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1-Improvable Discontinuous Functions

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1. Preliminaries

The word "function" will mean a bounded real function of a real variable. We consider functions f defined on a non-empty (metric subspace) $D \subset \mathbb{R}$. If $x \in D$ is an isolated point of D, we put $\lim_{t \to x} f(t) := f(x)$.

Definition 1. *For each function $f: D \to \mathbb{R}$, we denote*

$$C(f) = \left\{x \in D; \lim_{t \to x} f(t) = f(x)\right\};$$

$$U(f) = \left\{x \in D; \lim_{t \to x} f(t) \neq f(x)\right\};$$

$$L(f) = \left\{x \in D; \text{ there exists } \lim_{t \to x} f(t)\right\};$$

Definition 2. *A point $x_0 \in U(f)$ is called an improvable point of discontinuity of the function f.*

It is easy to see the following fact:

Remark 1. *Let $f: D \to \mathbb{R}$. Then $U(f) \cap C(f) = \emptyset$ and $L(f) = U(f) \cup C(f)$.*

The following proposition is well known (compare to [1]).

Proposition 1. *The set $U(f)$ is countable.*

We define the function $f_{(1)}$ as follows:

$$f_{(1)}(x) = \begin{cases} f(x) & \text{if } x \notin U(f), \\ \lim_{t \to x} f(t) & \text{if } x \in U(f). \end{cases}$$

The following easy remark will be very useful in the paper.

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Remark 2. *Let* $f : D \to \mathbb{R}$. *Then*

(i) $\{x \in D;\ f_{(1)}(x) \neq f(x)\} = U(f)$,

(ii) *if* $x \in L(f)$, *then* $\lim_{t \to x} f(t) = f_{(1)}(x)$,

(iii) $L(f) \subset C(f_{(1)})$.

Definition 3. *We denote*

$$\mathscr{A}_1 = \{f : D \to \mathbb{R};\ C(f_{(1)}) = D\}.$$

Of course, all continuous functions defined on D are in $\mathscr{A}_1$.

Definition 5. *Let* $f : D \to \mathbb{R}$. *For each interval* $I = (a, b) \cap D \neq \emptyset$, *the quantity* $\omega(f, I) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x)$ *is called the oscillation of* f *on* I. *For each fixed* $x \in D$, *the function* $\omega(f, (x - \delta, x + \delta) \cap D)$ *decreases with* $\delta > 0$ *and approaches a limit* $\omega(f, x) = \lim_{\delta \to 0} \omega(f, (x - \delta, x + \delta) \cap D)$ *called the oscillation of* f *at* x.

Theorem 1. *Let* $D \subset \mathbb{R}$ *be closed and* $f : D \to \mathbb{R}$. *If* $C(f_{(1)}) = D$, *then the set* $C(f)$ *is a dense subset.*

Proof. Suppose that $C(f)$ is not a dense subset of D. Then there is an open interval (a, b) such that $(a, b) \cap D \neq \emptyset$ and $(a, b) \cap D \cap C(f) = \emptyset$. Thus $(a, b) \cap D \subset \bigcup_{n=1}^{\infty} \{x \in D;\ \omega(f, x) \geq \frac{1}{n}\}$. Since $(a, b) \cap D$ is the set of the second category in $[a, b] \cap D$, and $\{x \in D;\ \omega(f, x) \geq \frac{1}{n}\}$ is closed, there exists an positive integer n_0 and an open interval (c, d) such that $(c, d) \cap D \neq \emptyset$ and $(c, d) \cap D \subset \{x \in D;\ \omega(f, x) \geq \frac{1}{n_0}\}$. Thus $(c, d) \cap D \subset D \setminus L(f)$. Since $U(f) \subset L(f)$, $C(f_{(1)}) \cap (c, d) = \emptyset$, a contradiction.

Definition 2. *Let* $K \subset D$. *We shall denote*

$$K^d = \{x \in D;\ x \text{ is an accumulation point of } K \text{ in } D\}$$

and $K^* = K \setminus K^d$.

Definition 7. *For* $A \subset D \subset \mathbb{R}$, *let*

$$\mathscr{M}(A) = \{f : D \to \mathbb{R};\ f(A) = \{0\} \text{ and, for each } x \in D, f(x) \geq 0\}.$$

The following auxiliary theorem is not difficult to prove.

