SOME NEW PROBLEMS IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

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In the presented paper some new applications of the theory of partial differential equations related to numerical analysis will be considered. This paper repeats to a certain extent the communication of the author which was published in the Proceedings of the Paris Symposium on the Theory of Partial Differential Equations, June 25 to 31 1962, and represents its further development.

Two important problems of numerical analysis are as follows: the problem of approximate integration and the problem of interpolation; they consist in determining the value of the integral of a given function

$$\int_{\Omega} f \, \mathrm{d}x \tag{1}$$

and in finding an appropriate procedure for the approximate evaluation of the function $f(x)$ at a given point z from its values on a finite set of points:

$$x^{(1)}, x^{(2)}, \ldots, x^{(N)} . \tag{2}$$

Integral (1) and the value of $f(z)$ is frequently expressed as a linear combination of the values of the function f at points (2). Moreover, the following formulas for errors are obtained:

$$(l, f) = \int f \, \mathrm{d}x - \sum_{k=1} c_k f(x^{(k)}) , \tag{3}$$

$$(j, f) = f(z) - \sum_{k=1}^{N} c_k(z) f(x^{(k)}) .$$

The kinds of approximate integration and interpolation can be studied from different points of view. Modern numerical analysis, concentrating in itself many new ideas and methods, approaches these problems as those of functional analysis aiming at the theory of approximation of compact sets in some functional spaces.

From this point of view both of these problems represent problems of approximation of functionals $\int f \, \mathrm{d}x$ or of an identical operator by means of linear combinations of the values of $f(x^{(k)})$.

In this connection, introducing different topologies in the space of functionals or in the space of operators yields essentially different results.

The first problem to study is an estimate of the quality of approximate formulas by means of introducing a corresponding topology in the space, the elements of which are

(l, f) and (j, f). The second problem is that of optimization of formulas, which is in general more complicated. The first problem, which will be discussed below, is directly related to partial differential equations.

There are two important approaches of the problem. One of them is as follows: One has to choose the coefficients c_k and the nodes $x^{(k)}$ such that (l, f) and (j, f) should vanish for a widest possible class of functions, e.g. for all polynomials of given degree. The second problem is reduced to an estimate of errors in a topological Banach space.

At the first approach a purely algebraical aspect of the problem seems to play the most important role. The functional aspect of the problem does not appear at this stage. The algebraical formulation of these problems is as follows: Consider the set of all monomials of degree $m - 1$ or less of the variable $x(x_1, \ldots, x_n)$

(4) $$\begin{pmatrix} x^{\alpha_1} \\ x^{\alpha_2} \\ \vdots \\ x^{\alpha_M} \end{pmatrix},$$

$\alpha_1, \alpha_2, \ldots, \alpha_M$ denoting the numbered set of all vectors with non-negative integer components:

(5) $$\alpha_j = (\alpha_j^{(1)}, \ldots, \alpha_j^{(n)}),$$
$$|\alpha_j| = \alpha_j^{(1)} + \ldots + \alpha_j^{(n)} \leqq m - 1$$

and

(6) $$x^{\alpha_j} = x_1^{\alpha_j(1)} x_2^{\alpha_j(2)} \ldots x_n^{\alpha_j(n)}.$$

Let us denote by S the matrix

(7) $$S = \begin{pmatrix} x^{(1)\alpha_1}, & x^{(2)\alpha_2}, & \ldots, & x^{(N)\alpha_1} \\ x^{(1)\alpha_2}, & x^{(2)\alpha_2}, & \ldots, & x^{(N)\alpha_2} \\ \ldots & \ldots & \ldots & \ldots \\ x^{(1)\alpha_M}, & x^{(2)\alpha_M}, & \ldots, & x^{(N)\alpha_M} \end{pmatrix}.$$

A polynomial of degree $m - 1$ of x can be represented as a scalar product of vectors $\bar{a} = (a_1, \ldots, a_M)$ and (4): $\bar{a}x^\alpha$ and its values at points (2) represent a row-vector

(8) $$\bar{a}S.$$

The problem of interpolation consists in finding coefficients $\bar{a}$, if values

(9) $$\bar{a}S = f$$

are given.

