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On Products of Pseudoradial Spaces

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Sequential compactness is shown to be equivalent to pseudoradiality for product of $\aleph_1$ compact semiradial T_2-spaces.

We shall prove that the product of $\aleph_1$ compact semiradial Hausdorff spaces is pseudoradial iff it is sequentially compact. Since it can be considered as a topic on products of pseudoradial spaces as well as a partial result concerning relation between sequential compactness and pseudoradiality, we shall repeat some known facts of these subjects first.

If nothing else is stated explicitly, all spaces are supposed to be Hausdorff.

I Introduction. Sequential compactness and pseudoradiality

The name 'sequence' will be used for a well ordered net, i.e. a transfinite sequence.

A subset A of a topological space is said to be *radially closed* if none sequence $\{x_\alpha; \alpha < \lambda\} \subset A$ converges to a point $x \notin A$.

A space X is called *radial* if for each $A \subset X$ and each $x \in \overline{A}$ there is a sequence $\{x_\alpha; \alpha < \lambda\} \subset A$ converging to x. X is called *pseudoradial* if each of its radially closed subsets is closed. Obviously, every radial space is pseudoradial.

It is well known and easy to prove that every countably compact pseudoradial T_1-space is sequentially compact.

The question of reverse implication in the class of compact T_2-spaces is much more complicated. Juhász and Szentmiklóssy have proved

Theorem A. ([JS], theorem 4) *If $\mathfrak{c} \leq \aleph_2$, then every compact sequentially compact space is pseudoradial.*

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Under MA + $\mathfrak{c} = \aleph_3$ + 'club filter on ω_1 has a base of cardinality $\aleph_2$', there exists a sequentially compact non-pseudoradial compactum ([JS], theorem 5). But $\mathfrak{c} = \aleph_\alpha$ with $\alpha \geq 3$ is not enough to determine it: in [DJSS] theorem 4 says that by adding any number of Cohen reals to a groundmodel of CH one obtains a model in which every sequentially compact compactum is pseudoradial.

II Products

The product of compact metric (hence radial) space and radial Lindelöf space may fail to be pseudoradial (Gerlits, Nagy; quoted in [FT]). This fact has led to question about behaviour of compact pseudoradial spaces under formations of product. Frolík and Tironi have proved

Theorem B. [FT] *The product of a compact pseudoradial space and a compact radial space is pseudoradial.*

The problem whether the same holds (in ZFC) for a pair of pseudoradial compacta remains open. However, theorem A implies consistency of the positive answer.

The class of semiradial spaces (situated between radial and pseudoradial ones) turned out to be useful for weakening assumptions of theorem B.

Let $\kappa \in \mathbf{Cn}$. A subset A of a topological space is called *κ-closed* if $\overline{B} \subset A$ for every $B \subset A$, $|B| \leq \kappa$. A is said to be *$< \kappa$-closed* provided it is λ-closed for all $\lambda < \kappa$.

A space X is called *semiradial* if for every infinite $\kappa \in \mathbf{Cn}$: $(\forall A \subset X)\, A$ is not κ-closed $\Rightarrow (\exists \lambda \leq \kappa)\,(\exists \{x_\alpha;\, \alpha < \lambda\} \subset A)\,(\exists x \notin A)\, x_\alpha \to x$.

So far the strongest generalization of theorem B is

Theorem C. [BG] *The product of two pseudoradial compact spaces is pseudoradial provided one of them is semiradial.*

Next we shall deal with products of infinitely many factors.

Given a product $X = \prod_{\gamma<\omega_1} X_\gamma$ and $\Gamma \subset \omega_1$, π_Γ will denote natural projection of X to $\prod_{\gamma\in\Gamma} X_\gamma$. The symbol π_γ will be used if $\Gamma = \{\gamma\}$.

Lemma 1. *Suppose X_γ $(\gamma < \omega_1)$ are regular spaces with $X = \prod_{\gamma<\omega_1} X_\gamma$ sequentially compact. Let A be radially closed non-closed subset of X. Then there exists $\tilde{A} \subset X$ radially closed such that for some $\gamma < \omega_1$, $\pi_\gamma[\tilde{A}]$ is not closed.*

Proof. Let X_γ $(\gamma < \omega_1)$, X and A satisfy assumptions of lemma 1, let $\mathbf{x} = \langle x_\gamma \rangle_{\gamma<\omega_1} \in \overline{A} \setminus A$. A is radially closed and $\mathbf{x} \notin A$, hence there is countable $\Gamma \subset \omega_1$ such that $\langle x_\gamma \rangle_{\gamma\in\Gamma} \notin \pi_\Gamma[A]$. By sequential compactness of X, there is finite $\tilde{\Gamma} \subset \Gamma$ with $\langle x_\gamma \rangle_{\gamma\in\tilde{\Gamma}} \notin \pi_{\tilde{\Gamma}}[A]$.

