H. Amann

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INVARIANT SETS FOR SEMILINEAR PARABOLIC AND ELLIPTIC SYSTEMS

H. Amann, Bochum

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and let $Q := \Omega \times (o,T)$ for some fixed $T > o$. Denote by $\partial/\partial t + A(x,t,D)$ a uniformly parabolic second order differential operator on $\overline{Q}$ with smooth coefficients, and let $B(x,D)$ be a (time independent) first order smooth boundary operator. We suppose that $B(x,D)$ is of the form $B(x,D)u = b(x)u + \delta(\partial u/\partial\beta)$, where either $\delta = o$ and $b(x) = 1$ (Dirichlet boundary operator) or $\delta = 1$ and $b(x) \geq o$ for all $x \in \partial\Omega$, and β is a smooth outward pointing, nowhere tangent vector field on $\partial\Omega$ (Neumann or regular oblique derivative boundary operator).

We denote by $f : \overline{Q} \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}^m$ a Lipschitz continuous function, and consider parabolic initial boundary value problems of the form

$$(1) \qquad \begin{aligned} \frac{\partial u}{\partial t} + A(x,t,D)u &= f(x,t,u,Du) && \text{in } \Omega \times (o,T] , \\ B(x,D)u &= o && \text{on } \partial\Omega \times (o,T] , \\ u(.,o) &= u_0 && \text{on } \overline{\Omega} , \end{aligned}$$

where $u = (u^1,\ldots,u^m)$. In other words, (1) is a "diagonal system" which is strongly coupled through the nonlinear function f. By a solution of (1) we mean a classical solution.

In order to obtain appropriate a priori estimates, we impose the following growth restriction for f, which we write in a self-explanatory symbolic form: namely we suppose that either

$$|f(x,t,u,Du)| \leq c(|u|)(1 + |Du|^{2-\varepsilon})$$

for some $\varepsilon > o$, or

$$|f^i(x,t,u,Du)| \leq c(|u|)(1 + |Du^i|^2)$$

for $i = 1,\ldots,m$, where $c \in C(\mathbb{R}_+,\mathbb{R}_+)$.

It is well known that (1) possesses a unique solution for every sufficiently smooth initial value u_0 satisfying appropriate compatibility conditions. However this solution may only exist for a small time interval and not in the whole cylinder Q. The existence of a *global* solution can be guaranteed provided an a priori bound for the maximum norm can be found. Unfortunately, establishing a priori bounds for the maximum norm is a rather difficult problem for systems since no good maximum principle is available.

Recently H. F. Weinberger [5] (and later Chueh, Conley and Smoller [3]) has given a weak substitute for a maximum principle which can be used for establishing a priori bounds. But these results presuppose a priori knowledge of the solution on the lateral boundary $\partial\Omega \times [o,T]$ of the cylinder Q which is, in general, only available for the case of Dirichlet boundary conditions.

In this paper we present a global existence and uniqueness theorem for problem (1) *without assuming any a priori knowledge on the solution for* $t > o$. We emphasize the fact that our results apply to the case of boundary conditions of the third kind which are of particular importance in applications (to problems of chemical engineering, for example).

For an easy formulation of our results we introduce the following hypotheses and notations. Let $\mathbb{D}$ be a compact convex subset of $\mathbb{R}^n$ such that $o \in \mathbb{D}$. For every $\xi_0 \in \partial\mathbb{D}$ let

$$N(\xi_0) := \{p \in \mathbb{R}^m \mid <p,\xi-\xi_0> \le o \ \forall \ \xi \in \mathbb{D}\} \quad ,$$

that is, $N(\xi_0)$ is the "set of outer normals" on $\partial\mathbb{D}$ at ξ_0 . Finally, for $k = 1,2$, we let

$$C^k_B(\overline{\Omega},\mathbb{D}) := \{u \in C^k(\overline{\Omega},\mathbb{R}^m) \mid Bu = o \ \text{ on } \ \partial\Omega \ \text{ and } \ u(\overline{\Omega}) \subset \mathbb{D}\} \quad .$$

Then we impose the following *tangency condition:*

For every $v \in C^1_B(\overline{\Omega},\mathbb{D})$ and for every $x_0 \in \overline{\Omega}$ with $v(x_0) \in \partial\mathbb{D}$, we suppose that

$$<p,f(x_0,t,v(x_0),Dv(x_0))> \le o \tag{Tg}$$

for all $t \in [o,T]$ and all $p \in N(\xi_0)$, where $<.,.>$ denotes the inner product in $\mathbb{R}^m$.

Condition (Tg) means geometrically that the vector $f(x_0,t,v(x_0),Dv(x_0))$, attached to $\partial\mathbb{D}$ at the point $v(x_0)$, lies in the cone which contains $\mathbb{D}$ and is described by the family of all supporting hyperplanes at $v(x_0)$. It is easily seen that (Tg) reduces to the condition introduced by Weinberger [5] (and also used by Chueh, Conley and Smoller [3]) in the case that f is independent of Du (the case studied in [3] and [5]). It is essentially the same condition as the one used by Bebernes [2] . We refer to [3] for a variety of examples satisfying (Tg). It is easy to give further examples in the case of nonlinear gradient dependence.

After these preparations we can give our basic existence and uniqueness theorem for problem (1) (cf. also [2] for the special case of Dirichlet and Neumann boundary conditions).

