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Generalisations of ε-Density

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We give several partial solutions to Fremlin's question DU about the existence of large homogeneous sets for ε-dense open families of finite sets of ordinals by introducing and considering some generalisations of the notion of ε-density. In particular we prove that every $\frac{1}{2}$-functionally dense open family has an infinite homogeneous set and that under $MA + \neg CH$ every $\frac{1}{2}$-dense open family satisfying an additional covering property is $\frac{1}{2}$-functionally dense and has a homogeneous set of size $\aleph_1$. Moreover, we prove that assuming $MA + \neg CH$ satisfying this covering property on a set of size $\aleph_1$, and that under the same assumptions functional density on a set of size $\aleph_1$ is another necessary and sufficient condition for such a homogeneous set to exist.

We also study the continuous version of ε-density and give some negative homogeneity results.

0 Introduction

Several years ago D. H. Fremlin formulated a question motivated by measure theory but quite combinatorial in nature, that has to do with the existence of large homogeneous sets for certain families of finite sets. The question, denoted DU on his list of problems, see [4], is formulated as follows.

Definition 0.1. *Let A be any infinite set and let $\varepsilon \in (0, 1)$.*

(1) A family $\mathcal{D} \subseteq [A]^{<\aleph_0}$ is said to be ε-dense on *A if for every $B \in [A]^{<\aleph_0}$ there is $D \in \mathcal{D}$ such that $D \subseteq B$ and $|D| \geq \varepsilon|B|$.*

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(2) *A family* $\mathscr{D} \subseteq [A]^{<\aleph_0}$ *is said to be* open *if it is closed under taking subsets.*
(3) *If* $\mathscr{D} \subseteq [A]^{<\aleph_0}$ *we say that* $A^* \subseteq A$ *is* homogeneous for $\mathscr{D}$ *if* $[A^*]^{<\aleph_0} \subseteq \mathscr{D}$.

Problem 0.2 (DU). *(i) If* $\mathscr{D}$ *is an* ε*-dense open family on* ω_1*, is there necessarily an infinite subset of* ω_1 *which is homogeneous for* $\mathscr{D}$*?*
(ii) Assume $MA(\aleph_1) + \neg CH$ *and that* $\mathscr{D}$ *is* ε*-dense open on* ω_1*. Is there necessarily a homogeneous set for* $\mathscr{D}$ *of size* $\aleph_1$*?*

It is well known that there is a $\frac{1}{2}$-dense open family on ω without an infinite homogeneous set. An example is provided by the *Schreier family*, consisting of all finite subsets F of $\omega \setminus \{0\}$ that satisfy $\min(F) \geq |F|$. It is also well known that under CH there is a $\frac{1}{2}$-dense open family on ω_1 without a homogeneous set of size $\aleph_1$. It was shown by Fremlin in [4] that in fact the answer to Problem DU will remain the same if one replaces ε throughout by 1/2. We shall hence concentrate on $\frac{1}{2}$-dense open families.

Question DU is still open but some partial solutions and relevant considerations appear in [1] and [4]. In this note we offer a further partial solution and some generalisations. We deal with two generalisations of ε-density which both give the original notion as a special case. The first one is the notion of functional density, and it is described in §1. As shown there the existence of infinite homogeneous sets for this kind of dense families can be proved outright in ZFC, and moreover the notion of measure precalibres gives rise to larger homogeneous sets. Hence this gives a partial affirmative answer to DU(i) under the additional assumption of functional density. In §2 we use functional density to give a partial answer to DU(ii), in the affirmative under an additional assumption formulated not as a density condition but as a covering property. This property says that in some weak sense the family is close to being closed under finite unions (in the latter case of course the homogeneous set is easily seen to exist). Our approach allows us to state a characterisation under $MA(\aleph_1)$ of those $\frac{1}{2}$-dense open families on ω_1 that admit a homogeneous set of size $\aleph_1$, phrased in terms of the covering property used in the main proof, and another one phrased in terms of functional density.

Section 3 is of quite a different flavour. In it we notice that ε-density may be understood as a specialisation to the counting measure of a general measure-theoretic density notion. We show some examples of families that are dense in this more general sense and do not admit homogeneous sets.

1 Functionally dense families

Definition 1.1. *Let* A *be any infinite set. A family* $\mathscr{D} \subseteq [A]^{<\aleph_0}$ *is said to be* $\frac{1}{2}$-functionally dense *or* $\frac{1}{2}$-FD on A *if for every* $B \in [A]^{<\aleph_0}$ *and every function* $g : B \to \omega$ *there is* $D \in \mathscr{D}$ *such that* $D \subseteq B$ *and* $\sum_{\alpha \in D} g(\alpha) \geq \frac{1}{2} \sum_{\alpha \in B} g(\alpha)$.

