Władysław Kulpa; Marian Turzański

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A Combinatorial Theorem for a Symmetric Triangulation of the Sphere S^2

WLADYSŁAW KULPA and MARIAN TURZAŃSKI

Katowice, Warsaw

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We shall prove a combinatorial lemma from which it follows that the set $g^{-1}(0)$ of zeros of a continuous and odd function $g : S^2 \to R$, $g(-x) = -g(x)$, from the 2-dimensional sphere S^2 contains a symmetric component.

In 1945 A. W. Tucker [5] discovered a combinatorial lemma which serves as a base for a direct proof of the Borsuk-Ulam antipodal theorem for $n = 2$. Ky Fan [1] in 1952 extended Tucker's result for arbitrary n and established some generalization of the Borsuk-Ulam theorem. In this note we shall present a combinatorial lemma which differs from Tucker's result. We assume that the reader is familiar with a notion of triangulation on the sphere S^2.

Let T be a symmetric meridional-latitudal triangulation of the sphere S^2 (i.e., $x \in T$ iff $-x \in T$, see Figure 1). Especially, we require that any symmetric triangulation with "small spherical triangles" induces a triangulation of the equator $E \subset S^2$ onto "small" segments. Such a triangulation will be called a proper symmetric triangulation. Fix an odd map $\alpha :\to \{-1, 1\}$ defined on the set of vertices of the triangulation T; $\alpha(-x) = -\alpha(x)$ for each $x \in V(T)$. For given two triangles $S_1, S_2 \in T$ define a relation "$\sim$";

$$S_1 \sim S_2 \quad \textit{iff} \quad \alpha(S_1 \cap S_2) = \{-1, 1\}$$

Observe that each maximal α-chain $S_0 \sim \dots \sim S_m$ of triangles from T must be α-cycle i.e., $S_0 \sim S_m$. Let us call an α-cycle to be symmetric if $S_0 \sim \dots \sim S_m = -S_m \sim \dots \sim -S_0$.

Main Lemma. *If $\alpha : V(T) \to \{-1, 1\}$ is an odd map (labeling) defined on the set of vertices of a proper symmetric triangulation T of the sphere S^2 then there exists at least one symmetric α-cycle*

Department of Mathematics, University of Silesia, ul. Bankowa 14, 40-007 Katowice, Poland

Cardinal Stefan Wyszyński University in Warsaw Faculty of Mathematics, ul. Dewajtis 5, 00-815 Warszawa, Poland