Theorem 2. *Let* $D \subset \mathbb{R}$, *let* A *be a dense subset of* D *and let* f *be 1-improvable function on* D *such that* $C(f) = A$. *Then* $g = |f - f_{(1)}| \in \mathscr{M}(A)$, $U(f) = U(g)$, $C(g) = A$ *and* g *is 1-improvable.*

2. 1-Improvable discontinuous functions

First, we shall give examples of discontinuous functions defined on $\mathbb{R}$ and belonging to the class $\mathscr{A}_1$ and one example of a function which does not belong to this class.

Example 1. Let $W = \{1/n; n \in \mathbb{N}\}$ and let f be the characteric function of the set W. Then $U(f) = W$, and 0 is not an improvable point of discontinuity of f. Note that $f_{(1)}(x) = 0$ for each $x \in \mathbb{R}$, so $f \in \mathscr{A}_1$. Observe that $f_{(1)}$ is a continuous function also at the point, which does not belong to the set $U(f)$.

Example 2. Let $K \subset [0, 1]$ be the Cantor set. Let K_1 be the set of all midpoints of all contiguous intervals of the Cantor set. Let h be the characteristic function of K_1. Observe that $U(h) = K_1$, and no point of K is an improvable point of discontinuity of h, but $h_{(1)}(x) = 0$ for each $x \in \mathbb{R}$, so $h \in \mathscr{A}_1$.

Example 3. Let W be as in Example 1. Let g be the characteristic function of $W \cup \{0\}$. Note now that $U(g) = W$, and 0 is not an improvable point of discontinuity of g, but $g_{(1)}$ is the characteristic function of $\{0\}$. The function g does not belong to $\mathscr{A}_1$ because $g_{(1)}$ is not continuous at the point 0.

Now, we establish necessary and sufficient conditions under which A is the set of all points of continuity of some 1-improvable discontinuous function. First, we give the conditions when A is an open subset of a complete space D and, next, when A is a $\mathscr{G}_\delta$ subset of a complete space D.

Lemma 1. *Let $D \subset \mathbb{R}$ be a closed set, $A \subset D$ be open in D and let $f \in \mathscr{M}(A) \cap \mathscr{A}_1$ be a function such that $C(f) = A$. Then $F^* = F \setminus F^d$ is dense in F, where $F = D \setminus A$.*

Proof. If $A = D$, then $f(x) = 0$ for each $x \in D$, and $F = \emptyset$. Assume that $A \neq D$ and let f fulfils the assumptions. Since, by Theorem 1, A is a dense subset of D and $f \in \mathscr{M}(A)$, we have that, for each $x \in F$, $\liminf_{t \to x} f(t) = 0$. Therefore, $U(f) = \{x \in D;\ f(x) > 0\}$ and we conclude that, for each $x \in D \setminus (U(f) \cup A)$, $f(x) = 0$ and $\limsup_{t \to x} f(t) > 0$.

We suppose that $cl\{x \in F; f(x) > 0\} \neq F$. Then there exists an open interval (a, b) such that $(a, b) \cap F \neq \emptyset$ and $(a, b) \cap F \cap \{x \in F; f(x) > 0\} = \emptyset$. Therefore, for each $x \in (a, b) \cap D, f(x) = 0$ and $(a, b) \cap D \subset C(f) = A$. This is impossible because $F \cap (a, b) \neq \emptyset$. Thus $cl\{x \in F; f(x) > 0\} = F$.

Suppose now that F^* is not a dense subset of F. Then there exists a closed interval $[a, b]$ such that $F^* \cap [a, b] = \emptyset$ and $F \cap (a, b) \neq \emptyset$. We may assume that $f(a) > 0$ and $f(b) > 0$. Let, for each $n \in \mathbb{N}$, $F_n = \{x \in [a, b];\ f(x) \geq \frac{1}{n}\}$. We claim that $F \cap [a, b] = \bigcup_{n=1}^{\infty} clF_n$. Let $x_0 \in F \cap [a, b]$. If $f(x_0) > 0$, then there exists $n \in \mathbb{N}$ such that $f(x_0) \geq \frac{1}{n}$ and $x_0 \in clF_n$. If $f(x_0) = 0$, then $x_0 \in (a, b) \cap F$ and $\limsup_{x \to x_0} f(x) > 0$. Then there exist $n \in \mathbb{N}$ and $(x_k)_{k=1}^{\infty} \subset D \cap (a, b)$, such that $\lim_{k \to \infty} x_k = x_0$ and, for each $k \in \mathbb{N}$, $f(x_k) \geq \frac{1}{n}$. Thus $(x_k)_{k=1}^{\infty} \subset F_n$, and $x_0 \in clF_n$.

