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A GAUGE THEORY FOR THE KADOMTSEV-PETVIASHVILI SYSTEM *

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Abstract

A Lagrangian formalism of scalar fields is considered and a new concept of "connection" is introduced. By this a gauge-theoretic understanding of the Sato theory on the K.-P. system is obtained. Our gauge group $\tilde{G}_-$ is the group consisting of pseudo-differential operators of non-positive orders with certain growth conditions. Then it can be concluded that the space R^* of elements of $\tilde{G}_-$ giving solutions of the K.-P. system defines the flat R^*-connection which we call the K.-P. connection. This connection can be regarded as a special gauge field.

Introduction

It is well known that various soliton equations can be obtained by using the theory of isospectral deformations of linear differential operators. A remarkable unification of soliton equations has been established by M. and Y. Sato [5] in terms of isospectral deformations of $D = d/dx$ in the category of pseudo-differential operators. This unified system of equations is called the Kadomtsev -Petviashvili system (= K.-P. system). They discovered the suppris-ing fact: The space of solutions of the K.-P. system makes the Grassmann manifold of infinite dimension and moreover, any solution of

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the K.-P. system can be reduced to that of a system of certain linear equations. Several attempts of undetstandings on the Sato theory and its generalizations have been presented. Some of them are the method of Riemann-Hilbert transforms [10], the method of group-decompositions [4], [7] and the field-theoretic method [1]. Our attempt which we present here is a new one, which we call a gauge-theoretic understanding.

In this paper, we see that the K.-P. system can be understood in the view point of Uchiyama's gauge theory [9]. We note that our gauge group is an infinite-dimensional Lie group. Hence our gauge theory for soliton equations is contrasted with that of Yang-Mills equations and nonlinear Heisenberg equation in dimensions of their gauge groups [3]. First, we consider the Lagrangian action:

$$\mathcal{L} = \int_{\mathbb{R}} \bar{\psi}\, D\psi\, dx \qquad (D = d/dx)$$

for scalar fields ψ, $\bar{\psi}$, i.e., wave functions on the real line $\mathbb{R}$. We analyse the symmetry of $\mathcal{L}$ and obtain as the gauge group of the first kind a group consisting of invertible pseudo-differential operators with constant coefficients of the form:

$$\cdots + c_n D^n + \cdots + c_1 D + c_0 + c_{-1} D^{-1} + \cdots + c_{-n} D^{-n} + \cdots .$$

Secondly, we apply the Uchiyama's gauge theory to our Lagrangian formalism. In this case, the gauge group of the second kind becomes a group consisting of invertible pseudo-differential operators with function coefficients of the form:

$$\cdots + u_n(x) D^n + \cdots + u_1(x) D + u_0(x) + u_{-1}(x) D^{-1} + \cdots + u_{-n}(x) D^{-n} + \cdots .$$

Then in order to obtain a new Lagrangian action which is invariant under this group, a connection, i.e., gauge field, necessarily arises in our consideration. It has a worth mentioning that pseudo-differential operators with negative orders, extended from usual differential operators, may be introduced as elements of the gauge group of the first or the second kind.

In Section 1, from a gauge group of pseudo-differential operators we introduce a new concept of "connection" on a fibre space over $\mathbb{R}$. Here we have to pay attention to the fact that our connection has been defined not only for a subgroup but also for a special subset R of the gauge group, although R does not admit a structure of subgroup. We prove that the decomposition law of pseudo-differential operators into the parts of non-positive and negative orders gives rise to the flat connection (Theorem 1). This is our first step to a gauge-theoretic understanding on the K.-P. system. In Section 2, we shall treat the Lagrangian action of scalar fields ψ, $\bar{\psi}$ with infinitely many parameters $t = (t_1, t_2, \ldots)$:

$$\mathcal{L}_t = \int_{\mathbb{R}} \bar{\psi}\, d\psi\, dx, \qquad d = \Sigma_{n=1}^{\infty} (\partial/\partial t_n) d\, t_n.$$

For this Lagrangian action we consider the gauge groups $\tilde{G}_o$, $\tilde{G}$ of the first and the second kind, and then $\tilde{G}$-connections. Then we can conclude that the space R^* of elements of $\tilde{G}$ giving solutions of the K.-P. system defines the flat R^*-connection which we call the K.-P. connection (Theorem 2).

Our discussions show that the space of solutions of soliton equations determines a special gauge field. Hence; we may expect to extend our discussions to the Yang-Mills equation and nonlinear Heisenberg equation by a gauge-theoretic version of the Sato theory on the Minkowski space-time [3].

