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Hyperspaces of Various Locally Connected Subcontinua

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Introduction

Over the last decade there have been discovered several deep characterizations of special subspaces of hyperspaces of euclidean spaces R^n or of n-cubes. Methods used involved a powerful technique of absorbers. Examples of fully recognized families of subsets of R^n or I^n are:

- locally connected subcontinua (for $n > 2$) [10];
- arcs $(n = 2)$ [3];
- pseudoarcs $(n = 2)$ [4];
- ANR's and AR's [5, 8].

There is, of course, a long list of interesting subspaces of the hyperspaces waiting for their turn. One can also consider hyperspaces of non-euclidean spaces. It seems that good candidates to study are, e.g., hyperspaces of Menger universal continua and their peculiar subspaces.

The first step leading to recognition is evaluation of an exact Borel class of a subspace, which is often not an easy task. The aim of this note is to gather some observations, mostly about Borel classes of some old and new examples of subfamilies of hyperspaces. In Theorem 1 a result from [10] is extended over a class of continua larger than cubes. In Theorem 2 the Borel class of simple closed curves in cubes is evaluated.

Most observations are inspired by ideas from [10]. I hope they can motivate further progress.

Preliminaries

All spaces in this paper are assumed to be metric separable. If d is a metric in a continuum X, then $C(X)$ denotes the hyperspace of all nonempty subcontinua of X with the Hausdorff metric d_H.

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If X is a locally connected continuum with no free (i.e., non-boundary) arcs, then $C(X)$ is homeomorphic to the Hilbert cube $Q = [-1, 1]^{\aleph_0}$ [7].

We set the following notation: M^n, $n \geq 1$ is the Menger n-dimensional universal continuum, M^n_{n-1} is the Sierpiński continuum in R^n universal for all $n-1$-dimensional compacta in R^n, D^n is the n-dimensional disk, S is the unit circle, $I = [0, 1]$.

If M is a continuum, then $\mathbf{M}(X)$ is the subspace of $C(X)$ consisting of all topological copies of M in X. We will also consider the subspaces

$$AR(X), ANR(X), LC^n(X), Den(X), L(X), L_1(X)$$

of $C(X)$ of all AR's, ANR's, LC^n-spaces, dendrites, locally connected subcontinua and locally connected 1-dimensional subcontinua in X, respectively.

It is well known that, for any compactum X, $L(X)$ is an $F_{\sigma\delta}$-set [12]. It is proved in [10] that $L([-1, 1]^n)$, $n > 1$, is not a $G_{\delta\sigma}$-set.

Let $\hat{c}_0 = \{(x_i) \in Q : \lim x_i = 0\}$. If $n > 2$, then $L([-1, 1]^n)$ is homeomorphic to $\hat{c}_0$ because it is an $F_{\sigma\delta}$-absorber in $C([-1, 1]^n)$ (see [10]).

At the beginning, we describe three auxiliary continua.

Harmonic fan F_v.

$$F_v = \left\{z \in R^2 : |z| \leq 1 \quad \text{and} \quad \arg z = \pm \tfrac{1}{i}, i = 1, 2, \ldots, \quad \text{or} \quad \arg z = 0\right\}.$$

Denote by v the vertex $z = 0$.

Harmonic comb C_B.

$$C_B = (I \times \{0\}) \cup \left(\left\{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\right\} \times [0, 1]\right) \subset R^2.$$

Denote by B the base segment $I \times \{0\}$.

Harmonic comb of n-disks C_n. Let s_i be the middle point of the interval $\left[\frac{1}{i+1}, \frac{1}{i}\right]$.

$$C_n = [-1, 0]^n \cup \left(\bigcup_{i=1}^{\infty} \left[s_i, \tfrac{1}{i}\right] \times [0, 1] \times [-1, 0]^{n-2}\right) \cup (\{0\} \times [0, 1] \times [-1, 0]^{n-2}) \subset [-1, 1]^n.$$

Results

For each $\mathbf{x} = (x_i) \in Q$ denote $\Gamma^+(\mathbf{x}) = \{i : x_i \geq 0\}$ and $\Gamma^-(\mathbf{x}) = \{i : x_i < 0\}$.