Suppose $k \in \omega$ is the smallest cardinality of $\Gamma \subset \omega_1$ such that for some radially closed $\tilde{A} \subset X$, $\pi_\Gamma[\tilde{A}]$ is not closed. Without loss of generality $\Gamma = \{0, \dots, k-1\}$. Let $\langle x_0, \dots, x_{k-1}\rangle \in \overline{\pi_\Gamma[\tilde{A}]} \setminus \pi_\Gamma[\tilde{A}]$; suppose $k > 1$.

$B = \tilde{A} \cap (\{x_0\} \times \prod_{0<\gamma<\omega_1} X_\gamma)$ is radially closed and for $\pi = \pi_{\Gamma\setminus\{0\}}$, $\langle x_1, \dots, x_{k-1}\rangle \notin \pi[B] = \overline{\pi[B]}$. Take a closed neighbourhood U of $\langle x_1, \dots, x_{k-1}\rangle$ satisfying $U \cap \pi[B] = \emptyset$. $C = \tilde{A} \cap (X_0 \times U \times \prod_{\gamma \geq k} X_\gamma)$ is radially closed, $x_0 \in \overline{\pi_0[C]} \setminus \pi_0[C]$ – contradicting minimality of k. □

It follows that the space $2^{\aleph_1}$ is pseudoradial iff it is sequentially compact. Let us generalize this.

Main Theorem. *The product of $\aleph_1$ semiradial compact spaces is pseudoradial iff it is sequentially compact.*

The proof includes further notion and two lemmas.

Recall that a sequence $\{x_\alpha; \alpha < \kappa\}$ in a space X is *free* provided $(\forall \alpha < \kappa)\, \overline{\{x_\beta; \beta < \alpha\}} \cap \overline{\{x_\beta; \alpha \leq \beta < \kappa\}} = \emptyset$.

Lemma D. ([Ny], lemma 5.6) *Let $\{x_\alpha; \alpha < \lambda\}$ be convergent sequence in a regular space such that $\overline{\{x_\beta; \beta < \alpha\}}$ does not contain the limit point for any $\alpha < \lambda$. Then $\{x_\alpha; \alpha < \lambda\}$ has a cofinal free subsequence.*

Observation in next lemma is a crucial point of Bella and Gerlits' proof of theorem C.

Lemma 2. *Let X be pseudoradial space, $\{C_\alpha; \alpha < \mu\}$ increasing sequence of closed subsets of X, μ regular. Suppose that $C = \bigcup_{\alpha<\mu} C_\alpha$ is not closed.*

Then there is $\{y_\alpha; \alpha < \mu\} \subset C$ converging outside C and a pair of functions $f, h : \mu \to \mu$ satisfying

$$(1) \qquad \beta_1 < \beta_2 < \mu \Rightarrow f(\beta_1) < h(\beta_1) < f(\beta_2)$$

such that $(\forall \alpha < \mu)\, y_\alpha \in C_{h(\alpha)} \setminus C_{f(\alpha)}$.

Proof. There is λ regular and $\{y_\alpha; \alpha < \lambda\} \subset C$, $y_\alpha \to y \notin C$. It is easy to check that neither $\lambda < \mu$ or $\mu < \lambda$. On replacing $\{y_\alpha; \alpha < \lambda\}$ by suitable cofinal subsequence, the values of f, h can be defined inductively. □

Proof of Main Theorem. Considering the Introduction, we only need to prove the right-left implication.

Let X_γ $(\gamma < \omega_1)$ be compact semiradial Hausdorff spaces, $X = \prod_{\gamma<\omega_1} X_\gamma$ sequentially compact. Assume for contradiction X contains a radially closed non-closed subset. We shall proceed with four steps.

(I) Pick the smallest cardinal κ and a radially closed $A \subset X$ (guaranteed by lemma 1) such that some projection of A (say the zero one) is not κ-closed. By semiradiality of X_0 and minimality of κ:

$$(\exists \{\mathbf{x}_\alpha; \alpha < \kappa\} \subset A)\,(\exists x^0 \notin \pi_0[A])\, \pi_0(\mathbf{x}_\alpha) \to x^0 .$$

κ is regular and as X is sequentially compact, κ is uncountable.