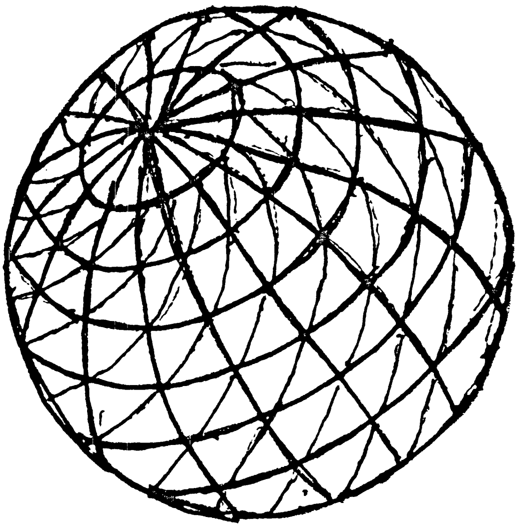

Figure 1

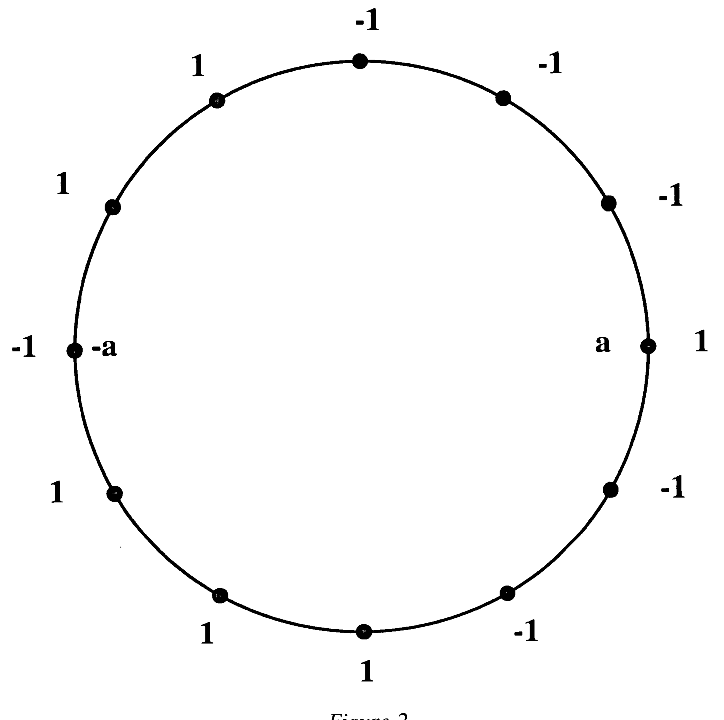

Figure 2

Proof. Let T_E be a triangulation of the equator E induced by the triangulation T (see Figure 2). The number of the all completely labeled segments (i.e., such that the function α assumes on its ends both values -1 and 1) is equal to $4s + 2$, where s is a natural number. To see this, choose an arbitrary vertex $a \in V(T_E)$. Since α is an odd map, we have $\alpha(\{-a, a\}) = \{-1, 1\}$. It is easy to observe that on "one half" of the equator E from $-a$ to a the number of completely labeled segments is odd, it is equal to $2s + 1$ (see Figure 3). This and the fact that α is odd immediately yields the number $4s + 2$.

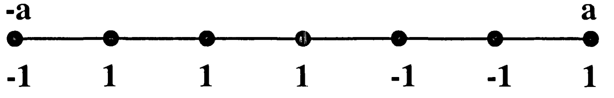

Figure 3

Now we can proceed to the proof of Main Lemma. Let us count the number M of non-symmetric α-cycles which occupy completely labeled segments from T_E. First observe that each α-cycle occupies even number of completely labeled segments from T_E. To see this, it suffices to observe that the trace of an α-cycle on the upper hemisphere is splitted onto disjoint family of α-chains such that each of them occupies exactly two completely labeled segments from T_E (see Figure 4). Since for each non-symmetric α-cycle its antipodal image is also an α-cycle therefore the number M is equal to $4k$. But the number of the all completely labeled segments is equal to $4s + 2$. Thus at least one of completely labeled segments should be occupied by a symmetric α-cycle.

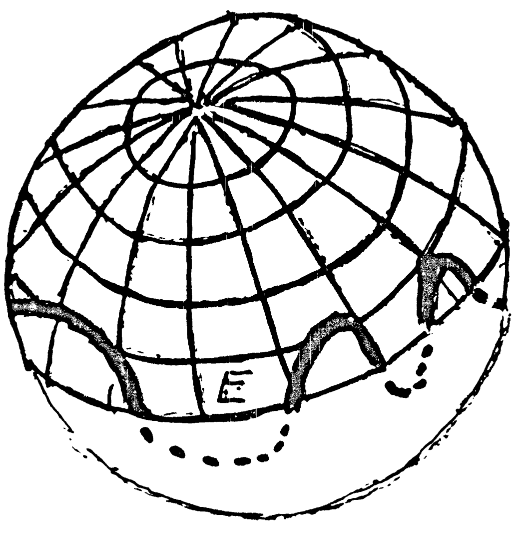

Figure 4

Corollary. *If a continuous map $g : S^2 \to R$ is odd then $g^{-1}(0)$ contains a connected and symmetric subset (component).*

Proof. Define an odd map $\alpha : S^2 \to \{-1, 1\}$ such that $\alpha(x) = -1$ if $g(x) < 0$, $\alpha(x) = 1$ if $g(x) > 0$ and for each pair of antipodal points $x, -x \in g^{-1}(0)$ define α by an arbitrary way preserving only the odd condition; $\alpha(-x) = -\alpha(x)$. Such defined map α has the following property; if $\alpha(x) = -1$ then $g(x) \leq 0$ and if $\alpha(x) = 1$ then $g(x) \geq 0$.

From the Main Lemma it follows that for each number $n > 0$ there exists a symmetric compact connected set $C_n \subset S^2$ being the union of spherical triangles belonging to a triangulation T_n consisting of simplices of diameter less than $\frac{1}{n}$.

Without loss of generality let us assume that there exist a converging sequence of points $c_n \in C_n$. Then according to Kuratowski's theorem [4, cf. 2] the upper limit $C = Ls\{C_n : n = 1, 2, \ldots\}$ is a symmetric and connected set. Since the continuous function g changes sign on each triangle contained in C_n and belonging to T_n, we infer that $g(C) = 0$.

This corollary can be served as a simple proof of the Borsuk-Ulam antipodal theorem;

For each continuous map $f : S^2 \to R^2$, $f = (f_1, f_2)$, there is a point $c \in S^2$ such that $f(c) = f(-c)$.

To see this let us put $g(x) := f_1(x) - f_1(-x)$ and $h(x) := f_2(x) - f(-x)$. Then the functions $g, h : S^2 \to R$ are odd and according to Corollary there is a connected and symmetric set $C \subset g^{-1}(0)$. The map h as an odd continuous map changes signs on the connected set C and therefore there is a point $c \in C$ such that $h(c) = 0$. Since $g(C) = 0$ we infer that $f(c) = f(-c)$.

