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Some Properties of Multiplication in the Algebra $C(X)$ Related to the Dimension of X

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We present some connections between a topological dimension of a compact Hausdorff topological space X and algebraical and topological properties of the algeba $C(X)$ of real-valued continuous functıons on X. In particular a new characterization of dimension of X in terms of properties of $C(X)$ is given.

Let X be a compact topological space (i.e. a compact Hausdorff topological space). We consider the algebra $C(X)$ of real-valued continuous functions defined on X. It is endowed with the norm $|f\| = \sup_{x \in X} |f(x)|$ and the operations of pointwise addition and multiplication: $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = = f(x) \cdot g(x)$.

The multiplication is a continous function from $C(X) \times C(X)$ to $C(X)$ but, in general, it is not open. We recall the definition of openness and introduce (following [1]) the definition of weak openness of the function mapping one topological space to another one.

Definition 1. *Let X and Y be topological spaces and let $f: X \to Y$ be a function. We say that f is* open *if it preserves the openness of sets, i.e. $f[U] \subset Y$ is open for every open set $U \subset X$. We say that f is* weakly open *if it preserves non-emptiness of the interior of sets, i.e. for every set $A \subset X$ we have* $\text{Int}(A) \neq \emptyset \Rightarrow \text{Int}(f[A]) \neq \emptyset$.

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M. Balcerzak, W. Wilczyński and A. Wachowicz proved ([1], [6], [7]), that the multiplication in $C([0,1])$ is weakly open. This result can be extended as follows (cf. [5]):

Theorem 2. *Let X be a compact topological space. The following equivalences hold:*

(1) *the multiplication* $\cdot : C(X) \times C(X) \to C(X)$ *is open iff* $\dim X < 1$,

(2) *the multiplication* $\cdot : C(X) \times C(X) \to C(X)$ *is weakly open and is not open iff* $\dim X = 1$,

(3) *the multiplication* $\cdot : C(X) \times C(X) \to C(X)$ *is not weakly open iff* $\dim X > 1$,

where $\dim X$ *denotes topological (covering) dimension of* X.

The first of these equivalences was suggested by D. H. Fremlin in January, 2004, during the 32nd Winter School in Abstract Analysis (oral communication).

One of the consequences of Theorem 2 is the following corollary:

Corollary 3. *Let X be a compact topological space. The following conditions are equivalent:*

- $\dim X \leq 1$.
- *For any subset $A \subset C(X)$ of the first category its preimage under the multiplication $(\cdot)^{-1}[A]$ is a set of the first category in $C(X) \times C(X)$.*

Corollary 3 extends the results obtained by A. Wachowicz, who showed that a preimage of a residual subset of $C([0,1])$ under multiplication is a residual subset of $C([0,1]) \times C([0,1])$.

We do not intend to prove Theorem 2 or Corollary 3 here. The detailed proofs can be found in [5]. We just wish to sketch the proof of the equivalence *the multiplication* $\cdot : C(X) \times C(X) \to C(X)$ *is weakly open iff* $\dim X \leq 1$ and to use presented ideas to characterize a dimension of a compact topological space X in terms of properties of the algebra of real-valued continuous functions on X. Similar characterizations were given by M. Katětov in 1950s (e.g.[4]).

Let us fix a notation. For $A, B \subset C(X)$ the image $\cdot[A \times B] = \{f \cdot g : f \in A, g \in B\}$ will be denoted by $A \cdot B$. If $f \in C(X)$ and $r > 0$ then $B(f, r) = \{g \in C(X) : \|g - f\| < r\}$ is the open ball with centre f and radius r.

Proof of the implication "The multiplication $\cdot : C(X) \times C(X) \to C(X)$ is weakly open $\Rightarrow \dim X \leq 1$".

Assume that $\dim X \geq 2$. By Hemmingsen Lemma (cf. [2]) there exist closed sets $A_1, B_1, A_2, B_2 \subset X$ such that $A_1 \cap B_1 = \emptyset$, $A_2 \cap B_2 = \emptyset$ and if L_1 is a partition between A_1 and B_1 and L_2 is a partition between A_2 and B_2 then $L_1 \cap L_2 \neq \emptyset$. (We recall that a closed set $L \subset X$ is a partition between disjoint closed sets $A, B \subset X$ if there exist two disjoint open sets $U, V \subset X$ satisfying $A \subset U$, $B \subset V$ and $L = X \setminus (U \cup V)$).

We define functions $f, g \in C(X)$ such that $f(x) = 1$ for $x \in A_1$, $f(x) = -1$ for $x \in B_1$, $g(x) = 1$ for $x \in A_2$ and $g(x) = -1$ for $x \in B_2$. For example, if we want to define function f we first define $f|_{A_1 \cup B_1}$ and then we use Tietze Theorem to extend f to whole X. The set $B(f, 1) \times B(g, 1) \subset C(X) \times C(X)$ is non-empty and open. We will show that $B(f, 1) \cdot B(g, 1)$ is nowhere dense (hence its interior is empty).