Observation 1.2. *Any family that is $\frac{1}{2}$ FD on A is $\frac{1}{2}$-dense on A, as can be seen by considering the constant 1 function. The converse is not true, since the Schreier family described in the Introduction is known to be $\frac{1}{2}$-dense on $\omega\setminus\{0\}$, but the function g assigning 2 to 1 and 1 to 2, 3, 4 shows that the family is not $\frac{1}{2}$-functionally dense on $\omega\setminus\{0\}$.*

The following theorem giving a positive homogeneity result shows that the notion of functional density corresponds to the usual Kelley's intersection number ([2]) in the sense that epsilon density corresponds to the weak Kelly intersection number (see [4] and [5] for the latter notion and its connection with epsilon density). Although we shall assume familiarity with the well known notions of Kelley's intersection number and Kelley's theorem, the notion of a measure precalibre might be less familiar so we remind the reader of its definition. An infinite cardinal κ is called *measure precalibre* iff for every family of κ sets in the measure algebra of 2^κ whose measure is uniformly bounded away from 0, there is a centred subfamily of size κ. One can consult [3] for further equivalent definitions and also to note that every cardinal that is a precalibre of measure algebras is a measure precalibre.

Theorem 1.3. *Suppose that $\mathscr{D} \subseteq [\kappa]^{<\aleph_0}$ is $\frac{1}{2}$-FD on κ and open. Then there is an infinite set $X \subseteq \kappa$ which is $\mathscr{D}$-homogeneous. If κ is a measure precalibre then there is a $\mathscr{D}$-homogeneous set of size κ.*

Proof. By identifying sets in $\mathscr{D}$ with their characteristic functions and ${}^\kappa 2$ with 2^κ we may treat $\mathscr{D}$ as a subset of 2^κ. Let $A_\alpha := \{D \in \mathscr{D} : \alpha \in d\}$ for $\alpha < \kappa$. By a straighforward argument we check that the family $(A_\alpha)_{\alpha<\kappa}$ has Kelley's intersection number at least $\frac{1}{2}$. The proof of Kelley's theorem [2] gives a finitely additive measure μ on 2^κ with $\mu(A_\alpha) \geq \frac{1}{2}$ for every α. Since $\aleph_0$ is a measure precalibre there is an infinite set $X \subseteq \kappa$ for which $(A_\alpha)_{\alpha\in X}$ has the finite intersection property. This exactly means that $[X]^{<\aleph_0} \subseteq \mathscr{D}$.

The last statement of the theorem follows similarly. $\bigstar_{1.3}$

It turns out that for any κ and a $\frac{1}{2}$-dense family $\mathscr{D} \subseteq [\kappa]^{<\aleph_0}$ one may define a $\frac{1}{2}$-FD family $\mathscr{C}(\mathscr{D})$ on κ that is highly connected to the original family.

Lemma 1.4. *Suppose that $\mathscr{D} \subseteq [\kappa]^{<\aleph_0}$ is $\frac{1}{2}$-dense on κ and that $h : \kappa \to \kappa \times \omega$ is a bijection. Let D be the family of all images under h of the elements of $\mathscr{D}$. Then*

$$\mathscr{C}(\mathscr{D}) := \{c \in [\kappa]^{<\aleph_0} : (\exists d \in D)\ c = \pi(d)\}$$

is $\frac{1}{2}$-FD on κ, where $\pi : \kappa \times \omega \to \kappa$ is the standard projection on the first coordinate.

Proof. Note that the definition of $\frac{1}{2}$-FD on κ does not change if we consider only functions $g : B \to \omega\setminus\{0\}$ for all $B \in [\kappa]^{<\aleph_0}$, and also note that D is $\frac{1}{2}$-dense on $\kappa \times \omega$. So let $B \in [\kappa]^{<\aleph_0}$ and $g : B \to \omega\setminus\{0\}$ be given. Consider the set

$$A = \bigcup_{\alpha\in B} \{\alpha\} \times \{1, 2, \ldots, g(\alpha)\}.$$

Since D is $\frac{1}{2}$-dense on $\kappa \times \omega$, we may find $d \in D$ such that $d \subseteq A$ and $|d| \geq \frac{1}{2}|A|$. Then, putting $c = \pi(d)$, we get $c \subseteq B$, and

$$\sum_{\alpha \in c} g(\alpha) \geq |d| \geq \frac{1}{2}|A| = \frac{1}{2}\sum_{\alpha \in B} g(\alpha). \quad \bigstar_{1.4}$$

It is not clear if one can use large homogeneous sets for $\mathscr{C}(\mathscr{D})$ to construct homogeneous sets for $\mathscr{D}$, but at least when $\aleph_1$ is a precalibre of measure algebras (in particular when MA($\aleph_1$) holds) the discussion above leads to a reformulation of the DU problem that appears to be simpler than the original.