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1. A Lagrangian formalism of scalar fields

We consider complex-valued functions defined on the real line $\mathbb{R}$ and a collection of operators including the differential operator $D = d/dx$. Let ψ and $\bar{\psi}$ denote two functions. Here $\bar{\psi}$ may <u>not</u> be the complex conjugate of ψ. Let S be an operator which maps a function ψ to the function $S\psi$. An operator $\bar{\psi} S$ formed with $\bar{\psi}$ and S is defined by $(\bar{\psi} S)\psi = \bar{\psi}(S\psi)$ for any ψ.

First, we deal with a Lagrangian action for ψ and $\bar{\psi}$ given by

$$(1.1) \qquad \int_{\mathbb{R}} \bar{\psi}\, D\psi\, dx.$$

We restrict ourselves to the case where there exist invertible operators W satisfying the following action law: For a function ψ and an operator $\bar{\psi}(= \bar{\psi} \cdot 1)$, identified with the function $\bar{\psi}$, an operator W acts on the pair as

(1.2) $\psi \to \psi' = W\psi, \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi} W^{-1}.$

Under this action the function $\bar{\psi}\psi$ is invariant. We are interested in a set of operators W which makes a group $\mathbf{G}_o$ and preserves $\bar{\psi} D \psi$ invariant, equivalently satisfies $WD = DW$. Choices of such groups are not unique. One of possible groups can be obtained by

(1.3) $G_o = \{W \mid W = \Sigma_{n=-\infty}^{n=+\infty} c_n D^n$ with constant coefficients$\}$.

Then the group G_o is a subgroup of $\mathbf{G}_o$. For an invertible operator W we put

(1.4) $\psi_W = W\psi, \qquad \bar{\psi}^W = \bar{\psi} W^{-1}.$

PROPOSITION (1.5). *The Lagrangian action*

(1.6) $\mathcal{L}_o = \int_{\mathbb{R}} \bar{\psi}^W D \psi_W \, dx, \qquad W \in G_o$

is invariant under the action of the group G_o.

Proof. We choose arbitrary elements W and W' of G_o and set ϕ by $W = \phi W'$, namely, $\phi = WW'^{-1}$. Since

(1.7) $\psi_W = \phi\, \psi_{W'}, \qquad \bar{\psi}^W = \bar{\psi}^{W'} \phi^{-1},$

we obtain $\bar{\psi}^W D \psi_W = \bar{\psi}^{W'} \phi^{-1} D \phi \psi_{W'} = \bar{\psi}^{W'} D \psi_{W'}$.

The group G_o is called *the gauge group of the first kind.* Next we proceed to a group

(1.8) $G = \{W \mid W = \Sigma_{n=-\infty}^{n=+\infty} u_n(x) D^n$ with function coefficients$\}$.

We call an element of G a formal pseudo-differential operator [5]. G is called *the gauge group of the second kind.* In order to obtain exact mathematical meanings, we have to restrict our considerations to special groups. For example, we may choose a group G consisting of elements W with the following condition: Every $u_n(x)$ is analytic function and there exists an integer n_o such that $\operatorname{ord} u_n(x) \geq n - n_o$ for any sufficiently large n ([4], [7], [8]). We have to pay attention to the fact that the Lagrangian action $\mathcal{L}_o$ is not invariant under G, because the commutator $[D, W] = DW - WD$ does not vanish identically. Here we note that the following equalities hold:

(1.9) $[D, W] = \Sigma(D u_n(x)) D^n$ for $W = \Sigma u_n(x) D^n$

and

(1.10) $W D W^{-1} = -[D, W] W^{-1} + D$ for $W \in G$.

The Uchiyama gauge theory [9] says that in order to get a new Lagrangian action which is invariant under the group of the second kind, a connection, i.e., a gauge field, has to be introduced. We call the disjoint union $\bigsqcup_{x \in \mathbb{R}} \{(\cdot)^W(x) \mid W \in G\}$ the *fibre space generated by* G *over* $\mathbb{R}$, or the *fibre* G-*space* simply. Then we can make the following definition:

<u>Definition (1.11)</u>. Let G be a group of the operators described in (1.8) and let R be a subset of G. A collection $\{\Omega(W) \mid W \in R\}$ of operators is called an R-*connection* (with a range R on the fibre G-space), if (1) there exists a pair (G_1, ρ) constituted with an injective set-map $\rho: G_1 \to G$ of a group G_1 to G such that $R = \rho(G_1)$ and (2) $L_\Omega(W) \equiv D - \Omega(W)$ satisfies

(1.12) $L_\Omega(W) = \phi L_\Omega(W') \phi^{-1}$ for $W, W' \in R$, where $W = \phi W'$.

