ as defined in section 2, where $\alpha(x,t) = (\alpha_1(x,t)\ ,\ \alpha_2(x,t)) = (0,0)$ and $\beta(x,t) = (\beta_1(x,t)\ ,\ \beta_2(x,t)) = (Ue^{V^\gamma t}, V)$, is a weakly positively invariant set by theorem 3. Hence, there exists at least one solution $(u(x,t)\ ,v(x,t)) \in (\alpha,\beta)$ for $(x,t) \in \bar{\pi}$. If $\gamma \geq 1$, then the solution to IBVP (15) - (16) is unique and one can obtain additional asymptotic properties.

Let $\varphi(x,t)$ be a solution of $L_2 v = 0$, the homogeneous heat equation, then $L_2\varphi(x,t) = 0 \leq u(x,t)\varphi^\gamma$ where $u(x,t)$ is the first component of the unique solution of (15) - (16). Hence, φ is an upper solution of $L_2 v = u(x,t)v^\gamma$, and $\varphi(x,t) \geq v(x,t)$ for $(x,t) \in \bar{\pi}$. By standard estimates for the heat equation, $v(x,t) \leq \varphi(x,t) \leq 4/\pi\ Ve^{-a_2\pi^2 t}$. From this, $v(x,t) \to 0$ uniformly in x as $t \to \infty$. For $v(x,t)$, there exists $T > 0$ such that, for all $t \geq T$, $|v^\gamma| < a_2\pi^2 - \varepsilon \equiv M$. Take $\psi(x,t)$ to be the solution of $L_1 u = -Mu$ with $\psi(x,T) = u(x,T)$, $\psi(1,t) = u(1,t)$ and $\psi(0,t) = u(0,t)$ for $t \geq T$. Then $L_1\psi = -M\psi < -(v(x,t))^\gamma\psi$ and ψ is an upper solution. Hence $u(x,t) \leq \psi(x,t)$ for $t \geq T$. By again standard estimates, $u(x,t) \leq \psi(x,t) \leq 4/\pi\ Ke^{-\varepsilon t}$ for $t \geq T$. We conclude that $(u(x,t)\ ,\ v(x,t)) \to 0$ as $t \to \infty$.

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