(II) Proceeding by induction on $\gamma < \omega_1$, let us construct sequences $\{\mathbf{x}_\alpha(\gamma); \alpha < \kappa\} \subset A$ with $\mathbf{x}_\alpha(\gamma) = \langle x_\alpha^{\gamma'}(\gamma)\rangle_{\gamma'<\omega_1}$, points $x^\gamma \in X_\gamma$ and for each $\alpha < \kappa$ an increasing function $f_\alpha : \omega_1 \to \kappa$ such that under the notation
$Const(\gamma) \Leftrightarrow |\{x_\alpha^\gamma(\gamma); \alpha < \kappa\}| = 1$,
$U_\alpha^\gamma = \overline{\{x_\beta^\gamma(\gamma); \beta < \alpha\}}$,
$V_\alpha^\gamma = \overline{\{x_\beta^\gamma(\gamma); \alpha \leq \beta < \kappa\}}$,
the following conditions are satisfied.

(a) $x_\alpha^\gamma(\gamma) \to x^\gamma$,

(b) $\neg Const(\gamma) \Rightarrow (\forall \alpha < \kappa) \quad U_\alpha^\gamma \cap V_\alpha^\gamma = \emptyset$,

(c) $(\forall \alpha < \kappa)(\forall \gamma' < \gamma)$
$(\{x_\alpha^{\gamma'}(\gamma''); \gamma' \leq \gamma'' < \gamma\} \subset U_{f_\alpha(\gamma)}^{\gamma'} \;\&\; \{x_\alpha^{\gamma'}(\gamma''); \gamma \leq \gamma'' < \omega_1\} \subset V_{f_\alpha(\gamma)}^{\gamma'})$.

We point out that from (c) follows $(\forall \alpha < \kappa)(\forall \gamma' < \gamma)\, x_\alpha^{\gamma'}(\gamma) \in V_\alpha^{\gamma'}$. Thus $x_\alpha^{\gamma'}(\gamma) \xrightarrow{\alpha<\kappa} x^{\gamma'}$.

$x^0 \notin \overline{\{\pi_0(\mathbf{x}_\beta); \beta < \alpha\}}$ for any $\alpha < \kappa$, so by lemma D, $\{\mathbf{x}_\alpha; \alpha < \kappa\}$ contains a subsequence $\{\mathbf{x}_\alpha(0); \alpha < \kappa\}$ with $\{x_\alpha^0(0); \alpha < \kappa\}$ free.

Put $f_\alpha(0) = \alpha$ for each $\alpha < \kappa$.

Let $0 < \gamma_0 < \omega_1$, suppose $\mathbf{x}_\alpha(\gamma)$, x^γ have been defined for all $\gamma < \gamma_0$, $\alpha < \kappa$ and functions f_α have domain γ_0.

We define an auxiliary sequence $\{\mathbf{z}_\alpha; \alpha < \kappa\} \subset A$, $\mathbf{z}_\alpha = \langle z_\alpha^\gamma\rangle_{\gamma<\omega_1}$, first. If γ_0 is isolated, let $\gamma_0 - 1$ be its predecessor and $\mathbf{z}_\alpha = \mathbf{x}_\alpha(\gamma_0 - 1)$ for each α.

Now assume γ_0 is limit, $\gamma_n \nearrow \gamma_0$. Fix $\alpha < \kappa$. $\{\mathbf{x}_\alpha(\gamma_n); n \in \omega\}$ contains a convergent subsequence; let $\mathbf{z}_\alpha$ be its limit point.

Put for all $\alpha < \kappa$

$$f_\alpha(\gamma_0) = \sup\big(\{f_\beta(\gamma_0); \beta < \alpha\} \cup \{\beta < \kappa; (\exists \gamma')(\exists \gamma) \\ (\gamma' \leq \gamma < \gamma_0 \;\&\; \neg Const(\gamma') \;\&\; x_\alpha^{\gamma'}(\gamma) \in V_\beta^{\gamma'})\}\big) + 1 .$$

$\kappa > f_\alpha(\gamma_0) > \alpha$ because $\neg Const(0)$ and $x_\alpha^0(0) \in V_\alpha^0$. And $f_\alpha(\gamma_0) > f_\alpha(\gamma)$ for each γ such that $1 \leq \gamma < \gamma_0$ since $x_\alpha^0(\gamma) \in V_{f_\alpha(\gamma)}^0$ according to (c).