The problem of approximate integration which is determined by the condition that all polynomials of degree $m - 1$ are to be integrated exactly, consists in finding such c that for any α,

(10) $$(l, x^{\alpha_j}) = 0.$$

However, from (10) it follows that

(11) $$S\bar{c} = \bar{\psi}$$

where

(12) $$\bar{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}$$

and

(13) $$\bar{\psi} = \int \begin{pmatrix} x^{\alpha_1} \\ \vdots \\ x^{\alpha_M} \end{pmatrix} \mathrm{d}x \,.$$

Thus the problems of integration and interpolation are adjoint.

The classical problem of interpolation consists of solving system (9) under the condition that S is a square matrix of rank

(14) $$\mathscr{C}(S) = M = N \,.$$

However, the problem is most interesting in the case that $N > M$ and

(15) $$\mathscr{C}(S) = M \,.$$

This problem will be overdetermined. Under the same hypotheses the problem of approximate integration will be indeterminate, i.e. it will have an infinite set of solutions.

Both problems are solved by constructing a right hand inverse matrix S_d^{-1} of S. By means of this matrix: first, every solution of (9) will be represented in the form

(16) $$\bar{a} = fS_d^{-1}$$

and, second, one succeeds in constructing the vector

(17) $$c = S_d^{-1}\psi$$

which will be a solution of system (11) (in general this solution is not unique).

The algebraical treatment of our problems cannot satisfy all natural requirements. The inverse matrix S_d^{-1} is not uniquely determined for $N > M$. On the other hand the insufficiency of the presented point of view is also apparent from the fact that constructing the formulas in the algebraical way one does not at all exploit a plentiful information on the behaviour of the function f, on the existence of its derivatives, an estimate of these derivatives etc.

Therefore, it is useful, as has been done for a long time in an implicit form, to determine a numerical estimate of the error inherent in the formulas of approximate integration or interpolation for a class of functions, and then to choose the parameters c_k and $x^{(k)}$ of the formula so that this error should be minimal. See [2].

The functional space B will be considered and the functionals (l, f) and (j, f) on the unit sphere of this space will be examined. In a natural way two problems arise: To find

(18) $$\min_{x^{(k)}, c_k,\, k \leqq N} \left[\max_{\|f\|_B = 1} (l, f) \right]$$

and

(19) $$\min_{x^{(k)}, c_k,\, k \leqq N} \left[\max_{\|f\|_B = 1} (j, f) \right].$$

Evidently the space B has to be such that the value of a function at a point is a linear functional of this function.

Both these problems possess two stages. In the present paper we confine ourselves to the first stage of these problems, i.e. to finding the maximum of functionals (l, f) and (j, f) on the unit sphere of some functional spaces.

Even in the case of one independent variable, when $x \in E_1$, the corresponding problems have been studied by several authors in the spaces C^m of functions with continuous (or bounded) derivatives of order m, or $L_p^{(l)}$ of functions whose derivatives of order l are integrable with power p. The extreme value of functionals (18) and (19) corresponded to the functions satisfying the equation

(20) $$\frac{\mathrm{d}^m y}{\mathrm{d}x^m} = 1 .$$

For our purposes it is interesting to examine three different functional spaces for functions f in the case $x \in E_n$:

i) $W_2^{(m)}(\Omega)$ is the space of functions whose derivatives of order m are square integrable on a bounded domain Ω. Evidently it is necessary to suppose here $m > n/2$ to assure that functionals (l, f) and (j, f) should be linear.

It is even convenient to study not the space $W_2^{(m)}$ but the space $L_2^{(m)}$ of classes of functions differing from each other by polynomials only. The norm in $L_2^{(m)}$ is given by the formula

(21) $$\|f\|^2_{L_2^{(m)}} = \int_\Omega \{ \sum_{|\alpha| = m} [\mathscr{D}^\alpha f\,]^2 \} \,\mathrm{d}x .$$

As a matter of fact, choosing the formula so that for two functions f_1 and f_2 differing from each other by polynomials of degree $m - 1$ equations

(22) $$(l, f_1) = (l, f_2)\,, \quad (j, f_1) = (j\,, f_2)$$

are valid, one obtains the same errors (l, f) and (j, f) for all functions of the same class.

ii) $U_2^{(m)}(\Omega)$ is the space of classes of functions defined in Ω with the norm

(23) $$\min \|\bar{f}\|_{L_2^{(m)}(\infty)}\,, \quad \bar{f} \doteq f \text{ in } \Omega .$$