Tending to the contradiction assume that $\hat{h} \in \mathrm{Int}\,\left(\overline{B(f,1) \cdot B(g,1)}\right)$. Then $\hat{h} \in \left(\overline{B(f,1) \cdot B(g,1)}\right)$ and $\hat{h} + \varepsilon \in \left(\overline{B(f,1) \cdot B(g,1)}\right)$ for some $\varepsilon > 0$. Let $\hat{f} \in B(f,1)$ and $\hat{g} \in B(g,1)$ be such that $\|\hat{h} - \hat{f} \cdot \hat{g}\| < \varepsilon/2$. If $x \in A_1$ then $\hat{f}(x) > 1 - 1 = 0$. If $x \in B_1$ then $\hat{f}(x) < -1 + 1 = 0$. It follows that $A_1 \subset \{x \in X : \hat{f}(x) > 0\}$, $B_1 \subset \{x \in X : \hat{f}(x) < 0\}$ and the set $L_1 = \{x \in X : \hat{f}(x) = 0\}$ is a partition between A_1 and B_1.Similarly, let $\tilde{f} \in B(f,1)$ and $\tilde{g} \in B(g,1)$ satisfy $\|(\hat{h} + \varepsilon) - \tilde{f} \cdot \tilde{g}\| < \varepsilon/2$. We have $A_2 \subset \{x \in X : \tilde{g}(x) > 0\}$ and $B_2 \subset \{x \in X : \tilde{g}(x) < 0\}$. The set $L_2 = \{x \in X : \tilde{g}(x) = 0\}$ is a partition between A_2 and B_2. It follows that there is $x_0 \in L_1 \cap L_2$. By the definition of L_1 one has $\hat{h}(x_0) + \varepsilon > \hat{f}(x_0) \cdot \hat{g}(x_0) - \varepsilon/2 + \varepsilon = \varepsilon/2$. On the other hand, by the definition of L_2 we have $\hat{h}(x_0) + \varepsilon < \tilde{f}(x_0) \cdot \tilde{g}(x_0) + \varepsilon/2 = \varepsilon/2$. The contradiction we obtained shows that $B(f,1) \cdot B(g,1)$ is nowhere dense. □

The key point of this proof is that the functions f and g satisfy $f^{-1}(0) \cap g^{-1}(0) \neq \emptyset$. Moreover, if we consider any function $\hat{f}$, which is close to f, and any function $\tilde{g}$, which is close to g, then we still have $\hat{f}^{-1}(0) \cap \tilde{g}^{-1}(0) \neq \emptyset$.

The implication *"The multiplication* $\cdot : C(X) \times C(X) \to C(X)$ *is weakly open* $\Leftarrow \dim X \leq 1$" is an easy consequence of the following two lemmas:

Lemma 4. *Let X be a compact topological space,* $\dim X \leq 1$ *and let U, $V \subset C(X)$ be non-empty and open. There exist $f \in U$ and $g \in V$ such that $f^{-1}(0) \cap g^{-1}(0) = \emptyset$.*

Lemma 5. *Assume that X is a compact topological space, U, $V \subset C(X)$ are non-empty and open and $f \in U$, $g \in V$ satisfy $f^{-1}(0) \cap g^{-1}(0) = \emptyset$. Then there exists $\delta > 0$ such that $B(f \cdot g, \delta) \subset U \cdot V$.*

We will not prove these lemmas here (see [5]). Instead, we will prove the following Lemma 6, which is more general than Lemma 4.

Lemma 6. *Let X be a compact topological space and let $n > 0$ The following conditions are equivalent*

- $\dim X < n$,
- *For every $g_1, g_2, \ldots, g_n \in C(X)$ and $\varepsilon > 0$ there exist $f_1, f_2, \ldots, f_n \in C(X)$ such that $| f_i - g_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$ and $\bigcap_{i=1}^{n} f_i^{-1}(0) = \emptyset$.*

Proof. Let X be a topological space, let (Y, d) be a metric space and let $\varphi : X \to Y$ be continuous. We say, following Hurewicz and Wallman ([3], Ch. VI), that a point $y \in Y$ is an unstable value of φ if for every $\varepsilon > 0$ there exists a continuous mapping $\psi : X \to Y$ satisfying $\forall_{x \in X}\, d(\varphi(x), \psi(x)) < \varepsilon$ and $y \notin \psi[X]$. Theorem VI 1 of [3] states that if $Y = \mathbb{R}^n$ and dim $X < n$ for some $n \in \mathbb{N}$ then every point of Y is an unstable value of φ. We will use this theorem in our context.