Let $y \in X_{\gamma_0}$ be a complete accumulation point of $\{z_\alpha^{\gamma_0}; \alpha < \kappa\}$.

Case (A)

$$(\forall \alpha < \kappa)(\exists \mathbf{y}_\alpha)\, \mathbf{y}_\alpha \in A \cap \Big(\prod_{\gamma<\gamma_0} V_\alpha^\gamma \times \{y\} \times \prod_{\gamma>\gamma_0} X_\gamma\Big).$$

Define $x^{\gamma_0} = y$ and for every $\alpha < \kappa$

$$\mathbf{x}_\alpha(\gamma_0) = \mathbf{y}_{f_\alpha(\gamma_0)} .$$

Case (B)

$$(\exists \alpha_0 < \kappa)\, A \cap \Big(\prod_{\gamma<\gamma_0} V_{\alpha_0}^\gamma \times \{y\} \times \prod_{\gamma>\gamma_0} X_\gamma\Big) = \emptyset .$$

Each $B_\alpha = A \cap (\prod_{\gamma<\gamma_0} V_\alpha^\gamma \times \prod_{\gamma\geq\gamma_0} X_\gamma)$ is radially closed. Denote for $\alpha < \kappa$

$$C_\alpha = \overline{\{z_\beta^{\gamma_0}; \alpha_0 \leq \beta < \alpha\}};$$
$$C = \bigcup_{\alpha<\kappa} C_\alpha .$$

By minimality of κ, each $\pi_\gamma[B]$ for a radially closed $B \subset X$ is $< \kappa$-closed. Hence $C \subset \pi_{\gamma_0}[B_{\alpha_0}]$, while $y \notin \pi_{\gamma_0}[B_{\alpha_0}]$. Applying lemma 2 to C_α, $C \subset X_{\gamma_0}$ pick $\{y_\alpha; \alpha < \kappa\} \subset C$, $y_\alpha \to x^{\gamma_0} \notin C$. Moreover, this sequence can be assumed free.

For the function h from lemma 2 and for each $\alpha < \kappa$, $y_\alpha \in C_{h(\alpha)} \setminus C_\alpha$. It means $y_\alpha \in \overline{\{z_\beta^{\gamma_0}; \alpha \leq \beta < h(\alpha)\}} \subset \pi_{\gamma_0}[B_\alpha]$; take $\mathbf{y}_\alpha \in B_\alpha$ from the preimage of y_α. Define for every α

$$\mathbf{x}_\alpha(\gamma_0) = \mathbf{y}_{f_\alpha(\gamma_0)}.$$

Finally denote $\mathbf{x} = \langle x^\gamma \rangle_{\gamma < \omega_1}$.

(III) If $\kappa = \omega_1$, consider

$$\{\mathbf{x}_\gamma(\gamma); \gamma < \omega_1\}.$$

Conditions (c) and (a) imply $\mathbf{x}_\gamma(\gamma) \to \mathbf{x}$ therefore $\mathbf{x} \in A$ — a contradiction.

(IV) If $\kappa > \omega_1$, denote $O_\alpha = \prod_{\gamma < \omega_1} V_\alpha^\gamma$. Now suffices to find for each $\alpha < \kappa$ a point $\mathbf{p}_\alpha \in O_\alpha \cap A$. Then $\mathbf{p}_\alpha \to \mathbf{x}$.

Fix α. According to (b) and (c), for every $\gamma < \omega_1$

$$\text{(2)} \qquad \text{either } Const(\gamma) \text{ or } \{x_\alpha^\gamma(\gamma'); \gamma \leq \gamma' < \omega_1\} \text{ is free.}$$

Proceeding by induction on $\gamma < \omega_1$, we shall define $\{\mathbf{z}_\beta(\gamma); \beta < \omega_1\} \subset A$ where $\mathbf{z}_\beta(\gamma) = \langle z_\beta^{\gamma'}(\gamma) \rangle_{\gamma' < \omega_1}$, $p^\gamma \in X_\gamma$ and $g^\gamma : \omega_1 \to \omega_1$ increasing such that