Since $F \cap [a, b]$ is closed, it follows that there exist an open interval $(c, d) \subset (a, b)$ and $n_0 \in \mathbb{N}$, such that

$$(c, d) \cap F \neq \emptyset \quad \text{and} \quad (c, d) \cap F \subset (c, d) \cap clF_{n_0}.$$

Therefore, for each $x \in (c, d) \cap F$,

$$\limsup_{t \to x} f(t) \geq \frac{1}{n_0} \quad \text{and} \quad (c, d) \cap F \subset D \setminus (A \cup U(f)).$$

Since $(c, d) \cap F \neq \emptyset$, there exists $x_0 \in (c, d) \cap F$ such that $f(x_0) > 0$. Therefore, $x_0 \in U(f)$, a contradiction.

Theorem 3. *Let A be an open subset of a complete space D. Then the following conditions are equivalent:*
(1) there exists a function $f \in \mathscr{A}_1 \cap \mathscr{M}(A)$ such that $C(f) = A$;
(2) $clA = D$ and if $F = D \setminus A$, then the set F^* is dense in F.

Proof. First we assume that $f: D \to \mathbb{R}$ satisfies condition (1). Thus the function f satisfies the assumptions of Lemma 1, so the set F^* is dense in F. Additionally, by Theorem 1, $clA = D$.

Now, we assume that condition (2) holds. If $F = \emptyset$, then we can put $f = 0$ on D.

Assume that $F \neq \emptyset$. Let f be the characteristic function of the set F^*. Since $D \setminus F$ is dense in D, we have that, for each $x \in F$, $\liminf_{t \to x} f(t) = 0$. Clearly $A \subset C(f)$.

Let $x_0 \in F$. We shall consider two cases:

1. $x_0 \in F^*$.
   Since x_0 is an isolated point of F, $\limsup_{x \to x_0} f(x) = 0$ and $f(x_0) = 1$. Therefore $x_0 \in U(f)$ and $x_0 \notin C(f)$.
2. $x_0 \in F^d$.
   Since F^* is dense in F, there exist $(x_n)_{n=1}^{\infty} \subset F^*$ such that

$$\lim_{n \to \infty} x_n = x_0 \quad \text{and} \quad \lim_{n \to \infty} f(x_n) = 1.$$

   Therefore $\limsup_{x \to x_0} f(x) > 0$ and $x_0 \notin U(f) \cup C(f)$.

Thus $C(f) = A$. Since $U(f) = F^*$, we obtain a function $f_{(1)}: D \to \mathbb{R}$ such that $f_{(1)}(x) = 0$ for each $x \in D$, so $C(f_{(1)}) = D$. Hence the function f satisfies condition (1) and the proof is completed.

Theorem 4. *Let A be an open subset of a complete space D. Then the following conditions are equivalent:*
(3) there exists a function $f \in \mathscr{A}_1$ such that $C(f) = A$;
(4) $clA = D$ and if $F = D \setminus A$, then the set F^* is dense in F.

Proof. Assume that condition (3) holds. Then, by Theorems 1 and 2, we have that there exists a function $g \in \mathscr{M}(A) \cap \mathscr{A}_1$ such that $C(g) = A$. Thus, by Theorem 3, we have condition (4). The reverse implication is obvious.

Lemma 2. *Let $A \subset D$, where D is a complete subspace of $\mathbb{R}$. The following condition are equivalent:*

(5) there exists a function $f \in \mathcal{M}(A) \cap \mathcal{A}_1$ such that $C(f) = A$;

(6) $clA = D$ and there exists an ascending sequence of closed sets $(A_n)_{n=1}^{\infty}$ such that

$$D \setminus A = \bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n=1}^{\infty} A_n^* \cap \bigcup_{n=1}^{\infty} A_n^d = \emptyset \quad \text{and} \quad D \setminus A \subset \bigcup_{n=1}^{\infty} clA_n^*.$$