By a theorem of Gagliardo any function from $W_2^{(m)}$ defined in the domain Ω with the boundary satisfying a Lipschitz condition, can be extended to the whole space without leaving $W_2^{(m)}$. If the norm in $W_2^{(m)}$ can be taken in the form

$$\|\bar{f}\|^2_{W_2^{(m)}} = \|\prod \bar{f}\|^2_{S_{m-1}} + \|\bar{f}\|^2_{L_2^{(m-1)}}$$

for the extended function it may be chosen finite, especially as $\|f\|_{L_2^{(m)}}$; moreover, the projection operator may be chosen so that it depends only on the values of $\bar{f}$ inside Ω.

For the norm $\|f\|_{U_2^{(m)}}$ all the axioms for norms are valid.

iii) $\tilde{W}_2^{(m)}$ is the space of periodical functions of n variables with the matrix of periods:

(24) $$H = (\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_n)\,.$$

The elements of this space are functions satisfying the relation

(25) $$u(x + H\beta) = u(x)\,,$$

where x is the column vector of coordinates of a given point and β is an arbitrary column vector with integer components.

The functions of this space will be examined in some fundamental domain Ω_0. This means that the denumerable set of domains

(26) $$\Omega_\beta = \Omega + H\beta$$

covers the whole space without repetition (modulo the points of the boundary the measure of which is zero). For the sake of simplicity it will be supposed henceforth that the volume of domain Ω_0 is equal to one. The space of classes of functions periodic in Ω_0 consists of functions differing from each other by a constant, as the constant represents the unique periodical polynomial.

Accordingly, the norm in $\tilde{W}_2^{(m)}$ can be defined as

(27) $$\|f\|_{\tilde{W}_2^{(m)}} = (a, f)^2 + \|f\|_{\tilde{L}_2^{(m)}}$$

where (a, f) is an arbitrary linear functional not equal to zero for a constant function. From the same reason as above one has to suppose $m > n/2$. In the problems presented it is again possible to examine $\max\limits_{\|f\|\tilde{L}_2^{(m)}} (l, f)$.

Now it will be shown that in all three cases there exist extreme functions for which the desired maximum of the functional on unit sphere is reached, and that these extreme functions represent solutions of some boundary value problems for some partial differential equations of elliptic type.

To do this let us first remark that both problems in all cases are essentially reduced to finding

(28) $$\max \frac{|(l, f)|}{\|f\|} \quad \text{or} \quad \max \frac{|(j, f)|}{\|f\|}$$

and, hence, the norm having been changed, they can be reduced to the problem of minimalization

$$I \min_{(l,f)=1} \|f\|^2 , \quad II \min_{(j,f)=1} \|f\|^2 .$$

The latter problems concerning a relative minimum of a square functional are examined as usual (see, e.g. [1], Chapter II, the Neumann problem).

Let the problems of relative extremes for different norms be denoted respectively: $I^{i)}$ and $II^{i)}$ for $\| \ \|_{L_2^m(\Omega)}$, $I^{ii)}$ and $II^{ii)}$ for $\| \ \|_{U_2^{(m)}}$, $I^{iii)}$ and $II^{iii)}$ for $\| \ \|_{\tilde{L}_2^{(m)}(H)}$.

Note that the problem $iii)$ shall be especially simple in both cases, since e.g. the interpolation can be carried out by means of one polynomial of degree zero, i.e. of a constant, and the validity of the cubature has also to be verified from the point of view of constants only. The minimalizing sequence for all examined problems having been constructed it is easy to establish that such a sequence will be fundamental and that the extreme will be realized by the limit function of this sequence. Then similarly as has been done in [1] one ascertains that the limit functions will satisfy the following equations respectively:

$$\int_\Omega \sum_{|\alpha|=m} \mathscr{D}^\alpha \mathscr{U} \mathscr{D}^\alpha \xi \, dx + \lambda \int_\Omega \xi \, dx - \lambda \sum c_k \xi(x^{(k)}) = 0 \tag{29}$$

for problems $I^{i)}$ and $I^{iii)}$,

$$\int_\Omega \sum_{|\alpha|=m} \mathscr{D}^\alpha \mathscr{U} \mathscr{D}^\alpha \xi \, dx + \lambda \xi(z) - \lambda \sum c_k \, \xi(x^{(k)}) = 0 \tag{30}$$