Theorem 1. *Let a continuum X topologically contain the harmonic fan F_v or comb C_B. Suppose a subspace $\mathscr{K} \subset C(X)$ satisfies*

$$\{C \in L(F_v) : v \in C\} \subset \mathscr{K} \cap C(F_v) \subset L(F_v)$$

in the case where X contains the harmonic fan or

$$\{C \in L(C_B) : B \subset C\} \subset \mathscr{K} \cap C(C_B) \subset L(C_B)$$

if X contains the harmonic comb. Then $\hat{c}_0$ embeds in $\mathscr{K}$ as a closed subset. In particular, $\mathscr{K}$ is not a $G_{\delta\sigma}$-set.

Proof. We slightly modify the proof of [10, Theorem 3.1]. Assume $F_v \subset X$. Let $\mathbf{x} = (x_i) \in Q$. Define

$$f(\mathbf{x}) = \{z \in F_v : \text{there is } i \text{ such that } x_i \neq 0 \text{ and } \arg z = \frac{x_i}{i|x_i|} \text{ and } |z| \leq |x_i|\} \cup \{z : |z| \leq 1 \text{ and } \arg z = 0\}.$$

Then $f: Q \to C(X)$ is an embedding satisfying $f(\hat{c}_0) = f(Q) \cap \mathscr{K}$. Hence, $f(\hat{c}_0)$ is a closed subset of $\mathscr{K}$.

Suppose now $C_B \subset X$. Put, for $t \in I$ and $i = 1, 2, \ldots$, $E_i(t) = \{\frac{1}{i}\} \times [0, t]$. Define an embedding $f: Q \to C(X)$ by

$$f(\mathbf{x}) = B \cup (\{0\} \times [0, 1]) \cup \bigcup_{i \in \Gamma^+(\mathbf{x})} E_{2i}(x_i) \cup \bigcup_{i \in \Gamma^-(\mathbf{x})} E_{2i-1}(|x_i|).$$

As before, it is clear that $f(\hat{c}_0)$ is a closed subset of $\mathscr{K}$.

Corollary 1. *If a continuum X contains a copy of F_v or C_B, then $L(X)$ is an absolute $F_{\sigma\delta}$-set but it is not a $G_{\delta\sigma}$-set. The subspaces $Den(X)$, $AR(X)$, $ANR(X)$, $L_1(X)$ and $LC^n(X)$ are not $G_{\delta\sigma}$-sets.*

If X is a locally connected continuum with no local separating point and no planar nonempty open subset, then, moreover, $L(X)$ is contained in a σZ-subset of $C(X)$.

Proof. The first part follows directly from Theorem 1. If X is locally connected with no local separating points and no planar nonempty open subsets, then $C(X)$ is a Hilbert cube [7] and X has the disjoint arcs property (see [11]). It means that continuous mappings from at most 1-dimensional compacta to X can be approximated by embeddings (see [6, p. 40]). This gives the possibility to use the method of [10] to show that $L(X)$ is contained in a σZ-set in $C(X)$.

We do not know whether $L(X)$ is $F_{\sigma\delta}$-strongly universal for locally connected X with no local separating points and no planar nonempty open subsets (even in the simplest case of $X = M^n$).

By a *Warsaw circle* we mean any continuum of the form $X = A \cup B$, where A is a compactification of $[0, 1)$ with the remainder a nondegenerate arc α and B is an arc with endpoints p, q such that p is an endpoint of α, $q = 0$ (i.e., q is an endpo nt of $[0, 1)$ in the compactification A) and $A \cap B = \{p, q\}$.

Theorem 2.

(1) $\mathbf{S}(X)$ *is $F_{\sigma\delta}$ for any continuum X.*

(2) *If $\mathscr{K} \subset C(X)$ contains $\mathbf{S}(X)$ and no Warsaw circle belongs to $\mathscr{K}$, then $\hat{c}_0$ embeds in $\mathscr{K}$ as a closed subset for any continuum X containing $[-1, 1]^2$. In particular, $\mathscr{K}$ is not $G_{\delta\sigma}$.*

(3) *If X is a locally connected continuum with no local separating point and no planar nonempty open subset, then* $\mathbf{S}(X)$ *is contained in a* σZ*-set in* $C(X)$.

Proof. In order to prove (1) recall the notion of a circle-like subcontinuum: a subcontinuum $Y \subset X$ is *circle-like* if, for every $\varepsilon > 0$, Y can be covered by a finite circular chain of open subsets of X of diameters less than ε. It is known that Y is a simple closed curve if and only if it is locally connected and circle-like.