Theorem 1.5. *Assume that $\aleph_1$ is a precalibre of measure algebras and that every family D that is $\frac{1}{2}$-dense open in $\omega_1 \times \omega$, and satisfies $\{\pi(d) : d \in D\} = [\omega_1]^{<\aleph_0}$ admits a homogeneous set of size λ. Then every $\frac{1}{2}$-dense open family on ω_1 admits a homogeneous set of size λ.*

Proof. Let $\mathscr{D}$ be a $\frac{1}{2}$-dense family on ω_1 and let h be any bijection between ω_1 and $\omega_1 \times \omega$. We construct $\mathscr{C}(\mathscr{D})$ as in Lemma 1.4, so a $\frac{1}{2}$-FD family on ω_1. By Theorem 1.3 and the assumption that $\aleph_1$ is a precalibre of measure algebras we then have a set $Z \in [\omega_1]^{\aleph_1}$ such that Z is homogeneous for $\mathscr{C}(\mathscr{D})$. Using a bijection between Z and ω_1 we can assume that $Z = \omega_1$. Then D as in the proof of Lemma 1.4 satisfies the assumptions of this theorem and any set that is homogeneous for it gives a homogeneous set for $\mathscr{D}$ of the same cardinality as the original. $\bigstar_{1.5}$

In the next section swe show that under MA($\aleph_1$) every $\frac{1}{2}$-dense open family on ω_1 satisfying a certain covering property is actually $\frac{1}{2}$-FD, and hence admits an uncountable homogeneous subset.

2 A covering property

When trying to construct homogeneous sets for dense open families the major difficulty is that except in trivial cases the family is not closed under finite unions. This means that one cannot assume too much in the way of a covering property of such a family. One may wonder if some weak covering property would suffice and in this section we consider a nontrivial such property. The property considered here is such that any $\frac{1}{2}$-dense open family on ω_1 satisfying it can be by a ccc forcing made into an $\frac{1}{2}$-FD open family on ω_1. This is interesting if one studies the behaviour of $\frac{1}{2}$-dense open families on ω_1 under $MA(\aleph_1)$.

Definiton 2.1. *For a Δ-system $\bar{F} = \langle F_\alpha : \alpha < \omega_1 \rangle$ of finite sets of ordinals with root r we say that $\bar{F}$ is* clean *iff*
(1) $\max(r) < \min(F_\alpha \setminus r)$ *for all* α,
(2) $\max(F_\alpha) < \min(F_\beta \setminus r)$ *for* $\alpha < \beta < \omega_1$,
(3) $|F_\alpha|$ *is a constant.*

Definition 2.2. *Suppose that $\mathscr{D}$ is a $\frac{1}{2}$-dense open family of finite subsets of ω_1. We say that $\mathscr{D}$ satisfies the* cv(4, 2)-condition *iff for every two clean Δ-systems $\bar{F}^l$ $(l < 2)$ of elements of $\mathscr{D}$, there is $\alpha \in [1, \omega_1)$ and $H_0, H_1 \in \mathscr{D}$ such that*

$$\left(F_0^0 \cup F_\alpha^0\right) \cup \left(F_0^1 \cup F_\alpha^1\right) \subseteq H_0 \cup H_1 .$$

Theorem 2.3. *Assume that $MA(\aleph_1)$ holds. Then for every $\frac{1}{2}$-dense open family $\mathscr{D}$ of finite subsets of ω_1 that satisfies the* cv(4, 2)*-condition, there is $A^* \in [\omega_1]^{\aleph_1}$ such that $[A^*]^{<\aleph_0} \subseteq \mathscr{D}$.*

Proof. Let $\mathscr{D}$ be as in the assumptions. We define a forcing notion $\mathbb{P}$ by letting its universe be the set $\{F_0 \cup F_1 : F_0, F_1 \in \mathscr{D}\}$ and ordering it by $p \leq q$ (q is stronger) iff $p \subseteq q$.