for problems $II^{i)}$ and $II^{iii)}$,

$$\int_\infty \sum_{|\alpha|=m} (\mathscr{D}^\alpha \mathscr{U}, \mathscr{D}^\alpha \xi) \, dx + \lambda \int_\Omega \xi \, dx - \lambda \sum c_k \, \xi(x^{(k)}) = 0 \tag{31}$$

for problem $I^{ii)}$ and finally

$$\int_\infty \sum_{|\alpha|=m} (\mathscr{D}^\alpha \mathscr{U}, \mathscr{D}^\alpha \xi) \, dx + \lambda \, \xi(z) - \lambda \sum c_k \, \xi(x^{(k)}) = 0 \tag{32}$$

for problem $II^{ii)}$.

It will be shown from the relations presented that the respective differential equations are valid:

$$(-1)^m \Delta^m u + \lambda [1 - \sum c_k \, \delta(x - x^{(k)})] = 0 \quad \text{in} \quad \Omega \tag{33}$$

for $I^{i)}$,

$$(-1)^m \Delta^m u + \lambda \, E_\Omega(x) - \lambda \sum c_k \, \delta(x - x^{(k)}) = 0 \tag{34}$$

for $I^{ii)}$,

where $E_\Omega(x)$ is the characteristic function of domain Ω, equal to zero outside Ω and to unity inside Ω and, finally,

(35) $$(-1)^m \Delta^m u + \lambda[1 - \sum c_k M(H, x - x^{(k)})] = 0$$

for $I^{iii)}$,

where function $M(H, x - x^{(k)})$ is a periodic distribution with the matrix of periods H, equal to $\delta(x - x^{(k)})$ in the fundamental domain Ω_0.

The corresponding equations for the interpolation problem are as follows:

(36) $$(-1)^m \Delta^m u + \lambda[\delta(x - z) - \sum c_k \delta(x - x^{(k)})] = 0$$

for $II^{i)}$ and $II^{ii)}$,

(37) $$(-1)^m \Delta^m u + \lambda[M(H, x - z) + \sum c_k M(H, x - x^{(k)})] = 0$$

for $II^{iii)}$.

Moreover, in problems $I^{i)}$ and $II^{i)}$ the solution has to satisfy some conditions of the type

(38) $$B_k u|_\Gamma = 0\,, \quad k = 1, 2, \ldots,$$

which are obtained by a formal integration by parts of formulas (33) and (36); however, in cases $I^{iii)}$ and $II^{iii)}$ it must be a periodic function with the matrix of periods H.

We will show the principal idea of the proof in any of the above mentioned cases, e.g. (32).

The difference $w = u - u_0$ of the function satisfying (29) and some particular solution u_0 of equation (32), as is easily verified, fulfils the following condition:

(39) $$\int_\Omega \sum_{|\alpha|=m} (\mathscr{D}^\alpha w\,, \mathscr{D}^\alpha \xi)\,\mathrm{d}x + \int_S \sum \frac{\partial^k \xi}{\partial n^k} B_1(w + u_0)\,\mathrm{d}\Gamma = 0\,.$$

Hence one obtains in a usual manner the equation for function w, namely equation $\Delta^m w = 0$ and as a consequence equation (32).

It remains to show how to choose the value of the Lagrange multiplier λ.

The solutions of equations (33), (34), (35) and (37) evidently contain a multiplier λ. There is

(40) $$u_\lambda = \lambda u_1$$

where u_1 is the solution corresponding to $\lambda = 1$. Consequently, one has

(41) $$\|u_\lambda\|^2 = \lambda^2 \|u_1\|^2\,,$$

(42) $$(l, u_\lambda) = \lambda(l, u_1)\,,$$

(43) $$(j, u_\lambda) = \lambda(j, u_1)\,,$$

$\|u_1\|$ and (l, u_1) satisfying another relation which is a consequence of equations (29)–(32).

Substituting function u_λ instead of ξ, which is evidently possible, one obtains

$$\|u_\lambda\|^2 = \lambda(l, u_\lambda),$$

i.e.