Put

$$U_m = \{C \in C(X) : C \text{ is covered by a finite circular chain of open subsets of } X \text{ of diameters less than } \tfrac{1}{m}\}.$$

Since U_m are open subsets of $C(X)$, the family $Ci(X) = \bigcap_m U_m$ of all circle-like subcontinua of X is a G_δ-set in $C(X)$. Hence $\mathbf{S}(X) = L(X) \cap Ci(X)$ is $F_{\sigma\delta}$.

To prove (2), let $\mathbf{x} = (x_i) \in Q$. Denote by $A_i(x_i)$ the segment from the point $(\frac{1}{i}, -1)$ to $(\frac{1}{i+1}, |x_i| - 1)$ in R^2 and put $B_i = \{\frac{1}{i+1}\} \times [-1, |x_i| - 1]$. Define an embedding $f : Q \to C([-1, 1]^2) \subset C(X)$ by

$$f(\mathbf{x}) = (\{0\} \times [-1, 1]) \cup \bigcup_{i \in \Gamma^+(\mathbf{x})} (A_{2i-1} \cup B_{2i-1}) \cup \bigcup_{i \in \Gamma^-(\mathbf{x})} (A_{2i} \cup B_{2i}) \cup ([0, 1] \times \{1\}) \cup (\{1\} \times [-1, 1]).$$

Observe that $f(\mathbf{x})$ is a Warsaw circle if and only if $\mathbf{x} \notin \hat{c}_0$ which means that $f(\hat{c}_0) = f(Q) \cap \mathcal{K}$.

Part (3) follows from Corollary 1.

Let us recall that the space $\mathbf{I}([-1, 1]^2)$ is an $F_{\sigma\delta}$-absorber in $C([-1, 1]^2)$, so it is homeomorphic to $\hat{c}_0$ [3]. It would be interesting to characterize $\mathbf{S}([-1, 1]^n)$.

By similar methods one can easily establish the following proposition.

Proposition 1. *Let a continuum X contain a copy of $[-1, 1]^n$, $n > 1$. Then each of the spaces $\mathbf{D}^n(X)$, $\mathbf{M}^n_{n-1}(X)$ and, for $n = 2k + 1 > 2$, $\mathbf{M}^k(X)$ contains a topological copy of $\hat{c}_0$ as a closed subset. In particular, the spaces are not $G_{\delta\sigma}$'s.*

Proof. The proof is very similar to that of Theorem 1. Concerning $\mathbf{D}^n(X)$, we take the harmonic comb C_n of n-disks which plays the role of the harmonic comb C_B considered in that proof.

In case of the other two spaces, we construct, for every $\mathbf{x} = (x_i) \in Q$, continua which look like combs built up from copies of M^n_{n-1} or M^n, respectively. More precisely, let Z_0 be the standard geometric model of M^n_{n-1} located in the cube $[-1, 0]^n$ (as constructed, e.g., in [9]). For $i > 0$, let Z_{2i-1} be such a model constructed in $[s_{2i-1}, \frac{1}{2i-1}] \times [0, x_i] \times [-1, 0]^{n-2}$, if $x_i > 0$, whereas Z_{2i} is a standard model of M^n_{n-1} in $[s_{2i-1}, \frac{1}{2i}] \times [0, |x_i|] \times [-1, 0]^{n-2}$, if $x_i < 0$. If $x_i = 0$, then we put $Z_{2i-1} = Z_{2i} = \emptyset$. Define an embedding $f : Q \to C(X)$ by

$$f(\mathbf{x}) = \bigcup_{i=0}^{\infty} Z_i \cup (\{0\} \times [0, 1] \times [-1, 0]^{n-2}).$$

It follows by known characterizations of M^n_{n-1} ([2] and [13]; see also [6]) that the countable union of Z_i's, $i = 0, 1, \ldots$, is homeomorphic to M^n_{n-1}, provided that the spaces Z_i's form a null-sequence. Thus, we again have $f(\hat{c}_0) = f(Q) \cap \mathbf{M}^{\mathbf{n}}_{\mathbf{n-1}}(X)$.

In the case of $\mathbf{M}^{\mathbf{k}}(X)$, $n = 2k + 1 > 2$, we define an embedding f similarly, replacing M^n_{n-1} by M^k in the construction above. The property $f(\hat{c}_0) = f(Q) \cap \mathbf{M}^{\mathbf{k}}(X)$ holds true because an appropriate characterization of M^k exists [1].

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