In particular, we call it a G-*connection* if in addition ρ is a group-isomorphism.

The following are examples of G-connections:

<u>EXAMPLES (1)</u> $\Omega(W) = D$. (2) $\Omega(W) = [D, W] W^{-1}$, in this case

(1.13) $L(W) \equiv L_\Omega(W) = W D W^{-1}$.

(3) Let G' be a subgroup of G and $\iota: G' \to G$ be the natural inclusion mapping. If $\Omega(W)$ $(W \in G)$ is a G-connection, then $\Omega(W)$ $(W \in G')$ becomes a G'-connection.

Immediatly from (1.12) we see that if $\Omega_1(W)$ and $\Omega_2(W)$ are R-connections, then the relation

(1.14) $\Omega_1(W) - \Omega_2(W) = \phi(\Omega_1(W') - \Omega_2(W')) \phi^{-1}$

holds for $W, W' \in R$, where $W = \phi W'$. This fact and Example (2) show that the following $\hat{\Omega}(W)$ given by

(1.15) $\hat{\Omega}(W) = W^{-1}([D, W] W^{-1} - \Omega(W)) W$ for $W \in R$

satisfy the condition $\hat{\Omega}(W) = \hat{\Omega}(W')$ for any pair of W and W' of R, namely $\hat{\Omega}(W)$ does not depend on a choice of $W \in R$. Therefore, we may write as $\hat{\Omega} = \hat{\Omega}(W)$. We call $\hat{\Omega}$ the *connection form* determined by $\Omega(W)$. An R-connection is called to be *flat* if its connection form vanishes identically, namely $\Omega(W) = [D, W] W^{-1}$.

By an application of Uchiyama theory to the Lagrangian action (1.6), we obtain

PROPOSITION (1.16). *Let* $\Omega(W)$ *be a* G-*connection. The Lagrangian action*

$$(1.17) \qquad \mathcal{L} = \int_{\mathbb{R}} \bar{\psi}^W (D - \Omega(W))\, \psi_W \, dx, \qquad W \in G$$

is invariant under the group G.

Proof. For arbitrary elements W and W', where $W = \phi W'$ in G we have

$$\bar{\psi}^W L(W)\, \psi_W = \bar{\psi}^{W'} \phi^{-1} (\phi\, L(W')\phi^{-1})\phi\, \psi_{W'} = \bar{\psi}^{W'} L(W')\, \psi_{W'} ,$$

which implies the invariance of $\mathcal{L}$ under G.

The following group is important for a study on the K.-P. system. We put

$$(1.18) \qquad G_- = \{ \Sigma_{n=0}^{\infty} v_n(x)\, D^{-n} \in G \,|\, v_o(x) = 1 \},$$

further, the space of operators $\mathcal{G} = \{ \Sigma_{n=-\infty}^{n=+\infty} u_n(x)\, D^n \}$ and its complementary subspaces $\mathcal{G}_+ = \{ \Sigma_{n=0}^{n=+\infty} u_n(x)\, D^n \}$ and $\mathcal{G}_- = \{ \Sigma_{n=1}^{n=+\infty} u_{-n}(x)\, D^{-n} \}$. Then any element S of $\mathcal{G}$ has the decomposition: $S = (S)_+ + (S)_-$ for $(S)_+ \in \mathcal{G}_+$ and $(S)_- \in \mathcal{G}_-$. Then we can prove

THEOREM 1. $\omega(W)$ $(W \in G_-)$ *is the flat* G_-*-connection if and only if*

$$(1.19) \qquad \omega(W) = -(L(W))_- \quad \text{for} \quad W \in G_-.$$

Proof. For $W, W' \in G_-$, where $W = \phi W'$, it holds that

$$\begin{aligned}
(L(W))_- &= (\phi(L(W')\, \phi^{-1})_- = (\phi(L(W')_+ \phi^{-1})_- + (\phi(L(W')_- \phi^{-1})_- \\
&= (\phi\, D\, \phi^{-1})_- + \phi(L(W'))_-\, \phi^{-1} \\
&= (-[D, \phi]\, \phi^{-1} + D)_- + \phi(L(W'))_-\, \phi^{-1} \qquad \text{(by (1.10))} \\
&= -D + \phi\, D\, \phi^{-1} + \phi(L(W'))_-\, \phi^{-1},
\end{aligned}$$