Claim 2.4. *For each $\beta < \omega_1$ the set*

$$\mathscr{E}_\beta \stackrel{\text{def}}{=} \{p \in \mathbb{P} : \max(p) \geq \beta\}$$

is dense in $\mathbb{P}$.

Proof of the Claim. Let $p \in \mathbb{P}$ be given, and let $\beta < \omega_1$. We commence by finding $F_0, F_1 \in \mathscr{D}$ such that $p = F_0 \cup F_1$. Let $n_l = |F_l|$ for $l < 2$. For each $l < 2$ we construct by induction on $\alpha < \omega_1$ a sequence $\bar{F}^\alpha = \langle F_\alpha^l : \alpha < \omega_1 \rangle$ of elements of $\mathscr{D}$ such that

(1) F_0^l is given,

(2) If $\alpha < \alpha'$, then $\max\left(F_\alpha^l\right) < \min\left(F_{\alpha'}^l\right)$,

(3) If $\alpha > 0$ then $\max\left(F_\alpha^l\right) \geq \beta$,

(4) $|F_\alpha^l| = n_l$.

Such sequences can be easily defined by using the $\frac{1}{2}$-density and openness of $\mathscr{D}$. Using the property cv(4, 2), there is $\alpha \geq 1$ and $H_0, H_1 \in \mathscr{D}$ such that $p \cup F_\alpha^0 \cup F_\alpha^1 \subseteq H_0 \cup H_1$. Letting $q = H_0 \cup H_1$ we obtain a condition in $\mathscr{E}_\beta$ that extends p. $\bigstar_{2.4}$

Claim 2.5. *$\mathbb{P}$ is ccc.*

Proof of the Claim. Suppose we are given $\{p_\alpha : \alpha < \omega_1\}$ from $\mathbb{P}$ and let us choose for each α a pair $\left(F_0^\alpha, F_1^\alpha\right)$ of elements of $\mathscr{D}$ such that $p_\alpha = F_0^\alpha \cup F_1^\alpha$. By applying the Δ-system lemma twice, we way assume that $\bar{F}_l = \langle F_\alpha^l : \alpha < \omega_1 \rangle$ forms a clean Δ-system, for $l < 2$. By the cv(4, 2) condition there is $\alpha < \omega_1$ and $H_0, H_1 \in \mathscr{D}$ such that $\left(F_0^0 \cup F_\alpha^0\right) \cup \left(F_0^1 \cup F_\alpha^1\right) \subseteq H_0 \cup H_1$, so p_0 and p_α are compatible in $\mathbb{P}$. $\bigstar_{2.5}$

By $MA(\aleph_1)$ we can find a filter G in $\mathbb{P}$ that intersects all $\mathscr{E}_\beta$. Let $A \stackrel{\text{def}}{=} \bigcup G$.

Claim 2.6. *$A \in [\omega_1]^{\aleph_1}$ and it satisfies that for every $F \in [A]^{<\aleph_0}$, there are $F_0, F_1 \in \mathscr{D}$ such that $F = F_0 \cup F_1$.*

Proof of the Claim. If $p \in G \cap \mathscr{E}_\beta$, then $\max(p) \geq \beta$, so by the genericity of G we obtain that $\sup(A) = \omega_1$.

Suppose that $F \in [A]^{<\aleph_0}$, hence there are $q_0, \dots, q_n \in G$ with $F \subseteq \bigcup_{i \le n} q_i$. As G is a filter, conditions $q_0, \dots, q_n$ have a common upper bound in G, so without loss of generality $n = 0$. Let $q_0 = H_0 \cup H_1$ for some $H_0, H_1 \in \mathcal{D}$, and let $F_l = H_l \cap F$. $\bigstar_{2.6}$

Now notice that $\mathcal{D}$ is $\frac{1}{2}$-FD on A, since if $F \in [A]^{<\aleph_0}$ and $g : F \to \omega$, choosing $F_0, F_1 \in \mathcal{D}$ so that $F = F_0 \cup F_1$, we obtain

$$\sum_{\alpha \in F} g(\alpha) \le \sum_{\alpha \in F_0} g(\alpha) + \sum_{\alpha \in F_1} g(\alpha),$$

so for at least one $l \in 2$ we have $\sum_{\alpha \in F_l} g(\alpha) \ge \frac{1}{2} \sum_{\alpha \in F} g(\alpha)$. Hence by Theorem 1.3 and the fact that $MA(\aleph_1)$ implies that $\aleph_1$ is a precalibre of measure algebras, we obtain that there is $A^* \in [A]^{\aleph_1}$ which is $\mathcal{D}$-homogeneous. $\bigstar_{2.3}$

It might be interesting to note that the full force of the covering condition used in Theorem 2.3 was not needed for the chain condition, but it was for the density argument. We may now use Theorem 2.3 and its proof to obtain a characterisation under $MA(\aleph_1)$ of those $\frac{1}{2}$-dense open families on ω_1 that have a homogeneous set of size $\aleph_1$.