(44)
$$\|u_1\|^2 = (l, u_1) = d_1 .$$

To solve the fundamental problem of $\max_{\|f\|=1}(l, f)$ one has to put $\lambda = 1/\sqrt{d_1}$, then

(45)
$$(l, u_\lambda) = \frac{d_1}{\sqrt{d_1}} = \sqrt{d_1}\,, \quad \|u_\lambda\| = 1\,;$$

the variational problem in question is solved with the aid of another norm by $\lambda = 1/d_1$, then

(46)
$$\|u_\lambda\| = \frac{1}{d_1^2}\,\|u_1\| = \frac{1}{d_1}$$

and

$$(l, u_\lambda) = 1\,.$$

Integration of equations (32) and (35) represents a rather involved boundary value problem. However, equations (33), (34), (36) and (37) can be integrated in a closed form. Consider equations (33) and (36). Let functions

$$G(z) = (-1)^{(n-1)/2}\,\frac{z^{2m-n}}{2^{2m}\pi^{(n/2)+1}(m - n/2 + 1)\,\Gamma(m)}$$

for n odd and

(47)
$$G(z) = (-1)^{n/2}\,\frac{z^{2m-n}\lg z}{2^{2m-1}\pi^{n/2}\,\Gamma(m - n/2 + 1)\,\Gamma(m)}$$

for n even, be called the elementary solution of the problem.

The solution of equation (33) can be written as

(48)
$$u = \int_\Omega G(x - y)\,\mathrm{d}y - \sum c_k\,G(x - x^{(k)})$$

and the solution of problem (36) as

(49)
$$u = G(x - t) - \sum c_k\,G(x - x^{(k)})\,.$$

These solutions will be unique modulo a polynomial of degree $m - 1$ in the class of functions with growth of finite order and will realize an extreme of the corresponding integral (29) or (30). It will be proved that the integrals presented will be elements of $L_2^{(m)}$.

To do this observe first of all that the derivatives of order m of the elementary solution will be square integrable in any finite domain.

As a matter of fact, the order of growth of these derivatives is r^{m-n} or $r^{m-n} \lg r$ near to its unique singular point, which secures integrability of an arbitrary power for $m > n$, and of a power less than $q = n/(n - m)$ for $m < n$. However, by condition $m > n/2$ one has $q > 2$, q.e.d.

Now consider the neighbourhood of infinity. The function $\mathscr{D}^\alpha G(x - y)$ will be expanded in a power series of y in a neighbourhood of the origin. The radius of convergence of this expansion is determined by $|y| < \varrho$, $\varrho = |x|$.

If x is sufficiently large one can write

$$\varrho > 2\mathrm{d}\Omega$$

where $\mathrm{d}\Omega$ is the diameter of domain Ω.

Computing the expansion of derivative $\mathscr{D}^\alpha G(x - y)$ one obtains:

$$\mathscr{D}^\alpha G(x - y) = \mathscr{D}^\alpha G(x) + \sum_{|\beta| \leqq m-1} (-1)^{|\beta|} \frac{\mathscr{D}^{\beta+\alpha} G(x)}{\beta!} y^\beta + R^\alpha_m(x, y) .$$

Here by $\beta!$ one denotes $\beta_1! \, \beta_2! \dots \beta_n!$. For the remaining member the following estimate valid in the circle $|y| < d$ is true:

$$|R^\alpha_m(x)| \leqq A|x|^{-n} \lg |x| .$$

Denote by Q^α the following polynomial of degree $m - 1$:

$$Q^\alpha(y) = \mathscr{D}^\alpha G(x) + \sum_{|\beta| \leqq m-1} (-1)^{|\beta|} \frac{\mathscr{D}^{\beta+\alpha} G(x - y)}{\beta!} y^\beta .$$

By hypothesis, for this polynomial the following cubature formula is true:

$$\int_\Omega Q^\alpha(y) \, \mathrm{d}y - \sum_{k=1}^N c_k \, Q^\alpha(x^{(k)}) = 0 .$$

Hence one obtains

$$\int_\Omega \mathscr{D}^\alpha G(x - y) \, \mathrm{d}y - \sum_{k=1}^N c_k \, \mathscr{D}^\alpha G(x - x^{(k)}) = $$
$$= \int_\Omega R^\alpha(x, y) \, \mathrm{d}y - \sum_{k=1}^N c_k \, R^\alpha(x, x^{(k)})$$

using the fact that all points of Ω and $x^{(k)}$ lie in the interior of $|y| < d$; using (45) one obtains

$$\left| \int_\Omega \mathscr{D}^\alpha G(x - y) \, \mathrm{d}y - \sum_{k=1}^N c_k \, \mathscr{D}^\alpha G(x - x^{(k)}) \right| < A . \tag{50}$$