which implies $D - \omega(W) = \phi(D - \omega(W'))\, \phi^{-1}$. Hence $\omega(W)$ is a G_--connection. Comparing the non-positive orders of the both sides of (1.10), we obtain $\omega(W) = -(L(W))_- = [D, W] W^{-1}$, i.e., $\omega(W)$ is flat. Conversely, if $\omega(W)$ $(W \in G_-)$ is the flat G_--connection, then $\omega(W)$

reduces to $\omega(W) = [D, W] W^{-1} = -(L(W))_-$ by (1.10).

2. A gauge theory for the K.-P. system

We consider a Lagrangian formalism for scalar fields, $\psi = \psi(x,t)$ and $\bar{\psi} = \bar{\psi}(x, t)$ defined on the real line $(x \in) \mathbb{R}$ with infinitely many parameters

$$t = (t_1, t_2, \dots),$$

and for some collections of operators including $D = d/dx$ and $D_n = \partial/\partial t_n$. The total differential operator with respect to the parameters is denoted by

(2.1) $$d = \Sigma_{n=1}^{\infty} D_n \, d\, t_n.$$

The Lagrangian action which we treat here is given by

(2.2) $$\mathcal{L}(t) = \int_{\mathbb{R}} \bar{\psi}(x, t)\, d\, \psi(x, t)\, dx$$

for functions ψ and $\bar{\psi}$. We proceed to our discussions analogous to the one done in the previous section. We are interested in invertible operators $W = W(x, t)$, consisting together with the action law for ψ and $\bar{\psi}$:

(2.3) $$\psi \to \psi' = W\, \psi (= \psi_W), \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\, W^{-1} (= \bar{\psi}^W).$$

Hence, the function $\bar{\psi}\, \psi$ is invariant under this action.

First, we consider a group

(2.4) $$\tilde{G}_o = \{W \mid W = \Sigma_{n=-\infty}^{n=+\infty} c_n(x)\, D^n\}.$$

In this case, we observe that coefficients $c_n(x)$ are constant with respect to t. Immediately, from $W D = d W$ for $W \in \tilde{G}_o$ we have

PROPOSITION (2.5). *The Lagrangian*

(2.6) $$\mathcal{L}_o = \int_{\mathbb{R}} \bar{\psi}^W d\, \psi_W\, dx, \qquad W \in \tilde{G}_o,$$

possesses the symmetry of the group $\tilde{G}_o$.

Following the Uchiyama theory, next we deal with a group

(2.7) $$\tilde{G} = \{W \mid W = \Sigma_{n=-\infty}^{n=+\infty} u_n(x, t)\, D^n \text{ with the property } (*)\}$$

(*) $u_n(x, t)$ $(n = 0, \pm 1, \pm 2, \dots)$ are analytic functions of x satisfying the following growth condition: There exists an integer n_o such that $\operatorname{ord} u_n(x, t) \geq n - n_o$ for any sufficiently large n

(see [4], [7], [8]).

The Lagrangian action (2.6) gives rise to a gauge group $\tilde{G}_o$ of the first kind and a gauge group $\tilde{G}$ of the second kind respectively. $\mathcal{L}_o$ is not invariant under $\tilde{G}$, since commutators

$$[D_m, W] = \Sigma (D_m u_n(x, t)) D^n \qquad (m = 1, 2, \ldots)$$

for $W = \Sigma u_n(x, t) D^n$, do not vanish identically, i.e., $[d, W] \neq 0$. Hence we have to make

Definition (2.8). Let $\tilde{R}$ be a subset of the group $\tilde{G}$ described in (2.7). A set $\{\Omega(W) | W \in \tilde{R}\}$ of operators is called a *multiconnection* (or *total connection*) *with a range* $\tilde{R}$ (or, simply $\tilde{R}$-*connection*) on the fibre $\tilde{G}$-space, if $\Omega(W)$ has the form $\Omega(W) = \Sigma_n \Omega_n(W) d t_n$ whose $\Omega_n(W)$ is a connection with a range $\tilde{R}$ with respect to D_n:

$$D_n - \Omega_n(W) = \phi(D_n - \Omega_n(W')) \phi^{-1}$$

for W and $W' \in \tilde{R}$, where $W = \phi W' (\phi \in \tilde{G})$. $\Omega_n(W)$ is call the *partial connection* of $\Omega(W)$.

