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A Generalization of Dubovitskij-Miliutin Theorem

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One from fundamental theorems of convex analysis is

Dubovitskij-Miliutin theorem. *Let X be a Hausdorff locally convex space, let $K_1, \ldots, K_n$ be convex cones in X (with the vertex at 0), all but one open, and let the intersection of all n cones is empty. Then there exist elements $x_1^* \in K_1^*, \ldots, x_n^* \in K_n^*$ not all zero, such that $x_1^* + \ldots + x_n^* = 0$.*

(Here K^* denotes the dual (polar) cone to K.) See e.g. [1].

This theorem is evidently non-symmetric: one of the cones stands by itself. Besides it is supposed in the theorem that almost all cones are "solid".

We give a generalization of the theorem, which is symmetric and works even in cases where none of the cones is solid:

Theorem 1 (a symmetric generalization of Dubovitskij-Miliutin theorem). *Let X be a Haussdorff locally convex space, and let $K_1, \ldots, K_n$ be convex cones in X with the empty intersection.*

Suppose that for any two subsets I, J of the set $\{1, \ldots, n\}$ such that both the intersections $K_I := \bigcap_{i \in I} K_i$ and $K_J := \bigcap_{i \in J} K_i$ are not empty, the following conditions are fulfilled:

1) the linear hull Y of the union of K_I and K_J is complementable in X;

2) tl e (arithmetical) difference $K_I - K_J$ has a non-empty interior in Y (equipped with the induced topology).

Then there exist elements $x_1^* \in K_1^*, \ldots, x_n^* \in K_n^*$, not all zero, such that their sum is equal to 0.

(We say that a vector subspace Y of a topological vector space X is *complementable* if there exist a vector subspace Z of X such that X is the direct sum

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of Y and Z and the topology of X is equal to the product of the induced topologies of Y and Z (by the canonical identification of the product space and the direct sum)).

The proof of Theorem 1 is based on the following generalization of a standard separation theorem (dealing with the case of two convex sets, one of them has an interior point; in [1] this standard theorem is named "the first separation theorem"):

Theorem 2 (a symmetric generalization of the standard separation theorem). *Let X be a topological vector space, and let A and B be two convex subsets of X with the empty intersection, such that the following conditions are fulfilled:*

1) the linear hull Y of the union of A and B is complementable in X;

2) the difference $A - B$ has a non-empty interior in Y.

Then there exists a non-zero continuous linear functional x^* on X that separates A and B (the supremum of the values of x^* on A is less or equal to the infimum of the values of x^* on B).

(For the proof we apply in Y the standard separation theorem to the singletone 0 and the difference $A - B$, and then we extend the resulting functional onto the whole space X, putting it be equal to 0 on a complementary subspace.)

In a similar way we can generalize other related theorems of convex analysis. For example by some analogical "symmetric" conditions on convex cones $K_1, \ldots, K_n$, the dual cone to their intersection is equal to the sum of their dual cones.

References

[1] ALEKSEEV, V. M., TIKHOMIROV, V. M. and FOMIN, S. V., Optimal Control, Consultants Bureau, New York and London, 1987.