In our notation the Tucker lemma can be stated as follows;

If $\alpha : V(T) \to \{-2, -1, 1, 2\}$ is an odd map from the set of vertices of a proper symmetric triangulation of the sphere S^2 then there are two points x, y being vertices of a triangle from T such that $\alpha(x) = -\alpha(y)$.

Now we shall show how to get directly from the Tucker lemma the Borsuk-Ulam theorem. Suppose that there is a continuous map $f : S^2 \to R^2$ such that $f(x) \neq f(-x)$ for each $x \in S^2$. Let $g(x) := f(x) - f(-x)$, $g = (g_1, g_2)$. Since $0 \notin g(S^2)$ there is an $\eta > 0$ such that $g(S^2) \subset (-\eta, \eta)^2$. The map g is uniformly continuous and therefore there exists a natural number n such that for each $x, y \in S^2$; $\|x - y\| < \frac{1}{n}$ implies $\|g(x) = g(y)\| < \eta$. Fix a proper symmetric triangulation T_n of S^2 consisting of triangles of diameter less that $\frac{1}{n}$ and define a map $\alpha : V(T) \to \{-2, -1, 1, 2\}$ satisfying the following condition; if $\alpha(x) = j$ then $g_j(x) \geq \eta$ and if $\alpha(x) = -j$ then $g_j(x) \leq \eta$. The map α can be defined by the formula; $\alpha(x) = j \operatorname{sgn} g_j(x)$, where $j = \min\{i : |g_i(x)| \geq \eta\}$.

Since g is odd, the map α is also odd. Applying the Tucker lemma we obtain two points x, y being vertices of a triangle from T such that $\alpha(x) = -\alpha(y)$. According to definition of α we have $\|g(x) - g(y)\| \geq 2\eta$, a contradiction.

Consider the sphere S^2 with a symmetric system consisting of finite number of meridian and latitudal lines as in Figure 1. This system induced a tiling T of S^2 onto spherical squares and triangles with a pole as a vertex. Fix an antisymmetric coloring $\alpha : T \to \{w, b\}$ into two colors; white and black (see Figure 5); $\alpha(P) = w$ iff $\alpha(-P) = b$, for each $P \in T$. A tiling T with a fixed antisymmetric coloring will be said to be antisymmetric tiling onto white and black tiles.

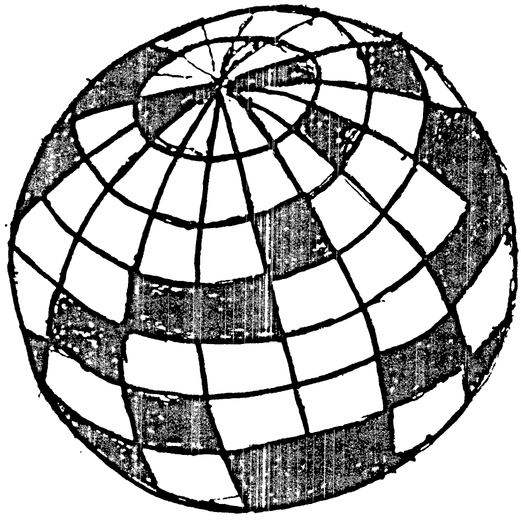

Figure 5

A sequence $P_0, \ldots, P_n$ of white [black] tiles is said to be rook's white [king's black] route if for each $i < n$ the intersection $P_i \cap P_{i+1}$ is a segment [a nonempty set]. Using similar reasoning as in the proof of Main Lemma as well as in [3] it is possible to prove the following chessboard theorem

If T is a antisymmetric tilıng of the sphere S^2 onto white and black tiles then there exists a symmetric path consisting of segments being the intersection of white and black tiles such that during a walk along the path on one hand there is a rook's white [or black] route and on the other hand — king's black [or white] route.

References

[1] FAN K., *A generalization of Tucker's combinatorıal lemma with topological applications,* Ann. of Ma h. 56 (1952) 431—437

[2] KULPA W., SOCHA L., TURZAŃSKI M., *Param tric extens on of the Poincaré theorem,* Acta Univ. Carol. Math. Phys. 41 2000), 39—46

[3] KULPA W., SOCHA L., TURZAŃSKI M., *Steinhaus chessboard theorem*, Acta Univ. Carol. Math. Phys. 41 (2000), 47 – 50.

[4] KURATOWSKI K., *Topology vol II*, Academic Press, New York 1968.

[5] TUCKER A. W., *Some topological properties of disk and sphere*, Proc. First Canadian Math. Congress, Montreal 1945, 285 – 309.