Proof. First, we assume that $f \in \mathcal{M}(A) \cap \mathcal{A}_1$, where $A = C(f)$. Then, by Theorem 1, $clA = D$. Thus we have that

$$D \setminus A \subset \left\{x \in D; \liminf_{t \to x} f(t) = 0\right\}$$

$$A = \left\{x \in D; \lim_{t \to x} f(t) = 0 \text{ and } f(x) = 0\right\}$$

$$U(f) = \left\{x \in D; \lim_{t \to x} f(t) = 0 \text{ and } f(x) > 0\right\}$$

$$D \setminus U(f) = \{x \in D; f(x) = 0\}$$

Let $A_n = cl\{x \in D \setminus A; f(x) \geq \frac{1}{n}\}$, for each $n \in \mathbb{N}$. We observe that $x_0 \in D \setminus A$ if and only if there exists $n \in \mathbb{N}$ such that $f(x_0) \geq \frac{1}{n}$ or $\limsup_{t \to x_0} f(t) > \frac{1}{n}$. Therefore $D \setminus A = \bigcup_{n=1}^{\infty} A_n$.

We suppose that there exists $x_0 \in \bigcup_{n=1}^{\infty} A_n^* \cap \bigcup_{n=1}^{\infty} A_n^d \neq \emptyset$. Let $n_1, n_2 \in \mathbb{N}$ be such that $x_0 \in A_{n_1}^*$ and $x_0 \in A_{n_2}^d$. Then

$$f(x_0) \geq \frac{1}{n_1} \quad \text{and} \quad \limsup_{t \to x_1} f(t) \geq \frac{1}{n_2}.$$

Since $x_0 \in D$, we have that $x_0 \in U(f)$ and $\limsup_{t \to x_0} f(t) > 0$, a contradiction. Therefore $\bigcup_{n=1}^{\infty} A_n^* \cap \bigcup_{n=1}^{\infty} A_n^d = \emptyset$.

Let $x_0 \in D \setminus A$. Then $\limsup_{t \to x_0} f_{|U(f)}(t) > 0$ or $x_0 \in U(f)$. Thus

$$x_0 \in \bigcup_{n=1}^{\infty} clA_n^* \quad \text{or} \quad x_0 \in \bigcup_{n=1}^{\infty} A_n^*.$$

Hence $D \setminus A \subset \bigcup_{n=1}^{\infty} clA_n^*$. Thus condition (6) holds.

Now, we assume that condition (6) is satisfied. If $A = D$, then we can put $f = 0$ on D. Assume that $A \neq D$. Let

$$f(x) = \begin{cases} 0 & \text{if} \quad \{m \in \mathbb{N};\ x \in A_m^*\} = \emptyset, \\ 1/n & \text{if} \quad x \in A_n^* \\ & \text{where } n = \min\{m \in \mathbb{N};\ x \in A_m^*\}. \end{cases}$$

Since $\bigcup_{n=1}^{\infty} A_n^* \cap \bigcup_{n=1}^{\infty} A_n^d = \emptyset$, $D \setminus A = \bigcup_{n=1}^{\infty} A_n$ and $D \setminus A \subset \bigcup_{n=1}^{\infty} clA_n^*$, we have that D is the following union of three disjoint sets

$$D = A \cup \bigcup_{n=1}^{\infty} A_n^* \cup \bigcup_{n=1}^{\infty} A_n^d,$$

$\bigcup_{n=1}^{\infty} A_n^d \subset D \setminus A \subset \bigcup_{n=1}^{\infty} cl A_n^* = \bigcup_{n=1}^{\infty} A_n^* \cup \bigcup_{n=1}^{\infty}(A_n^*)^d$ and $\bigcup_{n=1}^{\infty} A_n^d \subset \bigcup_{n=1}^{\infty}(A_n^*)^d$. Thus $\bigcup_{n=1}^{\infty}(A_n^*)^d = \bigcup_{n=1}^{\infty} A_n^d$. We observe that

$$\{x \in D;\ f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n^*$$

and $\{x \in D;\ \limsup_{t \to x} f(t) > 0\} = \bigcup_{n=1}^{\infty}(A_n^*)^d = D \setminus (A \cup \bigcup_{n=1}^{\infty} A_n^*)$.