We note that a multiconnection $\Omega(W)$ with a range $\tilde{R}$ implies

$$d - \Omega(W) = \Sigma_n \phi(D_n - \Omega_n(W')) \phi^{-1} = \phi(d - \Omega(W')) \phi^{-1}$$

for $W, W' \in R$ with $W = \phi W'$.

By use of Uchiyama's Theorem, we obtain

PROPOSITION (2.9). *Let* $\Omega(W)$ *be a* $\tilde{G}$-*connection. The Lagrangian*

$$\mathcal{L} = \int_{\mathbb{R}} \bar{\psi}^W (d - \Omega(W)) \psi_W \, dx \qquad \text{for } W \in \tilde{G}$$

is invariant under the group $\tilde{G}$.

We set

(2.10) $\tilde{G}_+ = \{ \Sigma_{n=0}^{n=+\infty} u_n D^n \in \tilde{G} | u_o \neq 0 \}$, $\tilde{G}_- = \{ \Sigma_{n=0}^{n=+\infty} u_{-n} D^{-n} \in \tilde{G} | u_o \equiv 1 \}$.

Corresponding to $\tilde{G}, \tilde{G}_+$ and $\tilde{G}_-$, we consider the spaces of operators $\tilde{\mathfrak{g}} = \{ \Sigma_{n=-\infty}^{n=+\infty} u_n D^n \}$, its complementary subspaces

(2.11) $\tilde{\mathfrak{g}}_+ = \{ \Sigma_{n=0}^{n=+\infty} u_n(x, t) D^n \}$, $\tilde{\mathfrak{g}}_- = \{ \Sigma_{n=1}^{n=\infty} u_{-n}(x, t) D^{-n} \}$,

that is the direct sum $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_+ \oplus \tilde{\mathfrak{g}}_-$. Hence, any element $X \in \tilde{\mathfrak{g}}$ is written as $X = (X)_+ + (X)_-$ for $(X)_+ \in \tilde{\mathfrak{g}}_+$ and $(X)_- \in \tilde{\mathfrak{g}}_-$.

Here we recall the K.-P. system. The operator $L = W D W^{-1}$ for

$W \in G_-$ derived from the flat connection implies that $L^n = W D^n W^{-1}$ and its decomposition $L^n = (L^n)_+ + (L^n)_-$. In this case, $(L^n)_+$ is the n-th differential operator. The *K.-P. system* is a system of equations defined by

$$\partial L/\partial t_n = [(L^n)_+, L] \qquad (n = 1, 2, \ldots). \tag{2.12}$$

When W ($\in \tilde{G}_-$) is an element described in the solution $L = W D W^{-1}$ of the K.-P. system, we shall say that W gives a solution of the K.-P. system. It is known ([1], [5], [6]) that *an element* W *of* G_- *gives a solution of the K.-P. system if and only if* W *satisfies*

$$\partial W/\partial t_n + (L^n(W))_- W = 0 \qquad (n = 1, 2, \ldots). \tag{2.13}$$

The following theorem is our main result:

THEOREM 2. *Let* R^* *be the space of all elements of* $\tilde{G}_-$ *each of which gives a solution of the K.-P. system. Then the set* $\{\Omega_{K.P.}(W) \mid W \in R^*\}$ *defined by*

$$\Omega_{K.P.}(W) = \Sigma_n \Omega_n(W)\, d t_n, \qquad \Omega_n(W) = -(L^n(W))_- \tag{2.14}$$

becomes the flat R^**-connection (say, the K.-P. connection) on the fibre* $\tilde{G}_-$*-space over* $\mathbb{R}$.

Remark. (1) The K.-P. connection is a direct generalization of the connection given in Theorem 1, when we identify t_1 with x and set $t_n = 0$ $(n = 2, 3, \ldots)$. (2) The flatness of the K.-P. connection is well known as the Zakharov-Shabat equation.

For the proof of this theorem we need the following two lemmas:

LEMMA 1 (Mulase's decomposition theorem [4]). *The group* $\tilde{G}$ *described in* (2.7) *can be decomposed into*

$$\tilde{G} = \tilde{G}_- \cdot \tilde{G}_+ ,$$

in a sense that any element $g \in \tilde{G}$ *determines the unique pair of elements* $g_1 \in \tilde{G}_-$ *and* $g_2 \in \tilde{G}_+$ *such that* $g = g_1 \cdot g_2$.