Theorem 1: Let the growth condition and the tangency conditions be satisfied. Then *the initial boundary value problem* (1) *has a unique global solution* u *for every initial value* $u_o \in C_B^2(\bar{\Omega}, \mathbb{D})$, *and* $u(\bar{Q}) \subset \mathbb{D}$.

Proof: By using the results of Kato, Tanabe, and Sobolevskii on abstract parabolic evolution equations as well as the results of Ladyzenskaja, Solonnikov, and Ural'ceva on the classical solvability of linear parabolic equations, it is shown that (1) is equivalent to the nonlinear evolution equation

$$\text{(2)} \qquad \dot{u} + A(t)u = F(t,u) \quad , \quad o < t \le T \quad , \qquad u(o) = u_o$$

in $X := L_p(\Omega, \mathbb{R}^m)$, where $p > 2$ is sufficiently large and $-A(t)$ is the infinitesimal generator of a holomorphic semigroup. We denote by X_α the domain of the fractional power $[A(o)]^\alpha$, $o < \alpha < 1$, and we let $\mathbb{M}_\alpha := L_p(\Omega, \mathbb{D}) \cap X_\alpha$, endowed with the topology of X_α , where α is sufficiently close to 1 . Then (2) is equivalent to the integral equation

$$\text{(3)} \qquad u(t) = U(t,o)u_o + \int_o^t U(t,\tau)F(\tau,u(\tau))d\tau$$

in $C([o,T], X_\alpha)$, where U denotes the linear evolution operator associated with (2).

The maximum principle implies that $U(t,\tau)(\mathbb{M}_\alpha) \subset \mathbb{M}_\alpha$ for $o \le \tau \le t \le T$, and it is shown that the tangency condition implies that

$$\text{(4)} \qquad \mathrm{dist}_X(y + hF(t,y), \mathbb{M}) = o(h) \quad \text{as} \quad h \to o+$$

for each $y \in \mathbb{M}_\alpha$. Hence we are left with the problem of solving the integral equation on the *closed* bounded subset $\mathbb{M}_\alpha$ of the Banach space X_α . By employing a discontinuous Euler method as developed by R. H. Martin (e.g. [4]), it can be shown that the Nagumo type condition (4) implies the existence of a unique local solution of (3) in $\mathbb{M}_\alpha$. Finally, by means of the growth condition, we obtain a priori estimates which guarantee that the local solution has a unique continuation to a global solution. □

Suppose now that A and f are independent of t . Then, as a consequence of Theorem 1, it follows that (1) defines a nonlinear semigroup $\{S(t) \mid o \le t < \infty\}$ on $\mathbb{M}_\alpha$, where $S(t)u_o$ denotes the solution at time t of the autonomous problem (1) with initial value $u_o \in \mathbb{M}_\alpha$. On the basis of the integral equation (3) and by using appropriate a priori estimates, it can be shown that, for every $t > o$, the nonlinear operator $S(t) : \mathbb{M}_\alpha \to \mathbb{M}_\alpha$ is continuous and has a relatively compact image.

For every $t \ge o$, let

$$\mathfrak{F}_t := \{u \in \mathbb{M}_\alpha \mid S(t)u_o = u_o\} ,$$

that is, $\mathfrak{F}_t$ is the fixed point set of $S(t)$. Then, by Schauder's fixed point theorem, $\mathfrak{F}_t \neq \emptyset$ for every $t > o$. Moreover, suppose that $t_1,\dots,t_m$ are positive numbers having $t > o$ as a common divisor. Then it is an easy consequence of the semigroup property (i.e., $S(t+\tau) = S(t)S(\tau)$) that

$$\mathcal{F}_t \subset \bigcap_{i=1}^{m} \mathcal{F}_{t_i} \quad .$$

This implies that the family $\{\mathcal{F}_t \mid t \in \mathbb{Q}_+\}$ has the finite intersection property. Hence, by compactness, $\cap\{\mathcal{F}_t \mid t \in \mathbb{Q}_+\} \neq \emptyset$. This shows that there exists an element $u_0 \in \mathbb{M}_\alpha$ such that $S(t)u_0 = u_0$ for all $t \in \mathbb{Q}_+$, that is, u_0 is a common fixed point of the family $\{S(t) \mid t \in \mathbb{Q}_+\}$. Finally, by using a continuity argument, it follows that $S(t)u_0 = u_0$ for all $t \geq o$, that is, u_0 is a rest point of the flow $\{S(t) \mid t \geq o\}$, hence a solution of the stationary equation.

By this argument we obtain

Theorem 2: Suppose that $A(x,D)$ *is a strongly uniformly elliptic second order differential operator with smooth coefficients. Suppose that* f *is independent of* t *and satisfies the growth condition and the tangency condition. Then the semilinear elliptic system*

$$(5) \qquad \begin{aligned} A(x,D)u &= f(x,u,Du) \quad &\textit{in} \quad \Omega \ , \\ B(x,D)u &= o \quad &\textit{on} \quad \partial\Omega \end{aligned}$$

has at least one solution u *such that* $u(\overline{\Omega}) \subset \mathbb{D}$.

It should be remarked that the assumption that in each single equation of the system (1) or (5) there occurs one and the same differential operator can be dropped if the conditions on $\mathbb{D}$ are strengthened. For further details, examples, and more detailed proofs we refer to [1] .

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Author's address: Institut für Mathematik, Ruhr-Universität, D-4630 Bochum, Germany