Therefore $A = \{x \in D;\ \lim_{t \to x} f(t) = 0$ and $f(x) = 0\}$ and

$$\bigcup_{n=1}^{\infty} A_n^* = \left\{x \in D;\ \lim_{t \to x} f(t) = 0 \text{ and } f(x) > 0\right\}.$$

Now, we know that

$$\bigcup_{n=1}^{\infty} A_n^d = \left\{x \in D;\ \limsup_{t \to x} f(t) > 0 \text{ and } f(x) = 0\right\}.$$

Hence $C(f) = A$ and $U(f) = \bigcup_{n=1}^{\infty} A_n^*$. Therefore, for each $x \in D$, $f_{(1)}(x) = 0$, and the proof is completed.

Theorem 5. *Let $A \subset D$, where D is a complete subspace of $\mathbb{R}$. Then the following conditions are equivalent:*

(7) there exists a function $f \in \mathscr{A}_1$ such that $C(f) = A$;

(8) $cl A = D$ and there exists an ascending sequence of closed sets $(A_n)_{n=1}^{\infty}$ such that

$$D \setminus A = \bigcup_{n=1}^{\infty} A_n,\quad \bigcup_{n=1}^{\infty} A_n^* \cap \bigcup_{n=1}^{\infty} A_n^d = \emptyset \quad \text{and} \quad D \setminus A \subset \bigcup_{n=1}^{\infty} cl A_n^*.$$

(9) $cl A = D$ and there exists a $\mathscr{G}_\delta$ set E such that $A \subset E$ and the set $C = E \setminus A$ is countable and dense in $D \setminus A$.

Proof. By Theorems 1 and 2, we may assume that $f \in \mathscr{A}_1 \cap \mathscr{M}(A)$. We observe that, by Lemma 2, conditions (7) and (8) are equivalent.

Put $C = \bigcup_{n=1}^{\infty} A_n^*$ and $E = A \cup C = D \setminus \bigcup_{n=1}^{\infty} A_n^d$. It is easy to see that the condition (8) implies (9).

Now, we assume that there exists a $\mathscr{G}_\delta$ set $E \supset A$ such that the set $C = E \setminus A$ is countable and dense in $D \setminus A$. Then $E = \bigcap_{n=1}^{\infty} E_n$where each of sets E_n is open in D.

Let $n \in \mathbb{N}$ and $E_n = D \cap U_n$, where $U_n = \bigcup_{n=1}^{\infty}(a_k^n, b_k^n)$ is an open subset of $\mathbb{R}$ and $((a_k^n, b_k^n))_{k=1}^{\infty}$ is the sequence of components of the set U_n.

We shall define three sets $P_1^{k,n}$, $P_2^{k,n}$, $P_3^{k,n}$ for each $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$.

If $a_k^n \notin cl(C \cap (a_k^n, b_k^n))$, then $P_1^{k,n} = \emptyset$, otherwise there exists

$(z_p^n)_{p=1}^{\infty} \subset C \cap (a_k^n, b_k^n)$ such that $\lim_{p\to\infty} z_p^n = a_k^n$, so we choose $P_1^{k,n} = \{z_p^n;\ p \in \mathbb{N}\}$.

If $b_k^n \notin cl(C \cap (a_k^n, b_k^n))$, then $P_2^{k,n} = \emptyset$, otherwise there exists $(z_p^n)_{p=1}^{\infty} \subset C \cap (a_k^n, b_k^n)$ such that $\lim_{p\to\infty} z_p^n = b_k^n$, so we choose $P_2^{k,n} = \{z_p^n;\ p \in \mathbb{N}\}$.

If $C \cap (a_k^n, b_k^n) = \emptyset$, then $P_3^{k,n} = \emptyset$, otherwise $P_3^{k,n} = \{z^n\}$ where $z^n \in C \cap (a_k^n, b_k^n)$.

Let $H_k^n = P_1^{k,n} \cup P_2^{k,n} \cup P_3^{k,n}$. Then $(H_k^n)^d \subset \{a_k^n, b_k^n\}$.

Put $F_n = \bigcup_{k=1}^{\infty} H_k^n$. Then, for each $k \in \mathbb{N}$, $F_n \cap (a_k^n, b_k^n) = H_k^n$.

We shall show that $F_n^d = D \setminus E_n$. Suppose that $x_0 \in F_n^d \cap E_n$. Since $x_0 \in E_n$, there exists $k \in \mathbb{N}$ such that $x_0 \in (a_k^n, b_k^n)$.

Thus $x_0 \in F_n^d \cap (a_k^n, b_k^n)$. Then there exists $(x_p)_{p=1}^{\infty} \subset F_n \cap (a_k^n, b_k^n) = H_k^n$ such that $\lim_{p\to\infty} x_p = x_0$. Therefore $x_0 \in (H_k^n)^d \subset \{a_k^n, b_k^n\}$, a contradiction.