From formula (50) the convergence of the integral

$$\int_{-\infty}^{\infty} \Big[\sum_{|\alpha|=m} |\mathscr{D}^\alpha u|^2 \Big]^{p/2} \, \mathrm{d}x$$

for an arbitrary positive p follows immediately. Thus one proves that u belongs to $L_2^{(m)}$. In the same manner one can prove an analogous result for the function u expressed by formula (49).

Consider the solution of problems $I^{iii)}$ and $II^{iii)}$ which also can be obtained in a finite form by means of Fourier series.

The solution of equations (35) and (39) is obtained in an elementary manner by means of the periodic solution of equation

$$\Delta^m \tilde{G} = 1 - M(H, x)\,. \tag{51}$$

For the solutions of (35) and (37) respectively the following formulas are evident:

$$u = (-1)^{m+1}\,\lambda \sum c_k\, G_0(x - x^{(k)})\,,$$
$$u = (-1)^{m+1}\,\lambda[G_0(x - z) - \sum c_k(z)\, G_0(x - x^{(k)})]\,, \quad \text{respectively}\,.$$

For the solution (51) the representation of function $M(H, x)$ will be found.

Introducing new variables by the formula

$$y = H^{-1}x$$

one sees that $x + H\beta$ is transformed in $y + \beta$.

Consequently, in coordinates y the function $M(H, x)$ will have periods equal to one with respect to each independent variable. The fundamental domain is thus transformed into the unit cube, from which it follows e.g. that the determinant of the matrix H is equal to one. The function

$$\Lambda(y) = M(H, Hy)$$

will also be periodic, all periods being integers which will be given by the formula

$$\Lambda(y_1) = \delta(y_1)\,\delta(y_2)\ldots\delta(y_n)$$

for

$$-\tfrac{1}{2} < y_k < \tfrac{1}{2}\,.$$

Now consider the function

$$\lambda(y_1)$$

which is a periodical distribution of one variable y_1 with period equal to one and which coincides with $\delta(y_1)$ for $|y_1| < \frac{1}{2}$.

This function can be expanded in a Fourier series of the form

$$\lambda(y_1) = \sum_{n=-\infty}^{\infty} e^{in\pi y_1} = 1 + \sum_{|n|\geqq 1} e^{in\pi y_1}\,.$$

Thus

$$\Lambda(y) = 1 + \sum_{|\gamma| \neq 0} e^{i\gamma\pi y}$$

and finally

$$M(H, x) = \Lambda(H^{-1}x) = 1 + \sum_{|\gamma| \neq 0} e^{i\pi(\gamma, H^{-1}x)} = 1 + \sum_{|\gamma| \neq 0} e^{i\pi(\gamma H^{-1}, x)} .$$

Substituting this expression into (51) one obtains as is easily seen:

$$\tilde{G}(x) = \sum_{|\gamma| \neq 0} \frac{e^{i\pi(\gamma H^{-1}, x)}}{(\gamma H^{-1})^{2m}} .$$

Now the case when in the region Ω_0 there is only one point at which the value f_k is given, is of interest. Supposing that this point lies at the origin of coordinates and taking into account that the integral of $\tilde{G}(x)$ is equal to zero one obtains the magnitude of the error

$$(l, G) = G(0) = \sum_{|\gamma| \neq 0} \frac{1}{(\gamma H^{-1})^{2m}} = d_1(H) .$$

The investigation of magnitude of this error, which is considered as a function of the matrix of periods, is concerned with very profound properties of the geometry of numbers.

Accordingly, to make this error as small as possible the matrix of the net must be chosen so that the net should represent a set of centers of the least dense covering of the space by spheres of a given radius. This question shall not be discussed here in more detail.

The purpose of this paper is only to indicate the deep lying connections existing between the theory of partial differential equations and numerical analysis. It is only a hint at several simple problems arising in this field which seems to us to be of a considerable interest and which can lead to essential practical applications.

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