Corollaray 2.7. *Assume $MA(\aleph_1)$. Then a $\frac{1}{2}$-dense open family $\mathcal{D}$ on ω_1 has a homogeneous set of size $\aleph_1$ iff for some set $X \in [\omega_1]^{\aleph_1}$ the family $\mathcal{D} \cap [X]^{<\aleph_0}$ satisfies the* cv(4, 2) *condition iff for some set $Y \in [\omega_1]^{\aleph_1}$ the family $\mathcal{D} \cap [Y]^{<\aleph_0}$ is $\frac{1}{2}$-FD on Y.*

Proof. Let us first prove the equivalence between the first two statements. In the forward direction, let X be the homogeneous set for $\mathcal{D}$ that has size $\aleph_1$. In the backward direction, using the same forcing as in the proof of Theorem 2.3 with $\mathcal{D} \cap [X]^{<\aleph_0}$ in place of $\mathcal{D}$ we may notice that the only relevant instances of the covering property are those applicable to $\mathcal{D} \cap [X]^{<\aleph_0}$, and hence the forcing argument will produce a $Y \in [X]^{\aleph_1}$ which is homogeneous for the family $\mathcal{D}$ restricted to X and hence for $\mathcal{D}$.

The second statement implies the third by the forcing argument in the proof of Theorem 2.3, and then the third implies the first by Theorem 1.3 and the fact that $MA(\aleph_1)$ is assumed, hence $\aleph_1$ is a measure precalibre. $\bigstar_{2.7}$

3 A continuous version of the problem

One way to understand problem DU is to view it in the context of the counting measure, as we shall explain below. This approach suggests a generalisation to a more general measure-theoretic context and we attempt such a thing here. For the rest of the section we work in the context of a given (X, Σ, ν) which is an infinite but semi-finite measure space. Hence we assume that $\nu(X) = \infty$ but for every $E \in \Sigma$ we have

$$\nu(E) = \sup\{\nu(A) : A \in \Sigma, \nu(A) < \infty\}.$$

Definition 3.1. *We say that* $\mathscr{D} \subseteq \Sigma$ *is* $\frac{1}{2}$*-dense open in* Σ *if it satisfies the following*
(i) if $C \subseteq D \in \mathscr{D}$ *and* $C \in \Sigma$ *then* $C \in \mathscr{D}$*;*
(ii) if $E \in \Sigma$ *and* $\nu(E) < \infty$ *then there is* $D \in \mathscr{D}$ *such that* $D \subseteq E$ *and* $\nu(D) \geq \frac{1}{2}\nu(E)$.

In this context we ask the following

Problem 3.2 (generalised DU). *Does every* ε*-dense family* $\mathscr{D} \subseteq \Sigma$ *contain a* $\subseteq$*-increasing sequence* (D_n) *with* $\lim \nu(D_n) = \infty$*?*

Taking $X = \kappa$, $\Sigma = \mathscr{P}(\kappa)$ and ν the counting measure on κ we obtain the original problem DU (about infinite homogeneous subsets) from the generalised DU. Since in this context we can only discuss homogeneous sets of size $\aleph_0$ the possible interest of the generalised DU lies in purely ZFC results. It turns out counterexamples are easier to produce in this setting. It is to be expected that the idea of Schreier family on ω which given a counterexample to the original DU setting (see the Introduction) may be adopted to another example, and that is what we shall do here.

Example 3.3. *Let* λ *to be the Lebesgue measure on* $X = [0, \infty)$ *and let* $\mathscr{D}$ *be a family of those Borel sets D, such that either* $D = \emptyset$ *or* $\lambda(D) \leq \inf(D)$*. Clearly* $\mathscr{D}$ *is open. For any Borel set* $E \subseteq X$ *of finite measure there is a point* x_0 *such that* $\lambda([0, x_0] \cap E) \geq \frac{1}{2}\lambda(E)$*. Writing* $D = [x_0, \infty] \cap E$ *we have* $D \in \mathscr{D}$*, so* $\mathscr{D}$ *is* $\frac{1}{2}$*-dense. Of course, there is no* $\subseteq$*-increasing sequence* $\langle D_n : n < \omega \rangle$ *with* $\lambda(D_n) \to \infty$*.*

The above example of course gives a measure of a countable Maharam type. The following example has Maharam type $\mathfrak{c}$.