Now, let $x_0 \in D \setminus E_n$. Then $x_0 \in D \setminus E$ and there exists a sequence $(x_p)_{p=1}^{\infty} \subset C \subset E_n \subset \bigcup_{k=1}^{\infty} (a_k^n, b_k^n)$ such that $\lim_{p\to\infty} x_p = x_0$. If there exist $p_0, k_0 \in \mathbb{N}$ such that, for each $p \geq p_0$, $x_p \in (a_{k_0}^n, b_{k_0}^n)$, then $x_0 \in cl(C \cap (a_{k_0}^n, b_{k_0}^n))$ and $x_0 = a_{k_0}^n$ or $x_0 = b_{k_0}^n$. Thus $x_0 \in (H_{k_0}^n)^d \subset F_n^d$. Otherwise, there exist subsequences $((a_{k_l}^n, b_{k_l}^n))_{l=1}^{\infty}$ and $(x_{p_l})_{l=1}^{\infty}$ such that, for each $l \in \mathbb{N}$, $x_{p_l} \in (a_{k_l}^n, b_{k_l}^n)$ and $\lim_{l\to\infty} x_{p_l} = x_0$. Therefore, for each $l \in \mathbb{N}$, $(a_{k_l}^n, b_{k_l}^n) \cap C \neq \emptyset$ and there exists $z_l \in F_n \cap (a_{k_l}^n, b_{k_l}^n) \neq \emptyset$. Then $x_0 = \lim_{l\to\infty} a_{k_l}^n = \lim_{l\to\infty} b_{k_l}^n = \lim_{l\to\infty} z_l^n$ and $x_0 \in F_n^d$. Thus $F_n^d = D \setminus E_n$.

We can suppose $C \neq \emptyset$, then we can write $C = \bigcup_{n=1}^{\infty} \{c_n\}$. For each $n \in \mathbb{N}$, let $B_n = cl\, F_n \cup \{c_n\}$. Then

$$D \setminus A = C \cup (D \setminus E) = C \cup \bigcup_{n=1}^{\infty} (D \setminus E_n) = C \cup \bigcup_{n=1}^{\infty} F_n^d = \bigcup_{n=1}^{\infty} B_n .$$

Since $B_n^d = F_n^d = D \setminus E_n$ and $F_n \cup \{c_n\} \subset E_n$, we have that

$$B_n^* = F_n \cup \{c_n\} \subset C .$$

Then $\bigcup_{n=1}^{\infty} B_n^d \subset D \setminus E$, $\bigcup_{n=1}^{\infty} B_n^* = C \subset E$ and $\bigcup_{n=1}^{\infty} B_n^d \cap \bigcup_{n=1}^{\infty} B_n^* = \emptyset$.

Let $x \in D \setminus A$. If $x \in C = \bigcup_{n=1}^{\infty} B_n^* \subset \bigcup_{n=1}^{\infty} cl\, B_n^*$ and if $x \in D \setminus E = \bigcup_{n=1}^{\infty} F_n^d$, then there exists $n \in \mathbb{N}$ such that

$$x \in F_n^d = (B_n^* \setminus \{c_n\})^d = (B_n^*)^d \subset cl\, B_n^* .$$

Thus $D \setminus A \subset \bigcup_{n=1}^{\infty} cl\, B_n^*$.

Let, for each $n \in \mathbb{N}$, $A_n = \bigcup_{k=1}^{n} B_k$. Then $(A_n)_{n=1}^{\infty}$ is ascending sequence of closed sets. We observe that, for each $n \in \mathbb{N}$, $A_n^d = (\bigcup_{k=1}^{n} B_k)^d = \bigcup_{k=1}^{n} B_k^d$. Therefore, for each $n \in \mathbb{N}$, $A_n^* = \bigcup_{k=1}^{n} B_n^*$. Hence $\bigcup_{n=1}^{\infty} A_n^d \cap \bigcup_{n=1}^{\infty} A_n^* = \emptyset$.

Since $B_n^* \subset A_n^*$, we have $D \setminus A \subset \bigcup_{n=1}^{\infty} cl\, A_n^*$. Since $D \setminus A = \bigcup_{n=1}^{\infty} A_n$, the proof of the theorem is completed.

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