Let X be a compact space with dim $X < n$ and let $\varepsilon > 0$ and $g_1, g_2, \ldots, g_n \in C[X]$ be given. The function $\varphi = (g_1, g_2, \ldots, g_n)$ maps X into R^n. Dimension of X is smaller than n, hence the point $(0, 0, \ldots, 0)$ is an unstable value of $(g_1, g_2, \ldots, g_n)$. It follows that there exists $\psi = (f_1, f_2, \ldots, f_n)$ such that $\| f_i - g_i| < \varepsilon$ and $(0, 0, \ldots, 0) \notin \psi[X]$, which is equivalent to $\bigcap_{i=1}^n f_i^{-1}(0) = \emptyset$.

Now, assume that dim $X \geq n$. By Hemmingsen Lemma there exist closed sets $A_1, B_1, A_2, B_2, \ldots, A_n, B_n \subset X$ such that $A_i \cap B_i = \emptyset$ for $i = 1, 2, \ldots, n$ and if the sets L_i are partitions between the sets A_i and B_i then $\bigcap_{i=1}^n L_i \neq \emptyset$. We define functions $g_1, g_2, \ldots, g_n \in C(X)$ such that $g_i(x) = \varepsilon$ for $x \in A_i$ and $g_i(x) = -\varepsilon$ for $x \in B_i$. Let $f_1, f_2, \ldots, f_n$ be arbitrary elements of $C(X)$ satisfying $\| f_i - g_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$. Since the sets $L_i := f_i^{-1}(0)$ are the partitions between A_i and B_i, we have $\bigcap_{i=1}^n f_i^{-1}(0) \neq \emptyset$. □

Lemma 6 allows us to write several conditions which are equivalent to dim $X < n$ for a fixed $n > 0$:

(1) For every $g_1, g_2, \ldots, g_n \in C(X)$ and $\varepsilon > 0$ there exist $f_1, f_2, \ldots, f_n \in C(X)$ such that $\| f_i - g_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$ and $\bigcap_{i=1}^n f_i^{-1}(0) = \emptyset$.

(2) For every $g_1, g_2, \ldots, g_n \in C(X)$ and $\varepsilon > 0$ there exist $f_1, f_2, \ldots, f_n \in C(X)$ such that $\| f_i - g_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n f_i^2$ is an invertible element of the algebra $C(X)$,

(3) The set $\{(f_1, f_2, \ldots, f_n) : \sum_{i=1}^n f_i^2$ is an invertible element of the algebra $C(X)\}$ is dense in $(C(X))^n$,

(4) The set $\{(f_1, f_2, \ldots, f_n) : f_1 \cdot C(X) + f_2 \cdot C(X) + \ldots + f_n \cdot C(X) = C(X)\}$ is dense in $(C(X))^n$,

Conditions (2), (3) and (4) use only the algebraical and topological properties of the algabra $C(X)$. During the the 33rd Winter School in Abstract Analysis K. P. Hart and R. Schipperus observed that condition (2) can be reformulated, so that it uses only the algebraical properties of $C(X)$:

(5) For every $g_1, g_2, \ldots, g_n \in C(X)$ and $\varepsilon > 0$ there exist $f_1, f_2, \ldots, f_n \in C(X)$ such that
- if $\delta \in \mathbb{R}$ satisfies $0 < |\delta| < \varepsilon$ then for every $i = 1, 2, \ldots, n$ the element $f_i - g_i - \delta$ is invertible in the algebra $C(X)$.
- $\sum_{i-1}^n f_i^2$ is an invertible element of the algebra $C(X)$.

References

[1] BALCERZAK, M., WACHOWICZ, A., WILCZYŃSKI, W., *Multiplying balls in the space of continuous functions on* [0, 1], preprint.

[2] ENGELKING, R., *Theory of dimensıons fi ite and infinite*, Sigma Series in Pure Mathemarics, 10. Heldermann Verlag, Lemgo, 1995.
[3] HUREWICZ, W., WALLMAN, H., *Dimension Theory*, Princeton Mathematical Series, v. 4, Princeton University Press, Princeton, 1948.
[4] KATĚTOV, M., *On rings of continuous function and the dimension of compact spaces*, Časopis Pěst. Mat. Fys. **75** (1950), 1 – 16.
[5] KOMISARSKI, A., *Some connections between properties of the multiplication in algebra $C(X)$ and a dimension of X*, preprint.
[6] WACHOWICZ, A., *Baire category and standard operations on pairs of continuous functions*, Tatra Mt. Math. Publ. **24** (2002), 141 – 146.
[7] WACHOWICZ, A., *On some residual sets*, PhD dıssertation, Łódź Technical University, Łódź, 2004 (in Polish).