Example 3.4. *Let* $X = \mathbb{R} \times \{0,1\}^{\mathfrak{c}}$ *and let* ν *be the product of the Lebesgue measure* λ *and the usual product measure* μ*, where* μ *is defined on the product* σ*-algebra of* $\{0,1\}^{\mathfrak{c}}$*. We let* Σ *be the domain of* ν*; then* Σ *is of size* $\mathfrak{c}$ *and every* $E \in \Sigma$ *is determined by* $\mathbb{R}$ *and countably many coordinates in* $\{0,1\}^{\mathfrak{c}}$*. Let* $\{E_\xi : \xi < \mathfrak{c}\}$ *be an enumeration of all sets in* Σ *of finite measure, and for each* $\xi < \mathfrak{c}$ *let* I_ξ *be a countable subset of* $\mathfrak{c}$ *such that* E_ξ *is determined by* $\mathbb{R}$ *and the coordinates in* I_ξ*.*
By induction on $\xi < \mathfrak{c}$ *we may define a function* $\theta : \mathfrak{c} \to \mathfrak{c}$ *such that*

$$\theta(\xi) \notin \bigcup_{\alpha \leq \xi} I_\xi \cup \{\theta(\alpha) : \alpha < \xi\}$$

for every $\xi < \mathfrak{c}$*. Considering elements of* X *as sequences with* $1 + \mathfrak{c}$ *entries and calling the first entry -1, we may define for* $\xi < \mathfrak{c}$ *sets* $C_\xi = \{x \in X : x(\xi) = 0\}$*. It follows from the definition of* μ *and* ν *that* $\nu(\bigcap_{k \geq 1} C_{\xi_k}) = 0$ *whenever* $\langle \xi_k : k < \omega \rangle$ *is a sequence of distinct elements of* $\mathfrak{c}$*.* (*)
Now let $D_\xi = E_\xi \cap C_{\theta(\xi)}$ *for every* $\xi < \mathfrak{c}$*, and define* $\mathscr{D}$ *to be a family of those* $D \in \Sigma$ *which are contained in some* D_ξ*. It follows that* $\mathscr{D}$ *is* $\frac{1}{2}$*-dense open, since*

$$\nu(D_\xi) = \nu(E_\xi \cap C_{\theta(\xi)}) = \frac{1}{2}\nu(E_\xi).$$

Note that the last conclusion uses the fact that $\theta(\xi) \notin I_\xi$. *Now we claim that there is no increasing sequence* $\langle A_n : n < \omega \rangle$ *in* $\mathscr{D}$ *with* $\lim \nu(A_n) = \infty$.

So suppose otherwise. Take n_1 *such that* $\nu(A_{n_1}) \geq 1$. *Since* $A_{n_1} \in \mathscr{D}$ *then* $A_{n_1} \subseteq D_{\xi_1}$ *for some* ξ_1. *But* $\nu(D_{\xi_1}) < \infty$ *so there is* $n_2 < n_1$ *such that* $\nu(A_{n_2}) > \nu(D_{\xi_1})$. *Now* $A_{n_2} \subseteq D_{\xi_2}$ *for some* ξ_2, *where* $\xi_2 \neq \xi_1$.

In this way we can define $n_1 < n_2 < \ldots$, *and distinct* $\xi_k < \mathfrak{c}$ *such that* $A_{n_k} \subseteq D_{\xi_k} \subseteq C_{\theta(\xi_k)}$. *Hence* $A_{n_1} \subseteq P$, *where* $P = \bigcap_k C_{\theta(\xi_k)}$. *But since* θ *is 1-1 it follows by* $(*)$ *above that* $\nu(P) = 0$, *which is a contradiction.*

The measure from Example 3.4 is of type $\mathfrak{c}$ and σ-finite. One might however modify the construction to get a non-σ-finite example (replacing λ on the first axis by any non-σ-finite measure on some σ-algebra of size $2^{\mathfrak{c}}$ and letting μ be the product measure on $\{0,1\}^{2^{\mathfrak{c}}}$). The examples above leave open the generalised DU question for those ν that are of countable type on every set of finite measure but not σ-finite.

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