(i) $z_\beta^\gamma(\gamma) \to p^\gamma$,

(ii) $(\forall \beta < \omega_1)(\forall \gamma' < \gamma)\, z_\beta^{\gamma'}(\gamma) \in W_\beta^{\gamma'} = \overline{\{z_{\beta'}^{\gamma'}(\gamma'); \beta \leq \beta' < \omega_1\}}$,

(iii) $(\forall \beta < \omega_1)(\forall \gamma' < \beta)\, z_\beta^{\gamma'}(\gamma) \in V_\alpha^{\gamma'}$,

(iv) $(\forall \beta < \omega_1)(\forall \gamma' < \omega_1)\, \gamma < \gamma' < \beta \Rightarrow$
$(\{z_{\beta'}^{\gamma'}(\gamma); \gamma' \leq \beta' < \beta\} \subset U_{f_\alpha g^\gamma(\beta)}^{\gamma'}\ \&\ \{z_{\beta'}^{\gamma'}(\gamma); \beta \leq \beta' < \omega_1\} \subset V_{f_\alpha g^\gamma(\beta)}^{\gamma'})$.

In step γ_0 we introduce auxiliary $\mathbf{w}_\beta = \langle w_\beta^\gamma \rangle_{\gamma < \omega_1}$ for $\beta < \omega_1$ in the following way.

If $\gamma_0 = 0$, put $\mathbf{w}_\beta = \mathbf{x}_\alpha(\beta)$. If γ_0 is isolated, let $\mathbf{w}_\beta = \mathbf{z}_\beta(\gamma_0 - 1)$.

Suppose γ_0 is limit, $\gamma_n \nearrow \gamma_0$. To preserve condition (iv), we will apply the 'countable diagonalization' to the positions $\beta < \omega_1$ chosen by increasing $g : \omega_1 \to \omega_1$.

Let $g(0) = \gamma_0$. For $\delta > 0$ put $\beta = \sup\{g(\delta'); \delta' < \delta\} + 1$. Choose countable $g(\delta) \geq \sup\{g^{\gamma_n}(\beta); n \in \omega\}$.

Fix $\delta < \omega_1$. $\{\mathbf{z}_{g(\delta)}(\gamma_n); n \in \omega\}$ contains a subsequence (determined by increasing $\Delta : \omega \to \omega$) converging to $\mathbf{w}_\delta \in A$. We will show analogy of (iv) holds for $\mathbf{w}_\delta$.

Consider $\{w_\delta^\gamma; \gamma \leq \delta < \omega_1\}$, $\gamma_0 \leq \gamma < \omega_1$. Choose δ, δ' such that $\gamma \leq \delta < \delta'$. $w_\delta^\gamma \xleftarrow{n < \omega} z_{g(\delta)}^\gamma(\gamma_{\Delta(n)}) \in U_{f_\alpha g^{\gamma_{\Delta(n)}}(\beta)}^\gamma$ where $\beta = \sup\{g(\bar\delta); \bar\delta < \delta'\} + 1$. $U_{f_\alpha g^{\gamma_{\Delta(n)}}(\beta)}^\gamma \subset U_{f_\alpha g(\delta')}^\gamma$ because $g(\delta') \geq g^{\gamma_n}(\beta)$ $(\forall n)$. Hence we have verified

$$\text{(3a)} \qquad \{w_\delta^\gamma; \gamma \leq \delta < \delta'\} \subset U_{f_\alpha g(\delta')}^\gamma.$$

Now take δ, δ' so that $\gamma < \delta' \leq \delta < \omega_1$. $w_\delta^\gamma \xleftarrow{n < \omega} z_{g(\delta)}^\gamma(\gamma_{\Delta(n)}) \in V_{f_\alpha g^{\gamma_{\Delta(n)}} g(\delta)}^\gamma$ (by (iv) for $\gamma_{\Delta(n)}$). For each n, $g^{\gamma_n} g(\delta) \geq g(\delta)$; with $\delta' \leq \delta$ this implies $V_{f_\alpha g^{\gamma_{\Delta(n)}} g(\delta)}^\gamma \subset V_{f_\alpha g(\delta')}^\gamma$. So we have for $\gamma < \delta'$

$$\text{(3b)} \qquad \{w_\delta^\gamma; \delta' \leq \delta < \omega_1\} \subset V_{f_\alpha g(\delta')}^\gamma.$$