LEMMA 2 ([4], [6]). *There exists a one to one correspondence between the space* R^* *and the space* Q *of solutions* U *of the initial value problem:*

$$\partial U/\partial t_n = [D^n, U], \qquad U|_{t=0} = U_0 \in G_-, \tag{2.15}$$

where G_- *is given in* (1.18). *The exact correspondence is described in the following manner: A solution* U *of* (2.15) *determines an* *element* W *of* G_- *by the decomposition* $U = W^{-1}V$ *in Lemma* 1. *Then* $L(W) = WDW^{-1}$ *gives a solution of* (2.12). *Conversely, for a solution* W *of* (2.12), *we can find a unique element* V *of* $\tilde{G}_+$ *such that* $V|_{t=0}$ = *identity and* $U = W^{-1}V$ *gives a solution of* (2.15).

The proof of Theorem 2. Let U_o be any element of G_-. U_o determines the unique solution U ($\in \tilde{G}_-$) of (2.15) by Lemma 2. U can be decomposed uniquely as $U = W^{-1}V$ with $W \in \tilde{G}_-$ and $V \in \tilde{G}_+$ by Lemma 1. This gives rise to a mapping $\rho : G_- \to \tilde{G}_-$ which maps U_o to W. This mapping ρ is injective ([4], [6]). Then we see that $R^* = \rho(G_-)$. Next we show that $\Omega_{K.P.}(W)$ becomes an R^*-connection. Let W and W′ be elements of R^* and set ϕ ($\phi \in \tilde{G}_-$) by $W = \phi W'$. It follows from

$$(\partial W/\partial t_n) = (\partial\phi/\partial t_n)\, W' + \phi(\partial W'/\partial t_n)$$

and from (2.13) that

$$-(L^n(W))_-\, W = -(\partial\phi/\partial t_n)\, W' - \phi(L^n(W'))_-\, W'.$$

Hence

$$\omega_n(W) = (\partial\phi/\partial t_n)\,\phi^{-1} + \phi\,\omega_n(W')\,\phi^{-1}$$

holds, which implies that $\omega_n(W)$ ($W \in R^*$) is a partial R^*-connection. Therefore, $\Omega_{K.P.}(W)$ ($W \in R^*$) is an R^*-connection. The flatness of the connection follows from (2.13):

$$0 = \Sigma_n(\partial W/\partial t_n + (L^n(W))_-\, W)\, dt_n = \Sigma_n(\partial W/\partial t_n - \omega_n(W))\, W)\, dt_n$$
$$= [d, W] - \Omega_{K.P.}(W)\, W.$$

References

[1] DATE, E., JIMBO, M., KASHIWARA, M., and MIWA, T.: *Solitons, τ-functions and Euclidean Lie algebras*, in Mathématique et Physique, Séminare de l'École Normale Supérieure 1979-1982. Ed. by Boutet de Monvel, A. Douady et J.-L. Verdier (Progress in Math. 37), Birkhäuser-Verlag, Boston-Basel-Stuttgart 1983, 261-279.

[2] KALINA, J., ŁAWRYNOWICZ, J., and SUZUKI, O.: *A differential geometric quantum field theory on a manifold II. The second quantization and deformations of geometric fields and Clifford*

groups, preprint, 1984.

[3] KRÓLIKOWSKI, W.: *On correspondence between equations of motion for Dirac particle in curved and twisted space-times*, preprint, 1985.

[4] MULASE, M.: *Complete integrability of the Kadomtsev-Petviashvili equation*, Advances in Math. 54, 57-66 (1984).

[5] SATO, M.: *Soliton equations and Grassmann manifolds*, lectures delivered at Nagoya Univ. (1982).

[6] SUZUKI, O., ŁAWRYNOWICZ, J., and KALINA, J.: *A geometric approach to the Kadomtsev-Petviashvili system* (*I*), preprint 1985.

[7] TAKASAKI, K.: *A new approach to the self-dual Yang-Mills equations*, Commun. Math. Phys. 94, 35-59 (1984).

[8] ———: *A new approach to the self-dual Yang-Mills equations* (*II*), Saitama Math. J., 3 (1985), 11-40.

[9] UCHIYAMA, R.: *Invariant theoretical interpretation of Interaction*, Phys. Rev., 101 (1956), 1597-1607.

[10] UENO, K. and NAKAMURA, Y.: *Transformation theory for anti-self--dual equations and the Riemann-Hilbert problem*, Phys. Lett. 109 B, 273-278 (1982).

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