Now we are ready to pass the step γ_0 in general. Let $y \in X_{\gamma_0}$ be a complete accumulation point of $\{w_\beta^{\gamma_0}; \gamma_0 \leq \beta < \omega_1\}$. For $\gamma < \omega_1$ denote

$$C_\gamma = \overline{\{w_\beta^{\gamma_0}; \gamma_0 \leq \beta < \gamma\}};$$
$$C = \bigcup_{\gamma < \omega_1} C_\gamma.$$

Since condition (iv) holds for $\gamma = \gamma_0 - 1$ (and (2) if $\gamma_0 = 0$ and (3a, b) in the limit case), it is either $C = \{y\}$ or $y \in \overline{C} \setminus C$. In each case there are $f, h : \omega_1 \to \{\gamma; \gamma_0 \leq \gamma < \omega_1\}$ satisfying (1), $p^{\gamma_0} \in X_{\gamma_0}$ and $\mathbf{z}_\beta(\gamma_0) \in A \cap \overline{\{\mathbf{w}_\delta; f(\beta) \leq \delta < h(\beta)\}}$ such that $z_\beta^{\gamma_0}(\gamma_0) \xrightarrow{\beta<\omega_1} p^{\gamma_0}$. Let $g^0 = f$, $g^{\gamma_0} = g^{\gamma_0 - 1} f$ if γ_0 is isolated, $g^{\gamma_0} = gf$ otherwise.

As in the part (III), it can be shown that $\mathbf{z}_\gamma(\gamma) \to \mathbf{p}_\alpha = \langle p^{\gamma'} \rangle_{\gamma' < \omega_1}$. Moreover, $z_\gamma^{\gamma'}(\gamma) \in V_\alpha^{\gamma'}$ whenever $\gamma > \gamma'$, hence $p^{\gamma'} \in V_\alpha^{\gamma'}$. It verifies $\mathbf{p}_\alpha \in O_\alpha \cap A$. □

As a consequence of the Main Theorem, we get the following result from [BM].

Corollary 3. *Product of countably many semiradial compact spaces is pseudoradial.*

Proof. Such space $\prod_{n \in \omega} X_n$ is sequentially compact and homeomorphic to $\prod_{\alpha < \omega_1} X_\alpha$, where X_α is a one-point space for $\omega \leq \alpha < \omega_1$. □

III Consistency results

Denote $[\omega]^\omega$ the set $\{A \subset \omega; |A| = \omega\}$.

The cardinal number $\mathfrak{h}$ (the *nondistributivity number of $\mathcal{P}(\omega)/fin$*) is defined as the smallest size of a collection $\mathfrak{A}$ of almost disjoint families on ω such that $(\forall B \in [\omega]^\omega)\ (\exists \mathscr{A} \in \mathfrak{A})\ (\exists A_1, A_2 \in \mathscr{A})\ A_1 \neq A_2\ \&\ |A_1 \cap B| = \omega = |A_2 \cap B|$.

It has been shown that $\mathfrak{h} = \min \{\kappa$; the product of some sequentially compact spaces $X_\gamma (\gamma < \kappa)$ is not sequentially compact$\}$ [Si]. In fact the spaces in Simon's proof are compact pseudoradial.

The number $\mathfrak{s}$, called the *splitting number*, is defined as $\min \{|\mathscr{S}|; \mathscr{S} \subset [\omega]^\omega\ \&\ (\forall A \in [\omega]^\omega)\ (\exists S \in \mathscr{S})\ |A \cap S| = \omega = |A \setminus S|\}$.

Let us remark that $\aleph_1 \leq \mathfrak{h} \leq \mathfrak{s} \leq \mathfrak{c}$ and for arbitrary regular cardinals $\kappa, \lambda, \aleph_1 \leq \kappa \leq \lambda$,

$$\mathfrak{h} = \mathfrak{s} = \kappa\ \&\ \mathfrak{c} = \lambda$$

is consistent with ZFC ([vD, theorem 5.1).

Corollary 4. $\mathfrak{h} > \aleph_1$. *Product of $\aleph_1$ semiradial compact spaces is pseudoradial.*

However, a completely different case is consistent too.

Example 5. $\mathfrak{s} = \aleph_1$. $2^{\aleph_1}$ is not sequentially compact ([vD], theorem 6.1), far less it can be pseudoradial.

The assumption of semiradiality in the Main Theorem can be consistently weakened. Theorem A gives

Corollary E. $\mathfrak{c} \leq \aleph_2$. *Product of less than* $\mathfrak{h}$ *compact pseudoradial spaces is pseudoradial.*

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